1. The Alexander polynomial in terms of homology, continued

Last time, we defined the Alexander ideal $E(A_K)$ to be the (principal) ideal generated by the maximal minors of a presentation matrix for the Alexander module $A_K$, and the Alexander polynomial to be a generator of this ideal. Here is another way to understand the Alexander polynomial. Recall that we have a Crowell exact sequence

$$0 \rightarrow H_1(X_\infty) \rightarrow A_K \xrightarrow{\partial_2} K \xrightarrow{\varepsilon_G} \mathbb{Z}[G^ab] \xrightarrow{\varepsilon[G^ab]} \mathbb{Z} \rightarrow 0$$

of $\mathbb{Z}[G^ab]$-modules, where $\theta_2$ is the homomorphism induced by $dg \mapsto \psi(g) - 1$ ($g \in G_K$, $\psi$ is the abelianization map) and $\varepsilon[G^ab](\sum_i a_i g_i) := \sum a_i$. We can separate this exact sequence into two short exact sequences, one of which is

$$0 \rightarrow H_1(X_\infty) \rightarrow A_K \rightarrow \ker \varepsilon[G^ab] \rightarrow 0.$$

**Exercise 1.1.** Show that $\ker \varepsilon[G^ab] \cong \mathbb{Z}[G^ab]$ as $\mathbb{Z}[G^ab]$-modules.

It follows that $\ker \varepsilon[G^ab]$ is a free $\mathbb{Z}[G^ab]$-module. We can define a section $\eta$ of $\theta_2$ by setting $\eta(\psi(g) - 1) = dg$ for some lift $g$ of $\psi(g)$ on a basis for $\ker \varepsilon[G^ab]$, and extending by linearity. Thus the short exact sequence $(\dagger)$ splits, that is, $A_K \cong H_1(X_\infty) \oplus \mathbb{Z}[G^ab]$ as $\mathbb{Z}[G^ab]$-modules.

From now on, we make use of the isomorphism $\Lambda := \mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[G^ab]$. From the direct sum $A_K \cong H_1(X_\infty) \oplus \Lambda$, we may assume that $Q_K = (Q_1 | 0)$ by some elementary operations if necessary, that is, $Q_1$ is a square presentation matrix for the $\Lambda$-module $H_1(X_\infty)$. Since this does not change the ideal generated by the maximal minors of $Q_K$, we have the following proposition.

**Proposition 1.2.** The Alexander ideal and Alexander polynomial are also given by $E(H_1(X_\infty)) = (\det(Q_1))$ and $\Delta_K(t) = \det(Q_1)$ respectively.

Moreover, since $\Lambda_\mathbb{Q} := \Lambda \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{Q}[t, t^{-1}]$ is a PID, we have a $\Lambda_\mathbb{Q}$-isomorphism

$$H_1(X_\infty) \otimes \mathbb{Q} \cong \bigoplus_{i=1}^s \Lambda_\mathbb{Q} / (f_i), \quad f_i \in \Lambda_\mathbb{Q}.\$$

Since a generator $\tau$ of $\text{Gal}(X_\infty / X_K)$ acts on the right hand side as multiplication by $t$, one thus obtains

$$\Delta_K(t) = f_1 \cdots f_s = \det(t \cdot \text{id} - \tau | H_1(X_\infty) \otimes \mathbb{Q}) \text{ mod } \Lambda_\mathbb{Q}^X.$$

2. Asymptotic formulas on the orders of the first homology groups of cyclic ramified coverings

The infinite cyclic covering $X_\infty$ and its first homology group are our main objects of study; however, the size of $X_\infty$, which allows for its richness of information, also makes it more difficult to study. Taking a leaf from the study of field extensions, we can attempt to simplify the problem by studying its finite subcoverings $X_n$ instead. The group $H_1(X_n)$ is still an infinite group since it has an infinite subgroup with generator given by the homology class of $\partial X_n$; however, we can remove this subgroup by filling in the tube enclosed by $\partial X_n$. This naturally leads us to consider the Fox completion $M_n$ of $X_n$. Recall that $M_n$ is constructed from $X_n$ by gluing a tube $V = D^2 \times S^1$ to $X_n$ along $\partial V$ and $\partial X_n$ in such a way that a meridian of $\partial V$ coincides with $na$.

We can summarize this discussion in the following diagram:

$$\begin{array}{cccc}
X_\infty & \downarrow & & \\
X_n & \subset & M_n & \\
X_K & \subset & S^3 & \\
\end{array}$$

The main result we shall prove is that $\#H_1(M_n)$ is finite and grows exponentially asymptotically; in fact, we shall have an explicit formula for the constant involved in terms of the Alexander polynomial of $K$. We can see an inkling of this in the following proposition.

