Student GA Seminar: C^0-Limit Theorem for Scalar Curvature

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0 Plan

Several things I would like to discuss:

- Description of the C^0-limit theorem.
- Bamler's proof via Ricci flow.
- Elements of Gromov's proof:
  1. Reduction to Dihedral theorem for cube
  2. Proof of the Dihedral theorem
  3. Weak notion of non-negative scalar curvature
- Motivation of the Dihedral Rigidity conjecture
- Applications in General Relativity
- Chao Li's proof in dimension 3
- Some related questions

1 Motivation

This seminar aims to focus on "C^0-characterization" of geometric objects. To be more precise, we are interested in using only the information of the metric to describe geometric quantities that usually involves taking derivative of the metric.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Then f is convex if and only if $f'' > 0$, but convexity could be described only using the function itself, i.e.:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

We could also give some examples in geometry:

Example. (Convexity in Geometry) The boundary of a Riemannian manifold M^n is convex $\iff$ $\mathrm{I\!I} > 0$.

A slightly trickier example:

Example. (Mean-Convexity) Suppose $\Sigma \subset M^n$ is a hypersurface, Σ is mean convex ($H > 0$) with respect to the normal vector ν if any normal deformation supported in a small region increases the area.

Proof. This can be seen from the first variation formula:

$$\frac{d}{dt}|_{t=0} \operatorname{Area}(\Sigma_t) = -\int_\Sigma H\langle X, \nu\rangle$$

□

So a natural question is to ask is if intrinsic curvature lower bounds can also be described via C^0 information. Here are something we known:

- Sectional curvature comparison: Topogonov Triangle comparison theorem.
- Ricci curvature lower bound: Cheeger-Colding-Naber Theory, Optimal transport methods via the convexity of entropy functional on $P(M)$, the space of probability measures on M with the Wasserstein metric.

Question. What about $\mathrm{scal} \geq \kappa$?

Note that the above two example makes sense since we were able to take limit (in some weak sense) of manifolds with curvature bounded below (i.e. the theory of Alexandrov spaces and RCD spaces), such that the limit space (which is in general singular) carries a structure of the curvature lower bound.
Therefore, in order to make the above question even make sense, one needs to look at what will happen if we take limit of a sequence of spaces with uniform scalar curvature lower bound. The first positive result is answered by the C^0-limit theorem from Gromov:

Theorem 1.1. (Gromov 2014[Gro14], Bamler 2015[Bam15]) Let M be a smooth manifold and $\kappa : M \to \mathbb{R}$ a continuous function. Suppose g_i is a sequence of C^2 metrics on M that converges to a C^2 Riemannian metric g in the C^0 topology. If $\mathrm{scal}_{g_i} \geq \kappa$ for all i, then $\mathrm{scal}_g \geq \kappa$.

2 Bamler's Ricci Flow Proof

We introduce Bamler's Ricci flow proof via by Ricci flow. The idea is to use the property of heat equation to smooth out everything and approximate it back. We first review some basic elements from Ricci flow:

- The Ricci flow equation is given by

$$\partial_t g_t = -2\mathrm{Ric}(g_t)$$

  But in our case, we are more interested in the Ricci-Deturck equation since it is strictly parabolic. We recall the construction: Fix a background Riemannian metric $\bar{g}$ and consider a family of diffeomorphism φ_t evolved under the harmonic map flow:

$$\partial_t \varphi_t = \Delta_{\tilde{g}_t, \bar{g}} \varphi_t$$

  where $\tilde{g}_t$ is evolved under Ricci flow. Then if we define g_t by $\varphi_t^*(g_t) = \tilde{g}_t$, then

$$\partial_t g_t = -2\mathrm{Ric}(g_t) - \mathcal{L}_{X_{\bar{g}}(g_t)}(g_t)$$

  where $X_{\bar{g}}(g_t)$ is the vector field generated by φ, i.e. $X_{\bar{g}}(g_t) := \Delta_{g(t),\bar{g}} \mathrm{Id}$. The Ricci-Deturck flow is strictly parabolic.

