Adjoint Selmer Groups as Iwasawa Modules

Deformation Rings, Hecke Algebras, and the Order of Trivial Zeros

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Abstract

This seminar provides a comprehensive and rigorous exposition of the Iwasawa theory for adjoint Selmer groups of Galois representations. We analyze the adjoint action of a Galois representation φ on the Lie algebra of the derived algebraic group S of a split classical group G. By establishing an exact base-change control theorem, we construct a canonical isomorphism between the Pontryagin dual of Greenberg's adjoint Selmer group and the module of Kähler differentials of the universal nearly ordinary deformation ring. Specializing to $G = GL(2)$ over a totally real field, Fujiwara's theorem enables the identification of this deformation ring with the p-adic Hecke algebra. We leverage this arithmetic-algebraic bridge to deduce the exact structure of the characteristic power series of the Selmer group at the trivial zero $s = 0$, identifying the precise multiplicity of the zeros via the Jacobian determinant of local Hecke operators.

1 Galois Representations and the Adjoint Action

Let $p \geq 5$ be a fixed prime. Let $\mathcal{O}$ be a discrete valuation ring, finite flat over $\mathbb{Z}_p$, with maximal ideal $\mathfrak{m}_{\mathcal{O}}$ and residue field $\mathbb{F}$. Let $G \subset GL(n)$ be a split classical group defined over $\mathcal{O}$. For instance, G could be $GL(n)$, or a split similitude group defined by a symmetric or symplectic form with unit discriminant.

Let $\mathbb{J}$ be a local complete noetherian integral domain in the category $CNL_{\mathcal{O}}$ (complete noetherian local $\mathcal{O}$-algebras) sharing the residue field $\mathbb{F}$. We consider a continuous Galois representation:

$$\varphi : \mathfrak{G} = \mathrm{Gal}(F^{(p,\infty)}/E) \to G(\mathbb{J}) \tag{1}$$

where F/E is a finite Galois extension with $p \nmid [F : E]$, and $F^{(p,\infty)}$ is the maximal extension of F unramified outside p and ∞.

Let S be the derived algebraic subgroup of G, and let $\mathfrak{s}_{/\mathbb{J}}$ be its Lie algebra. The group G acts on $\mathfrak{s}$ via the adjoint action. We define the adjoint representation $Ad_S(\varphi)$ acting on $\mathfrak{s}$:

$$Ad_S(\varphi) : \mathfrak{G} \to GL(\mathfrak{s}) \quad \text{via} \quad \sigma \mapsto \left(x \mapsto \varphi(\sigma)x\varphi(\sigma)^{-1}\right). \tag{2}$$

1.1 Near-Ordinarity and Local Filtrations

We assume φ is *nearly ordinary* at p. For each prime $\mathfrak{p} \mid p$ of F, the restriction $\varphi|_{D_{\mathfrak{p}}}$ to the decomposition group $D_{\mathfrak{p}}$ takes values in a split parabolic subgroup $P_{\mathfrak{p}}$ of G. This structure stabilizes a flag in the representation space $V(\varphi)$:

$$\mathcal{F}_{\mathfrak{p}} : 0 = V_0(\varphi) \subset V_1(\varphi) \subset \cdots \subset V_{m_{\mathfrak{p}}}(\varphi) = V(\varphi), \tag{3}$$

where each successive quotient $V(\varphi)/V_{j,\mathfrak{p}}(\varphi)$ is $\mathbb{J}$-free.

Let $\delta_{j,\mathfrak{p}}$ denote the representation of $D_{\mathfrak{p}}$ on the successive quotients $V_j(\varphi)/V_{j-1}(\varphi)$. Let $\mathfrak{n}_{\mathfrak{p}} \subset \mathfrak{s}$ be the Lie algebra of the unipotent radical of $P_{\mathfrak{p}} \cap S$. The adjoint representation space decomposes into filtration submodules governed by the flag:

$$V_{\mathfrak{p}}^{+}(Ad_S(\varphi)) = \{T \in V(Ad_S(\varphi)) \mid T(V_{j,\mathfrak{p}}) \subset V_{j-1,\mathfrak{p}} \text{ for all } j\}, \tag{4}$$

$$V_{\mathfrak{p}}^{-}(Ad_S(\varphi)) = \{T \in V(Ad_S(\varphi)) \mid T(V_{j,\mathfrak{p}}) \subset V_{j,\mathfrak{p}} \text{ for all } j\}. \tag{5}$$

2 The Greenberg Adjoint Selmer Group and Character Twists

For any Artin representation $\psi : \Delta = \mathrm{Gal}(F/E) \to GL(V)$ over $\mathcal{O}$ and a subfield $L \subset F^{(p,\infty)}$, Greenberg's Selmer group isolates the cohomology classes that satisfy the local near-ordinarity conditions.

