1. THE ALEXANDER POLYNOMIAL IN TERMS OF HOMOLOGY, CONTINUED

Last time, we defined the Alexander ideal $E(A_K)$ to be the (principal) ideal generated by the maximal minors of a presentation matrix for the Alexander module A_K, and the Alexander polynomial to be a generator of this ideal. Here is another way to understand the Alexander polynomial. Recall that we have a Crowell exact sequence

$$0 \to H_1(X_\infty) \to A_K \xrightarrow{\theta_2} \mathbb{Z}[G_K^{\mathrm{ab}}] \xrightarrow{\epsilon_{\mathbb{Z}[G_K^{\mathrm{ab}}]}} \mathbb{Z} \to 0$$

of $\mathbb{Z}[G_K^{\mathrm{ab}}]$-modules, where θ_2 is the homomorphism induced by $dg \mapsto \psi(g) - 1$ ($g \in G_K$, ψ is the abelianization map) and $\epsilon_{\mathbb{Z}[G_K^{\mathrm{ab}}]}(\sum a_g g) := \sum a_g$. We can separate this exact sequence into two short exact sequences, one of which is

$$0 \to H_1(X_\infty) \to A_K \to \ker \epsilon_{\mathbb{Z}[G_K^{\mathrm{ab}}]} \to 0. \tag{†}$$

Exercise 1.1. Show that $\ker \epsilon_{\mathbb{Z}[G_K^{\mathrm{ab}}]} \cong \mathbb{Z}[G_K^{\mathrm{ab}}]$ as $\mathbb{Z}[G_K^{\mathrm{ab}}]$-modules.

It follows that $\ker \epsilon_{\mathbb{Z}[G_K^{\mathrm{ab}}]}$ is a free $\mathbb{Z}[G_K^{\mathrm{ab}}]$-module. We can define a section η of θ_2 by setting $\eta(\psi(g)-1) = dg$ for some lift g of $\psi(g)$ on a basis for $\ker \epsilon_{\mathbb{Z}[G_K^{\mathrm{ab}}]}$, and extending by linearity. Thus the short exact sequence (†) splits, that is, $A_K \cong H_1(X_\infty) \oplus \mathbb{Z}[G_K^{\mathrm{ab}}]$ as $\mathbb{Z}[G_K^{\mathrm{ab}}]$-modules.

From now on, we make use of the isomorphism $\Lambda := \mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[G_K^{\mathrm{ab}}]$. From the direct sum $A_K \cong H_1(X_\infty) \oplus \Lambda$, we may assume that $Q_K = (Q_1 \mid 0)$ by some elementary operations if necessary, that is, Q_1 is a square presentation matrix for the Λ-module $H_1(X_\infty)$. Since this does not change the ideal generated by the maximal minors of Q_K, we have the following proposition.

Proposition 1.2. *The Alexander ideal and Alexander polynomial are also given by* $E(H_1(X_\infty)) = \langle \det(Q_1) \rangle$ *and* $\Delta_K(t) = \det(Q_1)$ *respectively.*

Moreover, since $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[t, t^{-1}]$ is a PID, we have a $\Lambda_{\mathbb{Q}}$-isomorphism

$$H_1(X_\infty) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i=1}^{s} \Lambda_{\mathbb{Q}}/(f_i), \quad f_i \in \Lambda_{\mathbb{Q}}.$$

Since a generator τ of $\mathrm{Gal}(X_\infty/X_K)$ acts on the right hand side as multiplication by t, one thus obtains

$$\Delta_K(t) = f_1 \cdots f_s = \det(t \cdot \mathrm{id} - \tau \mid H_1(X_\infty) \otimes_{\mathbb{Z}} \mathbb{Q}) \bmod \Lambda_{\mathbb{Q}}^{\times}.$$

2. ASYMPTOTIC FORMULAS ON THE ORDERS OF THE FIRST HOMOLOGY GROUPS OF CYCLIC RAMIFIED COVERINGS

The infinite cyclic covering X_∞ and its first homology group are our main objects of study; however, the size of X_∞, which allows for its richness of information, also makes it more difficult to study. Taking a leaf from the study of field extensions, we can attempt to simplify the problem by studying its finite subcoverings X_n instead. The group $H_1(X_n)$ is still an infinite group since it has an infinite subgroup with generator given by the homology class of ∂X_n; however, we can remove this subgroup by filling in the tube enclosed by ∂X_n. This naturally leads us to consider the Fox completion M_n of X_n. Recall that M_n is constructed from X_n by gluing a tube $V = D^2 \times S^1$ to X_n along ∂V_n and ∂X_n in such a way that a meridian of ∂V coincides with $n\alpha$.

We can summarize this discussion in the following diagram:

$$\begin{array}{ccc} X_\infty & & \\ \downarrow & & \\ X_n & \subset & M_n \\ \downarrow & & \downarrow \\ X_K & \subset & S^3 \end{array}$$

The main result we shall prove is that $\#H_1(M_n)$ is finite and grows exponentially asymptotically; in fact, we shall have an explicit formula for the constant involved in terms of the Alexander polynomial of K. We can see an inkling of this in the following proposition.

Proposition 2.1. $H_1(M_n) \cong H_1(X_\infty)/(t^n - 1)H_1(X_\infty)$ *for* $n \geq 1$.

