Seminar Notes: Main Theorems and Strategies of Proof

Hida's 1988 Paper on p-adic Hecke Algebras for GL_2 over Totally Real Fields

Seminar on Arithmetic Geometry and Automorphic Forms

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Abstract

These notes detail the main theorems and the architectural strategy of the proofs in Haruzo Hida's 1988 paper, *On p-adic Hecke algebras for* GL_2 *over totally real fields* (Annals of Mathematics, 128). This seminar is designed for an audience specializing in arithmetic geometry, specifically Iwasawa theory, Shimura varieties, and p-adic automorphic forms. We will explicitly unpack the statements of the Control Theorem and the Freeness Theorem, and subsequently delve into the cohomological machinery—specifically the interaction between the ordinary projector e, boundary cohomology, and Shapiro's Lemma—that underpins the proof.

Contents

1 Introduction and the Setup

Let F be a totally real field of degree $d = [F : \mathbb{Q}]$ and $\Sigma = \{\sigma_1, \ldots, \sigma_d\}$ the set of real embeddings $F \hookrightarrow \mathbb{R}$. Let $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$. Fix a prime p. Let K be a finite extension of $\mathbb{Q}_p$ containing all embeddings of F into $\overline{\mathbb{Q}}_p$, with valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$.

1.1 The Universal Hecke Algebra

Let $\mathfrak{N} \subset \mathcal{O}_F$ be an integral ideal prime to p. For any $r \geq 1$, let $U_r = U_1(\mathfrak{N}p^r)$ be the standard compact open subgroup of $G(\mathbb{A}_f)$. Let $\kappa = (k, w)$ be a classical weight, where $k \in \mathbb{Z}^\Sigma$ with $k_v \geq 2$ and $k_v \equiv w \pmod 2$. The parabolic cohomology group $\mathrm{H}^d_{\mathrm{par}}(Y(U_r), L_\kappa(\mathcal{O}))$ is a finitely generated $\mathcal{O}$-module carrying an action of the Hecke algebra $h_\kappa(U_r, \mathcal{O})$.

The ordinary projector $e = \lim_{n\to\infty} U(p)^{n!}$ cuts out the ordinary part:

$$\mathrm{H}^d_{\mathrm{ord}}(Y_r, L_\kappa(\mathcal{O})) = e \cdot \mathrm{H}^d_{\mathrm{par}}(Y_r, L_\kappa(\mathcal{O}))$$

which is a faithful module over the ordinary Hecke algebra $h^{\mathrm{ord}}_\kappa(U_r, \mathcal{O}) = e \cdot h_\kappa(U_r, \mathcal{O})$.

We define the Iwasawa algebra $\Lambda = \mathcal{O}[[\Gamma]]$, where Γ is a pro-p group isomorphic to the Galois group of the maximal multiple $\mathbb{Z}_p$-extension F_∞ of F localized at p, up to a finite torsion group. The universal ordinary Hecke algebra is:

$$\mathbf{h}^{\mathrm{ord}}(\mathfrak{N}) = \varprojlim_{r\to\infty} h^{\mathrm{ord}}_{\kappa_0}(U_r; \mathcal{O})$$

for a fixed base weight κ_0 (usually taken as $k = (2, \ldots, 2), w = 0$).

2 Statement of Main Theorems

Hida's 1988 paper fundamentally rests upon two interconnected theorems. The first asserts the geometric regularity of the universal Hecke algebra over the weight space $\mathrm{Spf}(\Lambda)$, and the second establishes that this single object systematically recovers classical Hecke algebras at specific geometric loci.

2.1 Theorem A: Freeness and Finite Generation

Theorem 2.1 (Hida 1988, Theorem 3.1 & 4.1)**.** *Let $V^\infty = \varprojlim_r \mathrm{H}^d_{\mathrm{ord}}(Y_r, \mathcal{O})$ be the universal ordinary cohomology module.*

1. *V^∞ is a finitely generated, free Λ-module.*
2. *The universal ordinary Hecke algebra $\mathbf{h}^{\mathrm{ord}}(\mathfrak{N})$ is a finitely generated, torsion-free Λ-module.*

The torsion-freeness of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{N})$ implies that it defines an equidimensional, flat scheme over the weight space $\mathrm{Spf}(\Lambda)$, known as the ordinary eigenvariety (or Hida family).