Definition 2.1 (Greenberg Selmer Group)**.** *Let $X^* = X \otimes_{\mathbb{J}} \mathbb{J}^*$ denote the Pontryagin dual, where $\mathbb{J}^* = Hom_{\mathbb{Z}_p}(\mathbb{J}, \mathbb{Q}_p/\mathbb{Z}_p)$. The Selmer group is the kernel of the restriction map to the inertia subgroups $I_{\mathfrak{p}}$:*

$$Sel(Ad_S(\varphi) \otimes \psi)_{/L} = \ker \left(H^1(\mathfrak{H}_L, (\mathfrak{s} \otimes_{\mathcal{O}} V)^*) \xrightarrow{res} \prod_{\mathfrak{p}|p} \frac{H^1(I_{\mathfrak{p}}, (\mathfrak{s} \otimes_{\mathcal{O}} V)^*)}{H^1(I_{\mathfrak{p}}, (\mathfrak{n}_{\mathfrak{p}} \otimes_{\mathcal{O}} V)^*)} \right) \tag{6}$$

where $\mathfrak{H}_L = Gal(F^{(p,\infty)}/L)$.

If φ originates from a pure, regular motive $M_{/E}$, the un-twisted Selmer group $Sel(Ad_S(\varphi))_{/E}$ is conjectured to be finite. Over a totally ramified $\mathbb{Z}_p$-extension E_∞/E, the Pontryagin dual $Sel^*(Ad_S(\varphi) \otimes \psi)_{/E_\infty}$ carries the structure of a finitely generated module over the Iwasawa algebra $\mathbb{J}[[\Gamma]] \cong \mathbb{J}[[T]]$, where $\Gamma = \mathrm{Gal}(E_\infty/E)$ and $T = \gamma - 1$.

2.1 Shapiro's Lemma and Twisted Selmer Groups

Let L/E be an extension linearly disjoint from F over E, and set $M = LF$. The representation $Ind_F^E\mathbb{F}$ decomposes as $\bigoplus_{\overline{\psi}} m(\psi)\overline{\psi}$ over absolutely irreducible representations of Δ. By establishing an isomorphism of $\mathcal{O}[\mathrm{Gal}(L/E)]$-modules via Shapiro's lemma applied to local and global Galois groups, we obtain:

$$Sel(\varphi_F)_{/M} \cong Sel(Ind_F^E \varphi_F)_{/L} \cong \bigoplus_{\psi} \left(Sel(\varphi_E \otimes \psi)_{/L}\right)^{\oplus m(\psi)}. \tag{7}$$

This profound decomposition allows us to study the twisted Selmer groups $Sel(\varphi_E \otimes \psi)_{/L}$ simply by studying the ψ-isotypic components $Sel(\varphi_F)_{/M}[\psi]$.

3 Universal Deformation Rings and Kähler Differentials

Let $\overline{\rho} = \varphi \pmod{\mathfrak{m}_{\mathbb{J}}} : \mathfrak{G} \to G(\mathbb{F})$ be the residual representation. We consider strict equivalence classes of deformations $\rho : \mathfrak{H}_L \to G(A)$ for $A \in CNL_{\mathcal{O}}$. To ensure the representability of the deformation functor $\Phi_{G,L}^{\phi}$, we enforce four strict cohomological and algebraic constraints:

- (AI_L) **Absolute Irreducibility:** $\overline{\rho}_L$ is absolutely irreducible over $\mathbb{F}$.
- (Z_L) **Centralizer Condition:** The centralizer of any deformation is exactly the scalar matrices in $G(A)$.
- (Reg_L) **Regularity:** $H^0(D_{\mathfrak{p}}, (\mathfrak{g}/\mathcal{P}_{\mathfrak{p}})(\mathbb{F})) = 0$ for all $\mathfrak{p} \mid p$, ensuring the vanishing of global obstructions to lifting.
- $(Z_{p,L})$ **Local Centralizer Condition:** The centralizer of deformations of local characters $\overline{\delta}_{j,\mathfrak{p}}$ over the inertia group is scalar.

