SEMINAR NOTES: CONGRUENCES OF CUSP FORMS (H. HIDA, 1981)

XIAORUN WU

Abstract. This is a summary Haruzo Hida's 1981 paper "Congruences of Cusp Forms and Special Values of Their Zeta Functions," structured by chapter. Key theorys and definitions are provided, where applicable.

Contents

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1. Introduction

1.1. **Motivation.** To relate the discriminant of a bilinear form on the space of cusp forms to the special values of zeta functions, and use this to prove congruences between modular forms.

1.2. **Definition: Zeta Function of a Primitive Cusp Form.** The zeta function $L(s,f)$ for a primitive cusp form f is defined by the Euler product:

$$L(s,f) = \prod_p [(1-\overline{\psi}(p)\alpha_p^2 p^{-s})(1-\overline{\psi}(p)\alpha_p\beta_p p^{-s})(1-\overline{\psi}(p)\beta_p^2 p^{-s})]^{-1} \quad (s \in \mathbb{C})$$

where ψ is the Dirichlet character of f, $\overline{\psi}$ is its complex conjugate, and $\alpha_p + \beta_p$ and $\alpha_p\beta_p$ are the eigenvalues of Hecke operators $T(p)$ and $T(p,p)$ respectively.

1.3. **Definition: The Bilinear Form $A(f,g)$.** For cusp forms $f, g \in S_k(\Gamma)$ of weight $k = n+2$, the $\mathbb{R}$-bilinear form is defined as:

$$A(f,g) = (2\sqrt{-1})^{n-1}[(f,g) + (-1)^{n+1}(g,f)]$$

where $(,)$ is the Petersson inner product.

1.4. **Definition: Discriminant $d(f)$.** Let W_f be the subspace spanned by conjugates of f and $L_f = W_f \cap L$ (where L is an integral lattice in $S_k(\Gamma)$). The discriminant is:

$$d(f) = \det\{(A(\delta_i,\delta_j))_{1\le i,j\le 2r}\} \in \mathbb{Q}^\times$$

where $\{\delta_i\}$ is a basis of L_f over $\mathbb{Z}$.

1.5. **Definition: The Rational Number $c(f)$.** The main theorem states $d(f) = e(N)^{-2r(f)}c(f)^2$. The number $c(f)$ is defined as:

$$c(f) = (u(f)\pi^{(k+1)r(f)})^{-1}\epsilon(N)^{r(f)}2^{-(k+2)r(f)}\cdot[(k-1)!]^{r(f)}[NN(\psi)\varphi(N/N(\psi))]^{r(f)}Z(k,f)$$

where $Z(k,f)$ is the product of $L(k,f^\sigma)$ over all conjugates, and $u(f)$ is a transcendental factor.

1.6. **Main Theorems Preview.**

1.6.1. *Theorem A:.* Establishes a formula relating the discriminant $d(f)$ to a rational number $c(f)^2$, which involves the zeta value $Z(k,f)$ (a product of $L(k,f^\sigma)$ over conjugates).

1.6.2. *Theorem B:.* States that if a prime p divides $c(f)^2$ (and satisfies certain conditions), there exists a cusp form g, not conjugate to f, such that $g \equiv f \pmod{\mathfrak{P}}$.

2. Parabolic Cohomology Groups

2.1. **Definition: Representation ρ_n.** For $n \geq 0$, ρ_n is a representation of $SL_2(\mathbb{Z})$ on $L(\mathbb{Z}) = \mathbb{Z}^{n+1}$ defined by:

$$\rho_n(\alpha)\begin{bmatrix} u \\ v \end{bmatrix}^n = \left[\alpha \begin{bmatrix} u \\ v \end{bmatrix}\right]^n \quad (\alpha \in SL_2(\mathbb{Z}))$$

where $\begin{bmatrix} u \\ v \end{bmatrix}^n$ is the vector of symmetric tensor powers ${}^t(u^n, u^{n-1}v, ..., v^n)$. Let $L(\mathbb{Z}) = \mathbb{Z}^{n+1}$ and $L(\Lambda) = L(\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda$. Similarly, we let $F(\Lambda) = (L(\Lambda) \times \mathfrak{H})/\overline{\Gamma}$, where $\overline{\Gamma} = \Gamma/\Gamma \cap \mathbb{Q}^\times$. Similarly, we define that for $F(\mathbb{Z})$.

