**Lie groups and representations, Fall 2009**

**Homework #5, due Wednesday, October 28.**

1. Show that the multiplication map $m : G \times G \longrightarrow G$ on a Lie group induces the addition map on its Lie algebra, $m_*(X,Y) = X+Y$, and the map of taking the inverse $g \longmapsto g^{-1}$ induces the map $X \longmapsto -X$.

2. Prove that the tangent bundle to a Lie group is trivial.

3. Let $L$ be the Lie algebra of a connected Lie group $G$.
(a) Prove that $[X,Y] = 0$ if and only if

$$\exp(tX)\exp(sY) = \exp(sY)\exp(tX) \qquad \text{for all} \quad t, s \in \mathbb{R}.$$

Here $X, Y \in L$.

(b) Show that $L$ is abelian iff $G$ is.

(c) Use (b) to classify connected abelian Lie groups and, more generally, abelian Lie groups (with finitely-many connected components).

4. Prove that $L$ is solvable iff there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset 0$ such that $L_{i+1}$ is an ideal of $L_i$ and the quotients $L_{i+1}/L_i$ are abelian.

5. Show that the sum of two nilpotent ideals is nilpotent and that any finite-dimensional Lie algebra has a unique maximal nilpotent ideal. Determine this ideal for the Lie algebra of upper-triangular $n \times n$ matrices.

6. Determine the adjoint action in the standard basis $e, h, f$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, compute the Killing form in this basis and find its signature.

7. Prove that $\mathfrak{sl}(n,\mathbb{R})$ is simple if $n > 1$, that is, nonabelian with no proper ideals.