2.2 Theorem B: The Control Theorem

Let $P_\kappa \subset \Lambda$ be an "arithmetic prime" corresponding to a classical weight $\kappa = (k, w)$. By definition, P_κ is the kernel of the structural homomorphism $\nu_\kappa : \Lambda \to \mathcal{O}_\kappa$ extending the character $\gamma \mapsto \gamma^{k-2} \operatorname{Nm}(\gamma)^{\frac{w-k+2}{2}}$ on a suitable open subgroup of Γ.

Theorem 2.2 (Hida 1988, Theorem 3.2 & 4.2)**.** *For any arithmetic prime P_κ of classical weight κ with $k_v \geq 2$, there is a canonical, surjective map of $\mathcal{O}_\kappa$-algebras:*

$$\phi_\kappa : \mathbf{h}^{\mathrm{ord}}(\mathfrak{N})/P_\kappa \mathbf{h}^{\mathrm{ord}}(\mathfrak{N}) \longrightarrow h_\kappa^{\mathrm{ord}}(U_1(\mathfrak{N}p^r); \mathcal{O}_\kappa)$$

which is an isomorphism, where p^r is the exact power of p dividing the conductor of the character associated to κ. Furthermore, there is a canonical isomorphism of Λ-modules for the cohomology:

$$V^\infty / P_\kappa V^\infty \xrightarrow{\sim} \mathrm{H}^d_{\mathrm{ord}}(Y_r, L_\kappa(\mathcal{O}_\kappa)).$$

3 Strategy of Proof: Cohomological Control

The core of the paper is proving the cohomological isomorphism in Theorem 2.2, from which the algebraic Control Theorem and Freeness Theorem follow via commutative algebra and duality.

3.1 Step 1: The Sheaf Theoretic Resolution

We wish to relate the local systems L_κ of non-trivial weights to the trivial local system at infinite level. Define the completed group algebra bundle $\mathscr{D}(\mathcal{O}) = \varprojlim_r \mathcal{O}[U_0/U_r]$. There is a canonical evaluation map of sheaves on Y_1:

$$\pi_\kappa : \mathscr{D}(\mathcal{O}) \longrightarrow L_\kappa(\mathcal{O})$$

induced by the algebraic representation of G defining L_κ. The kernel of this map, denoted $\mathcal{K}_\kappa$, yields a short exact sequence of sheaves on the fixed finite level manifold Y_r:

$$0 \longrightarrow \mathcal{K}_\kappa \longrightarrow \mathscr{D}(\mathcal{O}) \xrightarrow{\pi_\kappa} L_\kappa(\mathcal{O}) \longrightarrow 0 \tag{1}$$

By Shapiro's Lemma for locally symmetric spaces, the cohomology of Y_r with coefficients in $\mathscr{D}(\mathcal{O})$ is canonically isomorphic to the inverse limit of the cohomology with constant coefficients $\mathcal{O}$ over the tower $Y_m \to Y_r$:

$$\mathrm{H}^i(Y_r, \mathscr{D}(\mathcal{O})) \cong \varprojlim_{m \geq r} \mathrm{H}^i(Y_m, \mathcal{O}).$$

3.2 Step 2: The Long Exact Sequence and the Ordinary Projector

Taking the long exact sequence in parabolic cohomology associated to (1), we obtain:

$$\cdots \longrightarrow \mathrm{H}^d_{\mathrm{par}}(Y_r, \mathcal{K}_\kappa) \longrightarrow \mathrm{H}^d_{\mathrm{par}}(Y_r, \mathscr{D}(\mathcal{O})) \xrightarrow{(\pi_\kappa)_*} \mathrm{H}^d_{\mathrm{par}}(Y_r, L_\kappa(\mathcal{O})) \\ \longrightarrow \mathrm{H}^{d+1}_{\mathrm{par}}(Y_r, \mathcal{K}_\kappa) \longrightarrow \dots$$

To establish the isomorphism $V^\infty/P_\kappa V^\infty \xrightarrow{\sim} \mathrm{H}^d_{\mathrm{ord}}(Y_r, L_\kappa(\mathcal{O}))$, we must show two things after applying the exact functor e:

1. The map $e(\pi_\kappa)_*$ is surjective, which requires $e\,\mathrm{H}^{d+1}_{\mathrm{par}}(Y_r, \mathcal{K}_\kappa) = 0$.
2. The kernel of $e(\pi_\kappa)_*$ is exactly generated by the arithmetic prime ideal P_κ, i.e., $e\,\mathrm{H}^d_{\mathrm{par}}(Y_r, \mathcal{K}_\kappa) = P_\kappa \cdot V^\infty$.

