Lie groups and representations, Fall 2009

Homework #5, due Wednesday, October 28.

1. Show that the multiplication map $m : G \times G \rightarrow G$ on a Lie group induces the addition map on its Lie algebra, $m_*(X, Y) = X + Y$, and the map of taking the inverse $g \mapsto g^{-1}$ induces the map $X \mapsto -X$.

2. Prove that the tangent bundle to a Lie group is trivial.

3. Let $L$ be the Lie algebra of a connected Lie group $G$.
   (a) Prove that $[X, Y] = 0$ if and only if
       $$\exp(tX) \exp(sY) = \exp(sY) \exp(tX) \quad \text{for all} \quad t, s \in \mathbb{R}.$$ 
       Here $X, Y \in L$.
   (b) Show that $L$ is abelian iff $G$ is.
   (c) Use (b) to classify connected abelian Lie groups and, more generally, abelian Lie groups (with finitely-many connected components).

4. Prove that $L$ is solvable iff there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset 0$ such that $L_{i+1}$ is an ideal of $L_i$ and the quotients $L_{i+1}/L_i$ are abelian.

5. Show that the sum of two nilpotent ideals is nilpotent and that any finite-dimensional Lie algebra has a unique maximal nilpotent ideal. Determine this ideal for the Lie algebra of upper-triangular $n \times n$ matrices.

6. Determine the adjoint action in the standard basis $e, h, f$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, compute the Killing form in this basis and find its signature.

7. Prove that $\mathfrak{sl}(n, \mathbb{R})$ is simple if $n > 1$, that is, nonabelian with no proper ideals.