Proof. There is an exact sequence

$$H_1(X_\infty) \xrightarrow{t^n-1} H_1(X_\infty) \to H_1(X_n) \to \mathbb{Z} \to 0$$

that arises from the homology exact sequence associated to a short exact sequence of chain complexes

$$0 \to C_*(X_\infty) \xrightarrow{t^n-1} C_*(X_\infty) \to C_*(X_n) \to 0$$

(the map $(t^n-1)_* : H_0(X_\infty) \to H_0(X_\infty)$ is zero). Hence

$$H_1(X_n) \cong H_1(X_\infty)/(t^n-1)H_1(X_\infty) \oplus \mathbb{Z}.$$

Here $1 \in \mathbb{Z}$ corresponds to a lift $[\tilde{\alpha}^n]$ of $[\alpha^n]$ to X_n. (Since the image of α^n in $\mathrm{Gal}(X_n/X_K) \cong \mathbb{Z}/n\mathbb{Z}$ is 0, α^n can be lifted to X_n.) But $H_1(X_n) \cong H_1(M_n) \oplus \langle [\tilde{\alpha}^n] \rangle$, so we obtain the assertion. □

Note that by taking $n = 1$ in Proposition 2.1, we get

$$H_1(X_\infty)/(t-1)H_1(X_\infty) \cong H_1(M_1) = H_1(S^3) = 0,$$

that is, $H_1(X_\infty)$ is a torsion Λ-module.

The following lemma, whose proof we omit, guarantees that the groups $H_1(M_n)$ are finite under nice circumstances and gives us a way to compute them.

Lemma 2.2. *Let N be a finitely generated, torsion Λ-module and suppose that $E(N) = (\Delta)$. Then, for any $f(t) \in \mathbb{Z}[t]$, $N/f(t)N$ is a finite abelian group if and only if $\Delta(\xi) \neq 0$ for all nonzero roots $\xi \in \overline{\mathbb{Q}}$ of $f(t) = 0$. Moreover, if $f(t)$ can be decomposed as $\pm \prod_{j=1}^{s}(t - \xi_j)$, then*

$$\left| N/f(t)N \right| = \prod_{j=1}^{s} \left| \Delta(\xi_j) \right|.$$

Taking $N = H_1(X_n)$, $f(t) = t^n - 1$ and considering the Alexander polynomial as an integer polynomial with nonzero constant term, we see from Propositions 1.2 and 2.1 and Lemma 2.2 that if the equation $\Delta_K(t) = 0$ does not have a root that is a root of unity, then all the first homology groups $H_1(M_n)$ are finite and

$$\#H_1(M_n) = \prod_{j=0}^{n-1} \left| \Delta_K\left(\zeta_n^j\right) \right|, \tag{‡}$$

where ζ_n is a primitive nth root of unity. In this case, it makes sense to talk about the rate of growth of $\#H_1(M_n)$. This turns out to be a function of the Mahler measure of the Alexander polynomial.

Definition 2.3. For a nonconstant polynomial $g(t) \in \mathbb{R}[t]$, define the *Mahler measure* $m(g)$ of $g(t)$ by

$$m(g) := \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log\left| g\left(e^{i\theta}\right) \right| \, d\theta \right).$$

A question arises: how can one compute $m(g)$? A method is given by Jensen's formula in complex analysis.

Exercise 2.4. Show that if $g(t)$ splits over $\mathbb{C}$ as $g(t) = c\prod_{i=1}^{d}(t - \xi_i)$, then $m(g) = |c|\prod_{i=1}^{d}\max(\left|\xi_i\right|, 1)$. (Hint: Jensen's formula states that if f is a holomorphic function with no zeroes on the circle $|z| = r$, zeroes $a_1, \ldots, a_k$ in the open disk $|z| < r$ (and possibly other zeroes elsewhere), and $f(0) \neq 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log\left| f(re^{i\theta}) \right| \, d\theta = \log\left| f(0) \right| + \sum_{j=1}^{k} (\log r - \log\left| a_k \right|).)$$

We are now ready to state the main theorem of this section.

Theorem 2.5. *Assume that there is no root of $\Delta_K(t) = 0$ that is a root of unity. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \#H_1(M_n) = \log m(\Delta_K).$$

That is, $\#H_1(M_n)$ grows like $m(\Delta_K)^n$.

Proof. From Equation (‡), we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log \#H_1(M_n) &= \lim_{n\to\infty} \frac{1}{n} \prod_{j=0}^{n-1} \left| \Delta_K \left(\zeta_n^j \right) \right| \\
&= \int_0^1 \log \left| \Delta_K \left(e^{2\pi i x} \right) \right| \, dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \Delta_K \left(e^{i\theta} \right) \right| \, d\theta \\
&= \log m(\Delta_K).
\end{aligned}$$

□

Example 2.6. Let K be the figure eight knot. In one of the exercises, we computed the Alexander polynomial of the figure eight knot to be

$$\Delta_K(t) = t^2 - 3t + 1 = \left(t - \frac{3+\sqrt{5}}{2} \right) \left(t - \frac{3-\sqrt{5}}{2} \right).$$

Hence, by Exercise 2.4, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \#H_1(M_n) = \log m(\Delta_K) = \log \frac{3+\sqrt{5}}{2}.$$