PROGRAM FOR THE SEMINAR: PROOF OF LOCAL LANGLANDS

1. Introduction

The Local Langlands Correspondence (LLC) for GL_n is a generalization of local class field theory. This semester we will study the work of Harris and Henniart on the proof of LLC. The organizers are Sam Mundy and Yihang Zhu.

2. Main references

[Har98], [Hen93], [Hen00].

Seminar website: You can find a link on Sam's or Yihang's homepage.

3. Talks

The following content will be growing. Please refer to the updated version, which will be linked to the website.

Talk 1. Introductory talk by Michael Harris.

Talk 2. Review of both categories.

Reference: [Wed08]. You can find a link on the website of the seminar last semester.

This talk reviews the category of admssible representations of $GL_n(F)$ and the category of Weil-Deligne representations of F, excluding the discussion of L and epsilon factors. All the material is already covered in the seminar last semester, so the talk is really just a review.

GL_n side: 2.1-2.4. You can very quickly go through 2.1. Emphasize more on 2.2, and especially define the supercuspidal representations and state the Bernstein-Zelevinsky classification. Only state the Q-version and omit the Z-version. **Do omit** (2.2.11). Define the Steinberg representation as in (2.2.13).

For 2.3 and 2.4, you can very briefly define square-integrable, tempered, and generic, but say what they mean in terms of B-Z classification, which is possible to state extremely easily. (More on generic representations can be talked about in Talk 2, so don't worry if you have to make this brief.)

Galois side: 3.1. **Do omit** (3.1.9) and (3.1.10).

Please also talk about (4.2.2). At least state the formula there.

Now if you still have time, talk about 4.1. In doing so you should assume the Satake isomorphism, as people should already be familiar with it from many previous seminars.

Talk 3. Review of L and epsilon factors.

Reference: [Wed08, 2.5, 3.2].

First define generic representations and their Whittaker models following 2.4. Then talk about everything in 2.5 and 3.2. Emphasize the analogy between (2.5.1.1) and (3.2.1.1).

After this, state Theorem (1.2.2). If you still have time, explain why the unramified correspondence in 4.1 preserves the L and epsilon factors.

If you feel the above material is too little for a talk, you can talk about how Deligne proved the existence of epsilon factors on the Galois side. The reference is §4 of [Del73] (available on Deligne's homepage).

Talk 4. Characterization of the correspondence.

Reference: [Hen93].

There are three main results here, namely Theorem 1.1, its corollary, and Theorem 4.1. From Theorem 4.1 it follows that there is at most one local Langlands correspondence for GL_n (assuming the reduction to the supercuspidal case). Please state these results.

Next, please prove Theorem 4.1 assuming the corollary to Theorem 1.1. The proof is provided in Section 4.1.

Now please prove the corollary to Theorem 1.1, assuming that theorem. The proof is provided in Sections 3.3 and 3.4. Assume as a black box any results you need from Jacquet–Piatetskii-Shapiro–Shalika (which are recalled in 3.4).

(In the last six lines in 3.4, it seems that the cross references (3.3.4) and (3.3.3) should be interchanged.)

Finally, prove Theorem 1.1. You will first have to recall the theory of Whittaker models and L- and ϵ-factors of pairs from Jacquet–Piatetskii-Shapiro–Shalika, as Henniart does in Sections 2.1-2.3. This can be brief as it was already done in Talk 3. You will also need three technical results: Lemmas 2.1, 2.4.1, and 2.4.2. Please look up the references and prove one or more among these three results, if you have time. Then Sections 3.1 and 3.2 complete the proof of Theorem 1.1 pretty easily.

There seems to be a typo in the first equation on p. 346. Instead of $(\rho(\tilde{g})W)(h)$, it should probably be $(\widetilde{\rho(g)W})(h)$.

Talk 5. The paper [JPSS83].

Talk 6. Global objects.

Reference: [Har98, §1], [Clo90].

The goal of this talk is to introduce the global players in the proof of LLC, namely **regular algebraic automorphic representations**, and **compatible families of λ-adic representations**.

Make the following definitions from Clozel's Ann Arbor article [Clo90]:

- Isobaric automorphic representations [Clo90, Definition 1.2]
- The category *Alg* of algebraic automorphic representations [Clo90, 1.2.3]. For this you have to state the LLC for archimedean fields, recalled in loc. cit. Discuss the relation with Harish Chandra isomorphism and infinitesimal characters. For this you may look at Chao's notes of Jack Thorne's course.
- Infinite type. [Clo90, 3.3].
- Regular representations. [Clo90, 3.5].

Then discuss the definition of compatible families of λ-adic Galois representations and the weak association between such objects and automorphic representations. [Har98, §1]. Along the way you may want to recall the unramified LLC, aka Satake isomorphism (see [Wed08, 4.1]). State [Har98, Theorem 1.2] as a black box. State [Har98, Definition 1.3]. If there is time, go through [Har98, (1.4), Lemma 1.5].

References

[Clo90] Laurent Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 77–159. MR 1044819

[Del73] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, 501–597. Lecture Notes in Math., Vol. 349. MR 0349635

[Har98] Michael Harris, *The local Langlands conjecture for* $\mathrm{GL}(n)$ *over a p-adic field, $n < p$*, Invent. Math. **134** (1998), no. 1, 177–210. MR 1646587

[Hen93] Guy Henniart, *Caractérisation de la correspondance de Langlands locale par les facteurs ϵ de paires*, Invent. Math. **113** (1993), no. 2, 339–350. MR 1228128

[Hen00] ———, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. **139** (2000), no. 2, 439–455. MR 1738446

[JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464. MR 701565

[Wed08] Torsten Wedhorn, *The local Langlands correspondence for* $\mathrm{GL}(n)$ *over p-adic fields*, School on Automorphic Forms on $\mathrm{GL}(n)$, ICTP Lect. Notes, vol. 21, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008, pp. 237–320. MR 2508771