2.2. **Definition: Parabolic Cohomology Group.** Let $X = \bar{\Gamma}\backslash\mathfrak{H}$. The parabolic cohomology group is defined as the image of the cohomology with compact support:

$$H^1_p(X, F(\Lambda)) = \mathrm{Image}(H^1_c(X, F(\Lambda)) \to H^1(X, F(\Lambda)))$$

It is canonically isomorphic to the group cohomology $H^1_P(\bar{\Gamma}, L(\Lambda))$ defined via parabolic cocycles Z_P:

$$Z_P(\bar{\Gamma}, L(\Lambda)) = \{u \in Z(\bar{\Gamma}, L(\Lambda)) \mid u(\pi) \in (\pi - 1)L(\Lambda) \text{ for all } \pi \in P\}$$

where P is a set of representatives of parabolic elements (cusps).

2.3. **Theorems proven in this chapter:**

2.3.1. *Canonical Isomorphism for Parabolic Cohomology.* Establishes the isomorphism $H^1_p(X, F(\Lambda)) \cong H^1_P(\bar{\Gamma}, L(\Lambda))$ using group cohomology.

2.3.2. *Integrality (Theorem 1.2):* Under specific conditions on the level N and prime ℓ (e.g., $\ell \nmid N$), the quotient H^1/H^1_p is torsion-free, and $H^1_p(X, F(\mathbb{Z})) \otimes \mathbb{Z}_\ell \cong H^1_p(X, F(\mathbb{Z}_\ell))$.

3. A Theorem on Duality

3.1. **Pairings.** In this section, Hida discussed duality pairings on the sheaves and cup products in cohomology.

3.2. **Definition: The Matrix Θ_n.** A matrix $\Theta_n \in M_{n+1}(\mathbb{Z})$ is introduced to define duality, satisfying:

$${}^t\begin{bmatrix} u \\ v \end{bmatrix}^n \cdot \Theta_n \cdot \begin{bmatrix} x \\ y \end{bmatrix}^n = \left(\det \begin{bmatrix} u & x \\ v & y \end{bmatrix}\right)^n$$

This matrix induces a non-degenerate pairing on $L(\mathbb{Q})$.

3.3. **Definition: Bilinear Pairing $\langle x, y\rangle_\Lambda$.** Using the cup product $\mathcal{A}_\Lambda$ and the morphism θ_n induced by Θ_n:

$$\langle x, y\rangle_\Lambda = \mathcal{A}_\Lambda(x, \theta_n(y))$$

This provides a non-degenerate bilinear product on $H^1_p(X, F(\mathbb{Z}_\ell))$ when $\ell \nmid N$ and $\ell > n$.

4. Cohomology Groups and Spaces of Cusp Forms

4.1. **Definition: The Eichler-Shimura Map δ.** A map from cusp forms to cohomology classes defined by:

$$\delta(f) = \mathrm{Re}\left(\int_z^{z_0} f(z) \begin{bmatrix} z \\ 1 \end{bmatrix}^n dz\right)$$

This induces an isomorphism $S_k(\Gamma) \cong H^1_P(\bar{\Gamma}, L(\mathbb{R}))$.

4.2. **Definition: Cohomological Inner Product $\mathcal{A}_\Gamma(f, g)$.** Defined via the exterior product of the forms $\delta(f)$ and $\Theta_n\delta(g)$:

$$\mathcal{A}_\Gamma(f, g) = \int_{\Gamma\backslash\mathfrak{H}} {}^t\delta(f) \wedge \Theta_n\delta(g)$$

This matches the bilinear form defined in the introduction: $\mathcal{A}_\Gamma(f, g) = \langle\delta(f), \delta(g)\rangle^\Gamma_\mathbb{R}$.

Key formula: The cup product corresponds to the imaginary part of the Petersson inner product (up to constants). (see Pg 14).

From here, Hida described the action of Hecke operators on the cohomology groups, compatible with their action on cusp forms.

4.3. **Definition: Integral Lattice $V(N; \mathbb{Z})$.**

$$V(N; \mathbb{Z}) = \mathrm{Image}(H^1_p(\bar{\Gamma}, L(\mathbb{Z})) \to H^1_p(\bar{\Gamma}, L(\mathbb{R})))$$

This is the maximal free quotient of the integral cohomology.