Under these conditions, the functor $\Phi_{G,L}^{\phi}$ of strict equivalence classes of nearly ordinary deformations (with fixed similitude character and determinant ϕ) is representable by a universal couple $(R_{G,L}^{\phi}, \rho_{G,L}^{\phi})$. Furthermore, local deformation functors for the inertia groups $I_{\mathfrak{p}}$ and decomposition groups $D_{\mathfrak{p}}$ are representable by local rings $R_{G,L}^{I}$ and $R_{G,L}^{D}$. The global universal ring $R_{G,L}^{\phi}$ naturally possesses the structure of an algebra over the local deformation rings.

3.1 Mazur's Isomorphism via 1-Cocycles

To connect the arithmetic Selmer group with the algebraic deformation ring, we relate the continuous 1-differentials on $\mathrm{Spec}(R^{\phi}_{G,L})$ with 1-cocycles. For an R-module X, consider the algebra $R[X]$ with square-zero ideal X. An $\mathcal{O}$-algebra derivation $d_\xi : R \to X$ yields a deformation $\rho(\sigma) = \overline{\rho}(\sigma) \oplus u'_\rho(\sigma)$ in $G(R[X])$.

Defining $u_\rho(\sigma) = u'_\rho(\sigma)\rho(\sigma)^{-1}$, we observe that $u_\rho : \mathfrak{H}_L \to Ad_S(X)$ satisfies the 1-cocycle relation:

$$\begin{aligned} u_\rho(\sigma\tau) &= u'_\rho(\sigma\tau)\rho(\sigma\tau)^{-1} \\ &= \left[\rho(\sigma)u'_\rho(\tau) + u'_\rho(\sigma)\rho(\tau)\right]\rho(\tau)^{-1}\rho(\sigma)^{-1} \\ &= Ad_S(\rho)(\sigma)u_\rho(\tau) + u_\rho(\sigma). \end{aligned} \tag{8}$$

The near-ordinarity filtration condition is precisely verified if and only if $u_\rho|_{I_\mathfrak{p}}$ takes values in $V^-_\mathfrak{p}(Ad_S(X))^*$. This establishes the critical isomorphism.

Theorem 3.1 (Mazur's Isomorphism)**.** *Under the standard hypotheses (Z_L), (Reg_L), and $(Z_{p,L})$, the Pontryagin dual of the Selmer group is canonically isomorphic to the module of Kähler differentials:*

$$Sel^*(Ad_S(\varphi))_{/L} \cong \Omega_{R^{\phi}_{G,L}/R^I_{G,L}} \otimes_{R^{\phi}_{G,L}} \mathbb{J}. \tag{9}$$

Furthermore, the strict Selmer group fits into the exact sequence:

$$\Omega_{R^D_{G,L}/R^I_{G,L}} \otimes_{R^D_{G,L}} \mathbb{J} \longrightarrow Sel^*(Ad_S(\varphi))_{/L} \longrightarrow Sel^*_{st}(Ad_S(\varphi))_{/L} \longrightarrow 0. \tag{10}$$

4 Base Change and the Exact Control Theorem

We analyze the base-change morphism $\alpha_{L/L'} : R_{G,L} \to R_{G,L'}$ for a Galois extension L/L' with group Γ. An element $\rho \in \Phi^{\phi}_{L'}$ naturally restricts to $\Phi^{\phi,\Gamma}_{L}$. We bound the obstruction class $[\rho] \in H^2(\Gamma, \hat{G}_m(A))$ derived from Schur multipliers:

$$b_\rho(\sigma, \tau) = c(\sigma)c(\tau)c(\sigma\tau)^{-1} \in \hat{G}_m(A), \tag{11}$$

where $c : \mathfrak{H}_{L'} \to G(A)$ lifts the conjugate equivalences. Since $H^2(\Gamma, \hat{G}_m(A))$ is killed by the degree $[L : L']$, we deduce that if $p \nmid [L : L']$, the obstruction vanishes and $\Phi^{\phi}_{G,L'} \cong \Phi^{\phi,\Gamma}_{G,L}$.