**Proposition 2.1.** $H_1(M_n) \cong H_1(X_\infty) / (t^n - 1)H_1(X_\infty)$ for $n \geq 1$.  

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Proof. There is an exact sequence

\[ H_1(X_\infty) \xrightarrow{t^n-1} H_1(X_\infty) \rightarrow H_1(X_n) \rightarrow \mathbb{Z} \rightarrow 0 \]

that arises from the homology exact sequence associated to a short exact sequence of chain complexes

\[ 0 \rightarrow C_\ast(X_\infty) \xrightarrow{t^n-1} C_\ast(X_\infty) \rightarrow C_\ast(X_n) \rightarrow 0 \]

(the map \((t^n-1)_* : H_0(X_\infty) \rightarrow H_0(X_\infty)\) is zero). Hence

\[ H_1(X_n) \cong H_1(X_\infty)/(t^n-1)H_1(X_\infty) \oplus \mathbb{Z}. \]

Here \(1 \in \mathbb{Z}\) corresponds to a lift \(\tilde{\alpha^n}\) of \([\alpha^n]\) to \(X_n\). (Since the image of \(\alpha^n\) in \(\text{Gal}(X_n/X_K) \cong \mathbb{Z}/n\mathbb{Z}\) is 0, \(\alpha^n\) can be lifted to \(X_n\).) But \(H_1(X_n) \cong H_1(M_n) \oplus \langle [\tilde{\alpha^n}] \rangle\), so we obtain the assertion. \(\square\)

Note that by taking \(n = 1\) in Proposition 2.1, we get

\[ H_1(X_\infty)/(t-1)H_1(X_\infty) \cong H_1(M_1) = H_1(S^3) = 0, \]

that is, \(H_1(X_\infty)\) is a torsion \(\Lambda\)-module.

The following lemma, whose proof we omit, guarantees that the groups \(H_1(M_n)\) are finite under nice circumstances and gives us a way to compute them.

**Lemma 2.2.** Let \(N\) be a finitely generated, torsion \(\Lambda\)-module and suppose that \(E(N) = \langle \Delta \rangle\). Then, for any \(f(t) \in \mathbb{Z}[t]\), \(N/f(t)N\) is a finite abelian group if and only if \(\Delta(\xi) \neq 0\) for all nonzero roots \(\xi \in \overline{\mathbb{Q}}\) of \(f(t) = 0\). Moreover, if \(f(t)\) can be decomposed as \(\pm \prod_{j=1}^s (t - \xi_j)\), then

\[ |N/f(t)N| = \prod_{j=1}^s |\Delta(\xi_j)|. \]

Taking \(N = H_1(X_n)\), \(f(t) = t^n - 1\) and considering the Alexander polynomial as an integer polynomial with nonzero constant term, we see from Propositions 1.2 and 2.1 and Lemma 2.2 that if the equation \(\Delta_K(t) = 0\) does not have a root that is a root of unity, then all the first homology groups \(H_1(M_n)\) are finite and

\[ \#H_1(M_n) = \prod_{j=0}^{n-1} |\Delta_K(\zeta_n^j)|, \]

where \(\zeta_n\) is a primitive \(n\)th root of unity. In this case, it makes sense to talk about the rate of growth of \(\#H_1(M_n)\).

This turns out to be a function of the Mahler measure of the Alexander polynomial.

**Definition 2.3.** For a nonconstant polynomial \(g(t) \in \mathbb{R}[t]\), define the Mahler measure \(m(g)\) of \(g(t)\) by

\[ m(g) := \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |g(e^{i\theta})| \, d\theta \right). \]

A question arises: how can one compute \(m(g)\)? A method is given by Jensen’s formula in complex analysis.

**Exercise 2.4.** Show that if \(g(t)\) splits over \(\mathbb{C}\) as \(g(t) = c \prod_{i=1}^d (t - \xi_i)\), then \(m(g) = |c| \prod_{i=1}^d \max(|\xi_i|, 1)\). (Hint: Jensen’s formula states that if \(f\) is a holomorphic function with no zeroes on the circle \(|z| = r\), zeroes \(a_1, \ldots, a_k\) in the open disk \(|z| < r\) (and possibly other zeroes elsewhere), and \(f(0) \neq 0\), then

\[ \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta = \log |f(0)| + \sum_{j=1}^k (\log |r - \log |a_k||)\].

We are now ready to state the main theorem of this section.

**Theorem 2.5.** Assume that there is no root of \(\Delta_K(t) = 0\) that is a root of unity. Then

\[ \lim_{n \to \infty} \frac{1}{n} \log \#H_1(M_n) = \log m(\Delta_K). \]

That is, \(\#H_1(M_n)\) grows like \(m(\Delta_K)^n\).
Proof. From Equation (‡), we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \#H_1(M_n) = \lim_{n \to \infty} \frac{1}{n} \prod_{j=0}^{n-1} \left| \Delta_K \left( \zeta_n^j \right) \right|
\]
\[
= \int_0^1 \log \left| \Delta_K \left( e^{2\pi ix} \right) \right| \, dx
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \Delta_K \left( e^{i\theta} \right) \right| \, d\theta
\]
\[
= \log m(\Delta_K).
\]
\[\square\]

Example 2.6. Let \( K \) be the figure eight knot. In one of the exercises, we computed the Alexander polynomial of the figure eight knot to be
\[
\Delta_K(t) = t^2 - 3t + 1 = \left( t - \frac{3 + \sqrt{5}}{2} \right) \left( t - \frac{3 - \sqrt{5}}{2} \right).
\]
Hence, by Exercise 2.4, we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \#H_1(M_n) = \log m(\Delta_K) = \log \frac{3 + \sqrt{5}}{2}.
\]