Representations of finite groups

Homework #1, due Monday, September 21.

1. Describe all automorphisms of $\mathbb{Z}_5$, the cyclic group of order 5. How many are there? Identify the group of automorphisms.

2. The general linear group $GL(n, \mathbb{R})$ acts on vectors in $\mathbb{R}^n$ (an invertible matrix $A \in GL(n, \mathbb{R})$ takes $x \in \mathbb{R}^n$ to Ax). Describe the orbits of this action.

3. The center of a ring R consists of elements that commute with every element of R:

$$Z(R) := \{a \in R | ab = ba \quad \forall b \in R\}.$$

We proved in class that $Z(R)$ is a commutative unital ring.
a) Show that the center of the ring $\mathrm{Mat}_n(F)$ of $n \times n$ matrices with coefficients in a field F consists of multiples of the identity matrix.
b) Show that $Z(R_1 \times R_2) = Z(R_1) \times Z(R_2)$.

4. Let I, J be left ideals in the ring R. Which of the following sets are left ideals in R?

$$I \cap J, \quad I + J, \quad IJ, \quad R \setminus I, \quad RI, \quad IR, \quad I \setminus J.$$

IJ denotes the set of finite sums of products ij, over $i \in I, j \in J$.

5. Determine all $\mathbb{Z}$-module homomorphisms
(a) from $\mathbb{Z}$ to $\mathbb{Z}/4$,
(b) from $\mathbb{Z}/4$ to $\mathbb{Z}/3$,
(c) from $\mathbb{Z}/4$ to $\mathbb{Z}/2 \times \mathbb{Z}/2$.
For each of these homomorphisms describe its kernel and image.

6. Describe ideals of the ring $F[x]/(x^n)$, where F is a field.