For a totally ramified $\mathbb{Z}_p$-extension F_∞/F with $\Gamma_j = \mathrm{Gal}(F_\infty/F_j) = \langle\gamma_j\rangle$, the base-change map induces an exact quotient relation:

$$R^H_j \cong R^H_\infty/(\gamma_j - 1)R^H_\infty. \tag{12}$$

Applying this base change to the relative Kähler differentials provides the global Control Theorem.

Proposition 4.1 (Exact Control Theorem)**.** *The descent of the Selmer group from the Iwasawa level F_∞ to the base field F is exact up to a specific, calculable local error term. Writing the Kähler differentials explicitly, we obtain the fundamental exact sequence:*

$$\bigoplus_{\mathfrak{p}\in S_E} \mathbb{J}[S_\mathfrak{p}]^{m_\mathfrak{p}-1} \longrightarrow \frac{Sel^*(Ad_S(\varphi))_{/F_\infty}}{(\gamma_0 - 1)Sel^*(Ad_S(\varphi))_{/F_\infty}} \longrightarrow Sel^*(Ad_S(\varphi))_{/F} \longrightarrow 0, \tag{13}$$

where $S_\mathfrak{p}$ is the set of primes over $\mathfrak{p}$ and $m_\mathfrak{p}$ is the split rank of the center of the Levi-subgroup of $P_\mathfrak{p} \cap S$.

The exponent $m_\mathfrak{p} - 1$ appears because the fixed determinant condition $\det\rho = \phi_{\det}$ inherently annihilates one dimension of the Levi center's contribution.

5 Fujiwara's Theorem and p-Adic Hecke Algebras

We now specialize to $G = GL(2)$ over a totally real field F. Let $h^{n.ord}(p^\infty; \mathcal{O})_F$ be the universal nearly ordinary p-adic Hecke algebra, constructed via the projective limit $\lim_{\leftarrow\alpha} h(U_\alpha^{n.ord})$ acting on the middle cohomological degree $H^d_{cusp}(X(U_\alpha^{n.ord}), \mathcal{O})$ of the Hilbert modular variety $X(U_\alpha^{n.ord})$. The Hecke algebra is an algebra over the Iwasawa group ring $\mathcal{O}[[Cl_F(p^\infty) \times \mathcal{O}_p^\times]]$, with Hecke operators $\mathbb{T}(y)$ acting on the modular spaces.

For an arithmetic character κ_0, we consider the quotient $h^{\kappa_0} = h^{\phi,ord}$ corresponding to the specific weight and character.

Theorem 5.1 (Fujiwara / Wiles-Taylor)**.** *Assume* (NR_p) *(unramified at* p*, or* (LD_p) *if* $\overline{\rho}$ *is non-flat),* $(AI_{F(\sqrt{p^*})})$*,* (RG_F)*, and that* $\overline{\rho}$ *is* p*-ordinary. Then:*

1. ***Universality:*** *The local component of the Hecke algebra* $h^{\phi,ord}$ *and its associated Galois representation* ρ_{κ_0} *strictly represents the functor* $\Phi_F^{ord,\phi}$*.*
2. ***Isomorphism:*** *We have the canonical isomorphism* $R_F^{\phi,ord} \cong h^{\phi,ord}$ *as complete intersections over the Iwasawa algebra* $\mathcal{O}[[G_p^{ord}]]$*.*
3. ***Freeness:*** *The geometric module* $M_{\mathfrak{m}}^{\phi,ord}$ *is free over the Hecke algebra, providing a faithful algebraic manifestation of the Selmer group.*

Because of this profound identification $R_F \cong h_F$, the module of Kähler differentials of the Hecke algebra exactly computes the Pontryagin dual of the Selmer group.

6 Trivial Zeros and Characteristic Ideals

We analyze the behavior of the p-adic L-function at the trivial zero $T = 0$. Let $e_{\mathfrak{p}} = \mathrm{rank}_{\mathbb{J}} H^0(D_{\mathfrak{p}}, gr(Ad_S(\varphi))) = m_{\mathfrak{p}} - 1$ be the local dimension contributing to the trivial zero. For $G = GL(2)$, $m_{\mathfrak{p}} = 2$, so $e_{\mathfrak{p}} = 1$. Let $s = |S_F|$ be the total number of primes over p.