3.3 Step 3: Vanishing of Obstructions (The Crux)

This is the most technically demanding part of Hida's 1988 paper. Hida must prove that the ordinary projector annihilates cohomology outside the middle degree d. The parabolic cohomology $\mathrm{H}^i_{\mathrm{par}}$ is defined as the image of cohomology with compact support in the full cohomology:

$$\mathrm{H}^i_{\mathrm{par}}(Y_r, \mathcal{F}) = \mathrm{im}\left(\mathrm{H}^i_c(Y_r, \mathcal{F}) \to \mathrm{H}^i(Y_r, \mathcal{F})\right).$$

We embed this in the long exact sequence of the Borel-Serre compactification $\overline{Y}_r$ with boundary ∂Y_r:

$$\cdots \to \mathrm{H}^i_c(Y_r, \mathcal{F}) \to \mathrm{H}^i(Y_r, \mathcal{F}) \to \mathrm{H}^i(\partial Y_r, \mathcal{F}) \to \mathrm{H}^{i+1}_c(Y_r, \mathcal{F}) \to \dots$$

Lemma 3.1 (Boundary Vanishing Lemma)**.** *For all $i \geq d$ and any p-adic sheaf $\mathcal{F}$ of the form $L_\kappa(\mathcal{O})$ or $\mathcal{K}_\kappa$, the ordinary projector e kills the boundary cohomology:*

$$e\,\mathrm{H}^i(\partial Y_r, \mathcal{F}) = 0.$$

Strategy of Proof for Lemma 3.1. The boundary ∂Y_r is stratified by components corresponding to the standard parabolic subgroups of G. For $G = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2$, the proper parabolics are the Borel subgroups B. The boundary cohomology is computed via the cohomology of arithmetic subgroups of $B(\mathbb{Q})$. A careful matrix computation shows that the Hecke operator $U(p)$, which acts via the double coset $U_r \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U_r$, acts on the boundary cohomology components by multiplication by positive powers of p (originating from the modulus character of the parabolic). Since $e = \lim U(p)^{n!}$, this topologically nilpotent action forces e to act as the zero map on the boundary components for degrees $i \geq d$. □

By Lemma 3.1, for $i = d+1$, we have:

$$e\,\mathrm{H}^{d+1}_{\mathrm{par}}(Y_r, \mathcal{K}_\kappa) \subseteq e\,\mathrm{H}^{d+1}(Y_r, \mathcal{K}_\kappa) \cong e\,\mathrm{H}^{d+1}_c(Y_r, \mathcal{K}_\kappa).$$

Hida then employs a spectral sequence argument relating the cohomology of Y_r to group cohomology of Γ. Because Γ is virtually a p-adic Lie group of dimension at most $d+1$, its higher group cohomology vanishes in degrees $> d$. Combined with the boundary vanishing, this proves $e\,\mathrm{H}^{d+1}_{\mathrm{par}} = 0$.

Thus, $e(\pi_\kappa)_* : V^\infty \to \mathrm{H}^d_{\mathrm{ord}}(Y_r, L_\kappa(\mathcal{O}))$ is surjective.

3.4 Step 4: Identification of the Kernel

To finish the cohomological control, one shows that the kernel of $e(\pi_\kappa)_*$ is generated by the action of $\gamma - \nu_\kappa(\gamma)$ for $\gamma \in \Gamma$. Since the action of Γ is given by diamond operators which are central in the Hecke algebra, and by definition of $\mathscr{D}(\mathcal{O})$, one obtains an exact sequence of Λ-modules:

$$0 \to P_\kappa \mathscr{D}(\mathcal{O}) \to \mathscr{D}(\mathcal{O}) \to L_\kappa(\mathcal{O}) \to 0$$

Taking cohomology, applying e, and using the vanishing results, we get exactly:

$$V^\infty / P_\kappa V^\infty \cong \mathrm{H}^d_{\mathrm{ord}}(Y_r, L_\kappa(\mathcal{O})).$$

4 Strategy of Proof: Hecke Algebra Control

With the cohomological control established, Hida translates this geometric fact into the algebraic Control Theorem (Theorem 2.2).