- Evolution Equation of scalar curvature under Ricci flow:

$$\partial_t \mathrm{scal}_{g_t} = \Delta \mathrm{scal}_{g_t} + 2|\mathrm{Ric}(g_t)|^2$$

  and the evolution under Ricci-Deturck flow:

$$\partial_t \mathrm{scal}(g_t) = \Delta \mathrm{scal}(g_t) + 2|\mathrm{Ric}(g_t)|^2 - \nabla_{X_{\bar{g}}(g_t)} \mathrm{scal}(g_t)$$

- The heat kernel of the Ricci flow is denoted as by $\tilde{K}(x,t;y,s)$, which is defined as

$$\partial_t \tilde{K} = \Delta_{\tilde{g}_t, x} \tilde{K}$$

  where $\tilde{g}_t$ is evolved under Ricci flow, with the normalization condition $\tilde{K}(\cdot,t;y,s)$ the δ-function as $t \searrow s$. For (x,t) fixed, $\tilde{K}(x,t;\cdot,\cdot)$ solves the conjugate heat equation

$$-\partial_s \tilde{K} = \Delta_{\tilde{g}_s, y} \tilde{K} - \mathrm{scal}_{\tilde{g}_s}(y) \tilde{K}$$

  The heat kernel for the Ricci-Deturck flow is then simply defined by

$$K(x,t;y,s) := \tilde{K}(\varphi^{-1}(x), t; \varphi^{-1}(y), s)$$

  which satisfies

$$\partial_t K = \Delta_{g_t, x} K - \nabla_{X_{\bar{g}}(g_t)} K$$

  and

$$-\partial_s K = \Delta_{g_s, y} K - \mathrm{scal}_{g_s}(y) K + \nabla_{X_{\bar{g}}(g_s)} K$$

Next we discuss the main steps of the proof here. From now on, we will focus on the case where M is $\mathbb{R}^n$ and $\bar{g} = \delta_{ij}$, since M is locally Euclidean and we are considering a local problem.

Step 1: A result due to Koch-Lamm: It states that metrics that are sufficiently close to the Euclidean metric in the C^0-sense can be evolved by the Ricci DeTurck flow on a uniform time-interval. This flow becomes instantly smooth and depends continuously on the initial data.

Step 2: On a small time-interval and g C^0 close to the δ_{ij}, the Ricci-Deturck heat kernel $K(x, t; y, s)$ can be bounded from above by a standard Gaussian, and one can show

$$\int_{\mathbb{R}^n \setminus B(x,r)} K(x, t; y, s) dg_s(y) < C_1 e^{-\frac{r^2}{C_2(t-s)}}$$

Step 3: Since the scalar curvature is a supersolution for the associated heat equation on a Ricci DeTurck flow, we know that for any $x \in \mathbb{R}^n$ and any $0 < s < t$

$$\mathrm{scal}(x, t) \geq \int_{\mathbb{R}^n} K(x, t; y, s) \mathrm{scal}(y, s) dg_s(y)$$

From this and the inequality in Step 2, one can choose a sequence of $r_j \to 0$ and $t_j \to 1$ wisely to have a uniform estimate on $\mathrm{scal}_g(0, t)$ for any t in terms of the lower bound of $\mathrm{scal}_g(0, 1)$.

Step 4: Now suppose we are in the situation of the theorem, where a sequence of C^2 functions g_i converges to g in C^0, with $\mathrm{scal}_{g_i} \geq \kappa$ and $\mathrm{scal}_g(0) < \kappa' < \kappa$. Using a perturbation and cutoff trick, one can assume g_i, g are close to the Euclidean metric in $B(0, 1)$ without hurting too much the scalar curvature lower bound. Now one can evolve under the Ricci flow for each g_i, and from step 3 we can know that

$$\mathrm{scal}_{g_{i,t}}(0) > \kappa - \delta$$

for $t \in [0, \tau(\delta)]$. Step one shows that taking the limit $i \to \infty$ we have

$$\mathrm{scal}_{g_t}(0) > \kappa - \delta$$

for $t \in (0, \tau]$. Using Step 1 again, taking the limit as $t \to 0$ gives the contradiction.

3 Gromov's Proof

Gromov's approach is based on the following observations:

1. Since the problem is local, one can assume that κ is constant.

2. One can moreover assume that $\kappa = 0$. Since if $\kappa = n(n-1)$, one can define $(\tilde{M}, \tilde{g})$ where $\tilde{M} = M \times \mathbb{R}^+$ and $\tilde{g} = t^2 g + dt^2$, then the problem reduces to the case for $\kappa = 0$ on $(\tilde{M}, \tilde{g})$.

