Representations of finite groups

Homework #1, due Monday, September 21.

1. Describe all automorphisms of $\mathbb{Z}_5$, the cyclic group of order 5. How many are there? Identify the group of automorphisms.

2. The general linear group $GL(n, \mathbb{R})$ acts on vectors in $\mathbb{R}^n$ (an invertible matrix $A \in GL(n, \mathbb{R})$ takes $x \in \mathbb{R}^n$ to $Ax$). Describe the orbits of this action.

3. The center of a ring $R$ consists of elements that commute with every element of $R$:

   $$Z(R) := \{ a \in R | ab = ba \ \forall b \in R \}.$$

   We proved in class that $Z(R)$ is a commutative unital ring.
   a) Show that the center of the ring $\text{Mat}_n(F)$ of $n \times n$ matrices with coefficients in a field $F$ consists of multiples of the identity matrix.
   b) Show that $Z(R_1 \times R_2) = Z(R_1) \times Z(R_2)$.

4. Let $I, J$ be left ideals in the ring $R$. Which of the following sets are left ideals in $R$?

   $$I \cap J, \ I + J, \ IJ, \ R \setminus I, \ RI, \ IR, \ I \setminus J.$$

   $IJ$ denotes the set of finite sums of products $ij$, over $i \in I, j \in J$.

5. Determine all $\mathbb{Z}$-module homomorphisms
   (a) from $\mathbb{Z}$ to $\mathbb{Z}/4$,
   (b) from $\mathbb{Z}/4$ to $\mathbb{Z}/3$,
   (c) from $\mathbb{Z}/4$ to $\mathbb{Z}/2 \times \mathbb{Z}/2$.
   For each of these homomorphisms describe its kernel and image.

6. Describe ideals of the ring $F[x]/(x^n)$, where $F$ is a field.