4.1 Step 1: Faithfulness and Finite Generation

The algebra $\mathbf{h}^{\mathrm{ord}}(\mathfrak{N})$ acts faithfully on V^∞. By the cohomological control theorem, $V^\infty/\mathfrak{m}_\Lambda V^\infty \cong \mathrm{H}^d_{\mathrm{ord}}(Y_r, L_2(\mathbb{F}_p))$, where $\mathfrak{m}_\Lambda$ is the maximal ideal of Λ. Since $\mathrm{H}^d_{\mathrm{ord}}$ is a finite dimensional $\mathbb{F}_p$-vector space, Nakayama's Lemma for the local ring Λ implies that V^∞ is a finitely generated Λ-module.

By an application of the Auslander-Buchsbaum formula and analyzing the projective dimension of V^∞ over Λ, Hida deduces that V^∞ is in fact a free Λ-module (proving Theorem 2.1(1)). Because $\mathbf{h}^{\mathrm{ord}}(\mathfrak{N})$ embeds into $\mathrm{End}_\Lambda(V^\infty)$, it must be a torsion-free Λ-module, proving Theorem 2.1(2).

4.2 Step 2: Surjectivity and Perfect Pairings

We have a natural map:

$$\phi_\kappa : \mathbf{h}^{\mathrm{ord}}(\mathfrak{N})/P_\kappa \mathbf{h}^{\mathrm{ord}}(\mathfrak{N}) \longrightarrow h^{\mathrm{ord}}_\kappa(U_r, \mathcal{O}_\kappa).$$

Because Hecke operators at finite level are defined as the image of operators in the inverse limit, ϕ_κ is surjective.

To prove injectivity, Hida relies on the existence of a perfect pairing between the classical Hecke algebra and the space of cusp forms. Let $S^{\mathrm{ord}}_\kappa(U_r, \mathcal{O}_\kappa)$ be the space of ordinary Hilbert cusp forms. The pairing is defined by:

$$\langle \cdot, \cdot \rangle : h^{\mathrm{ord}}_\kappa(U_r, \mathcal{O}_\kappa) \times S^{\mathrm{ord}}_\kappa(U_r, \mathcal{O}_\kappa) \longrightarrow \mathcal{O}_\kappa, \quad \langle T, f \rangle = a_1(Tf)$$

where $a_1(f)$ is the first Fourier coefficient. This pairing is non-degenerate. By dualizing the cohomological isomorphism $V^\infty/P_\kappa V^\infty \cong \mathrm{H}^d_{\mathrm{ord}}$, and tracing the action of Hecke operators under this duality, one shows that the $\mathcal{O}_\kappa$-rank of $\mathbf{h}^{\mathrm{ord}}(\mathfrak{N})/P_\kappa \mathbf{h}^{\mathrm{ord}}(\mathfrak{N})$ must equal the $\mathcal{O}_\kappa$-rank of $h^{\mathrm{ord}}_\kappa(U_r, \mathcal{O}_\kappa)$. A surjective map between free finite rank modules of the same rank over a discrete valuation ring is an isomorphism. This completes the proof of Theorem 2.2.

5 Conclusions

Hida's 1988 strategy profoundly relies on shifting the perspective from modular forms to parabolic cohomology. The magic of the ordinary projector e is twofold: mathematically, it crushes the complicated boundary cohomology that normally prevents exact dimension-counting in higher degrees; philosophically, it selects exactly the locus in the weight space where p-adic interpolation is analytically rigid. The resulting isomorphism $V^\infty/P_\kappa \cong V_\kappa$ seamlessly intertwines the abstract commutative algebra of the Iwasawa algebra Λ with the concrete arithmetic geometry of Hilbert modular varieties.