3. Gromov shows that if a point x_0 has negative scalar curvature, then all sufficiently small neighbourhoods of x_0 contain (tiny) mean curvature convex cubical polyhedral domains $x_0 \in P$ with strictly acute (i.e. $< \frac{\pi}{2}$) dihedral angles. (Note that this is a C^0 property since mean convexity can be phrased using only C^0 information)

The last observation is crucial, in the sense that this gives a potential characterization of non-negative scalar curvature using only C^0 information, and the convergence theorem will follow from this. We rewrite Gromov's cubical comparison as follows

Theorem 3.1. Let $M = [0,1]^n$ with Riemannian metric g. The following cannot happen simultaneously:

- $\text{scal}_g > 0$ on M.
- Each faces of M is mean convex.
- Dihedral angle between faces is everywhere acute.

So now it suffices to show the above theorem. We also explain briefly about how Gromov approach this, suppose all of the three condition holds:

1. Along each faces one can consider a doubling procedure, which results in a cube such that every opposite faces are isometric.
2. One can glue the above cube into a torus $\mathbb{T}^n$ with a singular metric $\tilde{g}$, where $\tilde{g} > 0$ away from this singular set.
3. This singular set admits a natural stratification into different dimensions. The mean convexity of faces of M guarantees that the codimension one part will admit non-negative scalar curvature, the dihedral angle assumption guarantees the codimension 2 stratification to have non-negative scalar curvature. The rest will not affect the scalar curvature.
4. Therefore one can approximate $\tilde{g}$ by a metric that has non-negative scalar curvature. This will contradicts the Geroch conjecture on $\mathbb{T}^n$.

4 The Dihedral Rigidity Conjecture

The most important ingredient of Gromov's proof is the reduction to the characterization of $\text{scal} \geq 0$ in terms of cubes. So there are two natural following questions:

Question. (Polyhedral Comparison) Does the same theorem holds for other polyhedron?

Question. (Rigidity Conjecture) Does the rigidity situation hold? i.e. Suppose that M is a cube (or in general any polyhedron) with a Riemannian metric g such that

- $\text{scal} \geq 0$ on M.
- Each face of M is weakly mean convex.
- The dihedral angle between faces is not bigger than the model case. (The polyhedron in Euclidean space)

Then can we conclude that M is isometric to the model polyhedron?

Example. We examine the case where the dimension is 2. Suppose that (P, g) is a mean convex polygon with positive curvature and $(\tilde{P}, \delta_{ij})$ is the Euclidean model. We suppose α_j are the interior angle of P and $\tilde{\alpha}_j$ are the interior angle for $\tilde{P}$ with $\alpha_j \leq \tilde{\alpha}_j$ Then Gauss-Bonnet implies that

$$\int_P \frac{\text{scal}_g}{2} dg + \int_{\partial P} k_g + \sum (\pi - \alpha_j) = 2\pi$$

where k_g is the geodesic curvature (mean curvature). But we know that $\sum(\pi - \alpha_j) \leq \sum(\pi - \tilde{\alpha}_j) = 2\pi$ and hence

$$\int_P \frac{\text{scal}_g}{2} dg + \int_{\partial P} k_g = 0$$

Therefore P is flat and isometric to $\tilde{P}$.

Therefore, we are looking for a positive answer to this Dihedral Rigidity Conjecture in higher dimensions.

5 Relations to General Relativity

We also remark on some of the relations between Dihedral rigidity conjecture and mathematical general relativity.

Example (Quasi-Local Mass)**.** [ST03] Let $\Omega \subset \mathbb{R}^3$ be a convex domain with Euclidean metric, and (Ω_0, g) a Riemannian manifold with $\text{scal}_g \geq 0$ and $\partial\Omega_0$ isometric to $\partial\Omega$. Shi-Tam showed the theorem that

$$\int_\Omega H dA - \int_{\Omega_0} H_0 dA \geq 0$$

where H, H_0 are the corresponding mean curvatures of Ω, Ω_0. The quantity on the left hand side is called the Brown-York quasi-local mass. This suggests that quantities such as mass in general relativity might be detected in a way via polyhedron. Indeed, if we regard ∂M as a varifold, then the first variational formula is given by

$$\|\delta V\| := \int_{\partial M} H + \sum_{E = F_i \cap F_j} (\pi - \angle(F_1, F_2))|E|$$

which only evolves the faces and edges along the boundary but not lower dimensional stratification. Hence, this perfectly fits into the program of the Dihedral rigidity problem.

Example (ADM mass via Polyhedra)**.** [MP21] Miao and Piubellu are able to show for asymptotically flat manifold with dimension $n \geq 3$.

$$\mathbf{m}(g) = \lim_{P_k \to \infty} \frac{1}{(n-1)\omega_{n-1}} \left(-\int_{\mathcal{F}_k} H d\sigma + \int_{\mathcal{E}_k} (\alpha - \alpha_0) d\mu \right) + o(1)$$

where P_k is a sequence of polyhedron tends to infinity that satisfies some decay condition. Here $\mathcal{F}_k$ are the union of all faces in ∂P_k and $\mathcal{E}_k$ denotes the union of all the edges.