We parameterize the local inertia groups with $T_1, \dots, T_s \in \mathcal{O}[[I_F]]$ and define the Jacobian determinant of the Hecke operators $t_{\mathfrak{p}_i} = \mathbb{T}(\varpi_i) - \omega(\mathbb{T}(\varpi_i))$:

$$\mathcal{J}_{ac} = \det\left(\frac{\partial t_{\mathfrak{p}_i}}{\partial T_j}\right) \in \mathbb{I}. \tag{14}$$

Theorem 6.1 (Order of the Trivial Zero)**.** *Let* $\mathbb{J}$ *be an irreducible component of* $h_F^{\phi'}$*. Assume the universal properties hold for* F*, that* $\mathbb{J}$ *is a regular local ring, and that the Jacobian* $\mathcal{J}_{ac} \neq 0$ *under the projection to* $\mathbb{J}$*.*

Let $\Psi(T)$ *be the characteristic ideal of* $Sel^*(Ad(\varphi))_{/F_\infty}$ *over* $\mathbb{J}[[T]]$*. Then* $Sel^*(Ad(\varphi))_{/F_\infty}$ *has homological dimension 1 over* $\mathbb{J}[[T]]$ *and contains no non-null pseudo-null submodules. Furthermore, its characteristic ideal factors exactly as:*

$$\Psi(T) = \Phi(T)T^s \quad \textit{where} \quad \Phi(0) = \mathcal{J}_{ac} \cdot \eta \neq 0 \ \textit{(up to units)}, \tag{15}$$

where η *is the characteristic ideal of the finite residual module* $Sel^*(Ad(\varphi))_{/F}$*.*

6.1 Non-Vanishing of the Jacobian via Jacquet-Langlands

The non-vanishing of $\mathcal{J}_{ac}$ mathematically guarantees that the trivial zeros at $s = 0$ of the associated p-adic L-function occur with exact multiplicity $s = |S_F|$. The 'derivative' L-value (the Jacobian) intrinsically controls the arithmetic size of the residual Selmer group.

Proposition 6.2. *Suppose F_∞/F is the cyclotomic $\mathbb{Z}_p$-extension. If at least one arithmetic point $P : \mathbb{J} \to \mathcal{O}$ is associated to a p-divisible group of (potentially) multiplicative type at $\mathfrak{p}_1, \ldots, \mathfrak{p}_{s-1}$, then $\mathcal{J}_{ac} \neq 0$.*

Sketch of Proof. We use the Jacquet-Langlands correspondence applied to a carefully chosen quaternion algebra $B_{j/F}$. We construct $B_{j/F}$ such that it is division at $\mathfrak{p}_i$ for $i \leq j$ and split outside. For any arithmetic point Q, the associated modular form f_Q corresponds to an automorphic representation of $B_{j/F}^\times$ which is Steinberg at $\mathfrak{p}_i$ ($i \leq j$). This forces the eigenvalues $Q(t_i) = 0$. Using the density of arithmetic points and the modified Ramanujan bound $|Q(\mathbb{T}(\varpi_j))| = |\varpi_j|_p^{-n_\sigma - 1}$, we prove by induction that the parameters t_j are analytically independent over $\mathcal{O}[[I_0]]$, hence the Jacobian determinant cannot identically vanish. □

7 Application: Symmetric Powers of $GL(2)$

Let $\rho : \mathfrak{H} \to GL_2(\mathbb{J})$ be a nearly ordinary representation of Borel type. We consider the symmetric k-th tensor $\varphi = Sym^k(\rho) : \mathfrak{H} \to GL_{k+1}(\mathbb{J})$. The adjoint representation decomposes completely:

$$Ad_{SL(k+1)}(\varphi) \cong \bigoplus_{j=1}^{k} \det(\rho)^{-j} Sym^{2j}(\rho). \tag{16}$$

Writing $\varphi_j = \det(\rho)^{-j} Sym^{2j}(\rho)$, the Selmer datum of $Ad_{SL(k+1)}(\varphi)$ splits, yielding a profound isomorphism of Selmer groups:

$$Sel(Ad_{SL(k+1)}(\varphi))_{/L} \cong \bigoplus_{j=1}^{k} Sel(\varphi_j)_{/L}. \tag{17}$$

When ρ is associated to a rank 2 pure motive $M_{/E}$, the representation φ_j corresponds to the motive $M_j = \det(M^\vee)^j \otimes Sym^{2j}(M)$. M_j is critical if and only if j is odd. Our control theorems apply to each odd component φ_j, predicting that the corresponding p-adic L-functions $L_p(s, \varphi_j)$ possess trivial zeros of exact order $|S_F|$ at $s = 0$.