Hida proved that $V(N; \mathbb{Z}_\ell)$ is self-dual under the pairing $\langle, \rangle_N$ over $\mathbb{Z}_\ell$ under the condition that ℓ is prime to $e(N)N$ if $n > 0$.

5. Discriminants of Quadratic Forms

5.1. **General Setup.** Let K be a CM-field or a totally real field of finite degree, and V be a finite-dimensional vector space over K. Let T be a symmetric or skew-symmetric non-degenerate $\mathbb{Q}$-bilinear form on V such that:

$$T(ax, y) = T(x, a^\rho y) \quad \text{for every } x, y \in V \text{ and } a \in K$$

where ρ denotes the complex conjugation.

5.2. **Definition: Discriminant $d(T; L)$.** Let L be a lattice of V over $\mathbb{Z}$, and $\{x_1, ..., x_m\}$ be a basis of L over $\mathbb{Z}$ (where $m = \dim_\mathbb{Q}(V)$). The discriminant of T relative to L is defined as:

$$d(T; L) = \det(T(x_i, x_j))_{1\le i,j\le m}$$

The class $d(T)$ of $d(T; L)$ in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ depends only on T, not on the lattice L.

5.3. **Definition: Generalized Index $[L_1 : L_2]$.** For any lattices L_1 and L_2 in V, the generalized index is defined as:

$$[L_1 : L_2] = [L_1 : L_1 \cap L_2]/[L_2 : L_1 \cap L_2]$$

This generalizes the usual index definition to cases where L_1 does not necessarily contain L_2.

5.4. **Definition: Dual Lattice L^*.** For a lattice L, the dual lattice L^* under T over $\mathbb{Z}$ is defined as:

$$L^* = \{x \in V \mid T(x, y) \in \mathbb{Z} \text{ for all } y \in L\}$$

.

The main result is the following:

5.4.1. *Proposition 4.3 (Discriminant and Index).* The absolute value of the discriminant is equal to the index of the lattice in its dual:

$$|d(T; L)| = [L^* : L]$$

.

6. A Formula of the Petersson Inner Product of Cusp Forms

6.1. **Definition: Primitive Form.** A cusp form $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ (where $e(z) = \exp(2\pi\sqrt{-1}z)$) is called a primitive form of conductor N if it satisfies three conditions:f is exactly of level N (a new form in $S_k(\Gamma_1(N))$).f is a common eigenform of the Hecke operators $T(n)$ for all $n > 0$.The leading Fourier coefficient is normalized: $a(1) = 1$.

6.2. **Definition: The Dirichlet Series $D(s, f, g)$.** For two normalized eigenforms f (with coefficients $a(n)$) and g (with coefficients $b(n)$), the series is defined as:

$$D(s, f, g) = \sum_{n=1}^{\infty} a(n)b(n)n^{-s} \quad (s \in \mathbb{C})$$

.

6.3. **Definition: Euler Product of $D(s, f, g)$.** The series admits the following Euler factorization:

$$D(s, f, g) = \prod_p [(1 - \alpha_p\beta_p\alpha'_p\beta'_p p^{-2s})^{-1}(1- \alpha_p\alpha'_p p^{-s})^{-1}(1 - \alpha_p\beta'_p p^{-s})^{-1}(1 - \beta_p\alpha'_p p^{-s})^{-1}(1 - \beta_p\beta'_p p^{-s})^{-1}]$$

where α_p, β_p are roots associated with f and α'_p, β'_p are roots associated with g.

6.4. **Residue Formula: Rankin-Selberg Method.** The residue of $D(s, f, g)$ at $s = k$ is related to the Petersson inner product. Specifically, for $g = f_\rho$ (a "twisted" conjugate form defined in the text), the relation simplifies. The general residue relation is:

$$\zeta_N(2)\text{Res}_{s=k}D(s, f, g) = L(k, f, \bar{\psi})\text{Res}_{s=1}\zeta_N(s)$$

where ζ_N is the Riemann zeta function excluding Euler factors for primes dividing N.

6.5. **Theorem 5.1: Explicit Formula for $L(k, f, \bar{\psi})$.** Let f be a primitive form of conductor N and character ψ. Then:

$$L(k, f, \bar{\psi}) = 2^{2k}\pi^{k+1}(k-1)!^{-1}\{\delta(N)NN(\psi)\varphi(N/N(\psi))\}^{-1}(f, f)_\Gamma$$

where:$N(\psi)$ is the conductor of ψ.φ is the Euler totient function.$\delta(N) = 2$ if $N \leq 2$, and 1 if $N \geq 3$.$(f, f)_\Gamma$ is the Petersson inner product.