Example (Positive Mass Theorem)**.** [Li19] We explain that the dihedral rigidity problem can be realized as a local positive mass theorem:
Fix a Euclidean polyhedron P, we suppose the following conjecture is true

Conjecture. For any Riemannian polyhedron (M, g) admitting a degree one map onto P with non-negative scalar curvature, weakly mean convex faces and dihedral angle not larger than the angle of P. Then M is flat.

Then we show that this implies the positive mass theorem:
Consider (X^n, g) an asymptotically flat Riemannian manifold, $X \backslash K$ diffeomorphic to $\mathbb{R}^n \backslash B_r(0)$ with K compact and negative ADM mass.

Step 1: By Lohkamp's observation, there exists a non-flat metric $\tilde{g}$ on X with $\mathrm{scal}_{\tilde{g}} \geq 0$, and $\tilde{g}$ is isometric to the Euclidean metric on $X \backslash K$.

Step 2: We rescale P by a large factor such that ∂P lies entirely inside $\mathbb{R}^n \backslash B_R(0)$, therefore, ∂P can be isometrically embedded as a boundary in $X \backslash K$, which bounds a Riemannian polyhedron in M.

Step 3: Construct a degree one map from X to P, which takes K to $\{0\}$ and $X \backslash K$ to $P \backslash \{0\}$. Now $(X, \tilde{g})$ should be flat by the conjecture, therefore we obtain a contradiction.

6 Dihedral Rigidity for 3-Manifolds

In this section we will briefly discuss Chao Li's proof [Li17] of the polyhedron comparison and Dihedral Rigidity conjecture for cone and prisms in dimension three.

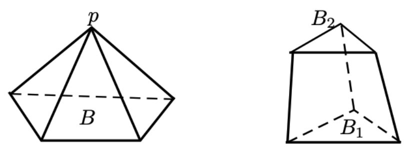

FIGURE 1. A (B, p)-cone and a (B_1, B_2)-prism.

Theorem 6.1. The polyhedron comparison conjecture holds for non-obtuse prism and cone in dimension 3.

Let (M^n, g) be a Riemannian manifold. We will use $F_1, ..., F_k$ to denote the side faces of M, and $F_1', ..., F_k'$ the faces for the corresponding Euclidean model P. Moreover, we let γ_j be the angle between F_j' and the base face. We will also use the following notations:

- Let Σ^{n-1} be an hypersurface with non-empty boundary $\partial\Sigma \subset \partial M$.
- Assume Σ separates the interior of M into two components, denote one of them E. (We will let E be the part that contains the cone vertex p if P is a cone).
- Let X be the outward unit normal vector field of ∂M in M.
- Let N denotes the unit normal vector field of Σ in E.
- Let ν be the out ward pointing unit normal vector field of $\partial\Sigma$ in Σ
- Let $\bar{\nu}$ be the out ward pointing unit normal vector field of $\partial\Sigma$ in ∂M pointing outward E.
- Let A be the second fundamental form of $\Sigma \subset E$.
- Let $\mathrm{I\!I}$ be the second fundamental form of $\partial M \subset M$.
- Let H be the mean curvature of $\Sigma \subset E$.
- Let $\bar{H}$ be the mean curvature of $\partial M \subset M$.

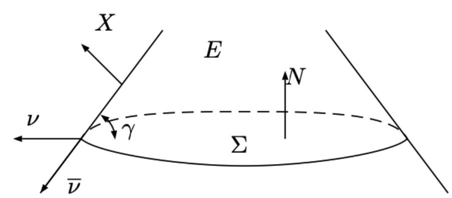

FIGURE 2. Capillary surfaces

Remark. Before we start describing the proof, it is important to know that we need to make the non-obtuse assumption here ($\gamma_j < \frac{\pi}{2}$) for the regularity problem of free boundary minimal surface. This is harmless in the non-rigid case for dimension three, but will be crucial in the rigidity problem. So we will put the assumption here.

Consider the following functional:

$$\mathcal{F}(E) = \mathcal{H}^{n-1}(\partial E \cap M^\circ) - \sum_{j=1}^{k} (\cos\gamma_j)\mathcal{H}^{n-1}(\partial E \cap F_j)$$

where E ranges over the set that $p \in E$ (or the upper face contained in E) and $E \cap B = \emptyset$. We provide the first and second variation formula:
First Variation: For Y a normal variation that is tangential to ∂M along $\partial\Sigma$ with $f = \langle Y, N\rangle$, then

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{F}(E_t) = -\int_\Sigma H f d\mathcal{H}^{n-1} + \sum_{j=1}^{k} \int_{\partial\Sigma\cap F_j} \langle Y, \nu - (\cos\gamma_j)\bar{\nu}\rangle d\mathcal{H}^{n-2}$$

Therefore, E is stationary point of $\mathcal{F}$ if and only if $H = 0$ and $\nu - (\cos\gamma_j)\bar{\nu}$ is normal to F_j, which means the angle between N and X (or ν and $\bar{\nu}$), is everywhere equal to γ_j. In general, the resulting hypersurface Σ is called a minimal capillary surface.
Second Variation: Now if Σ is minimal capillary, we have the second variational formula:

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{F}(E_t) = -\int_\Sigma (f\Delta f + (|A|^2 + \mathrm{Ric}(N,N))f^2) d\mathcal{H}^{n-1} + \sum_{j=1}^{k} \int_{\partial\Sigma\cap F_j} f(\frac{\partial f}{\partial\nu} - Qf) d\mathcal{H}^{n-2}$$

where

$$Q = \frac{1}{\sin\gamma_j}\mathrm{I\!I}(\bar{\nu},\bar{\nu}) + \cot\gamma_j A(\nu,\nu)$$

Existence: By varifold maximum principle, the minimizer exists. If Σ is regular, then it is a minimal surface meeting F_j at constant angle γ_j.
Regularity: The minimizer is a $C^{1,\alpha}$ graph over its tangent plane. It is smooth away from the corner.
From now on, for simplicity, we assume P is a prism and each γ_j is exactly $\frac{\pi}{2}$. Notice that in this case Σ is just a free boundary minimal hypersurface. Then for variation that is supported away from the corner, we have the stability inequality:

$$\int_\Sigma |\nabla f|^2 - (|A|^2 + \mathrm{Ric}(N,N))f^2 - \int_{\partial\Sigma} \mathrm{I\!I}(\bar{\nu},\bar{\nu})f^2 \geq 0$$

(Notice that $|A|^2$ is integrable since Σ is $C^{1,\alpha}$)
Re-arrangement: Recall the Schoen-Yau rearrangement:

$$|A|^2 + \text{Ric}(N,N) = \frac{1}{2}(\text{scal}_M - \text{scal}_\Sigma + |A|^2)$$

and hence if we assume $f = 1$ using an approximation argument in the stability inequality, we will obtain:

$$-\int_\Sigma \frac{1}{2}(\text{scal}_M - \text{scal}_\Sigma + |A|^2) - \int_{\partial\Sigma} \mathrm{II}(\bar{\nu},\bar{\nu}) \geq 0$$

Using the fact that the Gauss curvature $K_\Sigma = \frac{1}{2}\text{scal}_\Sigma$, it yields

$$\int_\Sigma \frac{1}{2}(\text{scal}_M + |A|^2) - \int_\Sigma K_\Sigma + \int_{\partial\Sigma} \mathrm{II}(\bar{\nu},\bar{\nu}) \leq 0$$

Gauss-Bonnet: We can now apply the Gauss-Bonnet formula for $C^{1,\alpha}$ surface with piecewise smooth boundary components:

$$\int_\Sigma K_\Sigma + \int_{\partial\Sigma} k_g + \sum_{j=1}^{l}(\pi - \alpha_j) = 2\pi\chi(\Sigma) \leq 2\pi$$

where α_j are the interior angles of Σ at the corner.
Contradiction Argument: By the hypothesis about the interior angle and using the fact that Σ is capillary, we know that

$$\sum_{j=1}^{l}(\pi - \alpha_j) \geq 2\pi$$

from some Euclidean geometry argument, and hence

$$-\int_\Sigma K_\Sigma \geq \int_{\partial\Sigma} k_g = \int_{\partial\Sigma} \mathrm{II}(T,T)$$

where T is the unit tangent vector along $\partial\Sigma$. Plug in to the stability inequality gives

$$0 \geq \int_\Sigma \frac{1}{2}(\text{scal}_M + |A|^2) + \int_{\partial\Sigma} \mathrm{II}(\bar{\nu},\bar{\nu}) + \mathrm{II}(T,T) = \int_\Sigma \frac{1}{2}(\text{scal}_M + |A|^2) + \int_{\partial\Sigma} \bar{H} > 0$$

This is the contradiction.

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