7. Discriminants Associated with Primitive Forms (and $u(f)$)

7.1. **Definition: The Subspace $W_f(\mathbb{R})$.** Let $S(f) = \sum_{\sigma \in I} \mathbb{C} f^\sigma$ be the space spanned by conjugates of a primitive form f. $W_f(\mathbb{R})$ is the image of $S(f)$ in the real cohomology space $V(N; \mathbb{R})$ under the isomorphism ϕ.

7.2. **Definition: The Lattice L_f.**

$$L_f = W_f(\mathbb{R}) \cap V(N; \mathbb{Z})$$

This is the intersection of the "f-typical" real space with the integral cohomology lattice.

7.3. **Definition: Basis ω_ν for $W_f(\mathbb{R})$.** Let $\{f_1, ..., f_r\}$ be the set of all conjugates of f (where $r = [K_f : \mathbb{Q}]$). A basis $\{\omega_\nu\}_{1 \leq \nu \leq 2r}$ for $W_f(\mathbb{R})$ is defined using the isomorphism ϕ:

$$\omega_\nu = \begin{cases} \phi(f_\nu) & \text{if } \nu \leq r \\ \phi(\sqrt{-1} f_{\nu - r}) & \text{if } r < \nu \leq 2r \end{cases}$$

.

7.4. **Mathematical Definition of the Transcendental Factor $u(f)$.** To define $u(f)$, compare the rational lattice structure with the analytic structure:

(1) Take a basis $\{\delta_\nu\}_{1 \leq \nu \leq 2r}$ of the lattice L_f over $\mathbb{Z}$.
(2) Form the real invertible matrix $U \in M_{2r}(\mathbb{R})$ that transforms the lattice basis into the vector space basis $\{\omega_\nu\}$:

$$(\delta_1, ..., \delta_{2r})U = (\omega_1, ..., \omega_{2r})$$

.
(3) Define $u(f)$ as the absolute value of the determinant of this change-of-basis matrix:

$$u(f) = |\det(U)|$$

This factor measures the "volume" difference between the integral structure provided by cohomology and the analytic structure provided by the cusp forms.

8. A Criterion of Congruences Among Cusp Forms

8.1. **The main Theorem.** Theorem 7.1: Let f be a primitive form of weight $k \geq 2$. Let p be a prime factor of the special value part $C(f)$ (derived from $c(f)$) such that $p > k - 2$. Then there exists a normalized eigenform g (not conjugate to f) such that $g \equiv f \pmod{\mathfrak{P}}$.

8.2. **Detailed Proof Steps (Regarding Hecke Ring Action):**

8.2.1. *Index Formula and Prime Divisibility:* From Theorem 6.1, we have the relation $e(N)^{-2r(f)}c(f)^2 = [L_f^* : L_f]$, where L_f^* is the dual lattice of L_f under the pairing T_f. Let M_f be the projection of the total lattice $L = V(N;\mathbb{Z})$ into the subspace $W_f(\mathbb{R})$. It is established that $M_f = L_f^*$. Therefore, the prime p (being a factor of $c(f)^2$) divides the index $[M_f : L_f]$.

8.2.2. *Orthogonal Decomposition and Projections:* The space decomposes as $V(N;\mathbb{R}) = W_f(\mathbb{R}) \oplus Y$, where Y is the orthogonal complement.Let $L_Y = Y \cap V(N;\mathbb{Z})$ and M_Y be the projection of L into Y.The projections induce natural isomorphisms of modules:

$$r_f : L/(L_f \oplus L_Y) \xrightarrow{\sim} M_f/L_f$$

$$r_Y : L/(L_f \oplus L_Y) \xrightarrow{\sim} M_Y/L_Y$$

Thus, $M_f/L_f \cong M_Y/L_Y$.

8.3. **Hecke Algebra Action:** Let $R = R(\Gamma, \Delta_1)$ be the Hecke algebra over $\mathbb{Z}$. Since f is primitive, the submodules L_f, M_f, L_Y, M_Y are stable under the Hecke action. Let R_f and R_Y be the restrictions of the Hecke algebra R to the spaces $W_f(\mathbb{R})$ and Y respectively. We have surjective homomorphisms:

$$\phi_f : R \to R_f \quad \text{and} \quad \phi_Y : R \to R_Y$$

.

8.3.1. *Localizing at the Prime p:* Since p divides the index $[M_f : L_f]$, the module M_f/L_f is non-trivial. There exists a maximal ideal $\mathfrak{p}_f$ of R_f with residue characteristic p that contains the annihilator of M_f/L_f. Pull this ideal back to the full algebra R to define $\mathfrak{p} = \phi_f^{-1}(\mathfrak{p}_f)$.

8.3.2. *Establishing the Congruence:* Define $\mathfrak{p}_Y = \phi_Y(\mathfrak{p})$. Since $M_f/L_f \cong M_Y/L_Y$ as R-modules, the ideal $\mathfrak{p}_Y$ is a non-trivial maximal ideal in R_Y (it contains the annihilator of M_Y/L_Y).The action of $T(n)$ on $(M_f/L_f) \otimes_{R_f} \kappa$ (where $\kappa = R_f/\mathfrak{p}_f$) is scalar multiplication by $a(n) \pmod{\mathfrak{P}}$ (the eigenvalue of f). Because of the isomorphism, the action of $T(n)$ on $(M_Y/L_Y) \otimes \kappa$ is also scalar multiplication by $a(n) \pmod{\mathfrak{P}}$.This implies the existence of an eigenform in the space Y (which is orthogonal to f) that shares these eigenvalues modulo $\mathfrak{P}$. Since R_Y is commutative and acts on $M_Y \otimes \mathbb{Q}$, we can lift this representation to find a cusp form $g \in Y$ such that for all n:

$$b(n) \text{ (eigenvalue of } g) \equiv a(n) \text{ (eigenvalue of } f) \pmod{\mathfrak{P}}$$

.Since $g \in Y$ and Y is the orthogonal complement to the space spanned by conjugates of f, g cannot be conjugate to f.

9. L-functions of an Imaginary Quadratic Field

9.1. **Setup: Field and Character.** Let M be an imaginary quadratic field with discriminant $-d$. Let λ be a primitive Hecke character of M of conductor $\mathfrak{c}$ satisfying the infinity type condition:

$$\lambda((a)) = a^{k-1} \quad \text{for } a \equiv 1 \pmod{\mathfrak{c}} \quad \text{with } 1 < k \in \mathbb{Z}$$

.

9.2. **Definition: Associated Cusp Form f_λ.** The primitive form f_λ of weight k and conductor $N = N(\mathfrak{c})d$ is defined by:

$$f_\lambda(z) = \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) e(N(\mathfrak{a})z)$$

where the sum runs over all integral ideals $\mathfrak{a}$ of M prime to $\mathfrak{c}$. The character ψ of f_λ is given by $\psi(m) = \chi(m)\lambda((m))m^{1-k}$, where χ is the quadratic residue symbol corresponding to M.

9.3. **Definition: L-function of the Character $L(s, \lambda)$.**

$$L(s, \lambda) = \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s} \quad (s \in \mathbb{C})$$

.

9.4. **Relation to Zeta Values (Formula 8.3).** The special value $Z(k, f_\lambda)$, which is the product of L-values over conjugates, is given by:

$$Z(k, f_\lambda) = (2\pi)^{r(\lambda)} (d^{-\frac{1}{2}} h w^{-1})^{r(\lambda)} \cdot \prod_{p|N} (1 - \chi(p) p^{-1})^{r(\lambda)} \cdot \prod_{\sigma \in I_f} L(k, \bar{\psi}^\sigma \lambda^{2\sigma})$$

where w is the number of roots of unity in M, h is the class number, and $r(\lambda) = [K_{f_\lambda} : \mathbb{Q}]$.

9.5. **Definition: Hurwitz Numbers H^μ_ν.** In the context of $M = \mathbb{Q}(\sqrt{-3})$, numbers H^μ_ν related to the special values are defined as:

$$H^\mu_\nu = w(\mu - 1)! d^{\delta_\mu} L(\mu, \lambda_\nu) \pi^{-\mu} \Omega^{-\nu} \quad \text{for } \frac{\nu}{2} < \mu \leq \nu$$

where Ω is a specific period of the elliptic curve $y^2 = 4x^3 - 4$

References

[1] H. Hida, *Congruences of cusp forms and special values of their zeta functions*, Inventiones Math. **63** (1981), 225–261.

Email address: xiaorunw@math.columbia.edu