Seminar Notes: Adjoint Modular Galois Representations and their Selmer Groups

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Abstract

This seminar provides a detailed exposition of the 1997 colloquium paper by Hida, Tilouine, and Urban. We trace the formulation of class number formulas for the Selmer groups of the adjoint representation $ad(\phi)$ associated to a two-dimensional modular Galois representation ϕ. The progression spans the classical elliptic curve case, the one-variable Iwasawa theoretic setting over weight space, and the two-variable Main Conjecture involving the cyclotomic extension. Finally, we review the strategy for proving the two-variable Main Conjecture utilizing p-adic Siegel modular forms on $GSp(4)$.

1 Galois Representations and the Selmer Group

Let $p \geq 5$ be a fixed prime. Let $G = \mathrm{Gal}(\mathbb{Q}^{(p)}/\mathbb{Q})$ be the Galois group of the maximal extension unramified outside $\{p, \infty\}$. Let $\mathcal{O}$ be a finite flat valuation ring over $\mathbb{Z}_p$ with residue field $\mathbb{F}$. Consider a complete noetherian local $\mathcal{O}$-algebra A. Let $\phi : G \to GL_2(A)$ be a continuous two-dimensional Galois representation acting on $V = A^2$. The group G acts on $\mathrm{End}(V)$ by conjugation, and we isolate the three-dimensional trace-zero subspace $V(ad(\phi))$, affording the adjoint representation $ad(\phi) : G \to GL_3(A)$.We impose three structural conditions on ϕ:

1. **(AI)** Absolute Irreducibility over G (equivalent to the absolute irreducibility of ϕ restricted to the specified quadratic field).
2. **(Ord)** p-Ordinarity: For the decomposition group D_p, the restriction is upper-triangular with an unramified character δ:
$$\phi|_{D_p} \cong \begin{pmatrix} \delta & * \\ 0 & \epsilon \end{pmatrix}$$
3. **(Reg)** Regularity: The residual characters $\delta \pmod{\mathfrak{m}_A}$ and $\epsilon \pmod{\mathfrak{m}_A}$ are distinct.

Definition 1.1 (Selmer Group). *Let $V(\delta) \subset V$ be the δ-eigenspace. Define $V_+ = \{\xi \in V(ad(\phi)) \mid \xi(V(\delta)) = 0\}$. Following Greenberg, the Selmer group is defined as the kernel of the restriction map to inertia I_p:*
$$Sel(ad(\phi)) = \ker\left(H^1(G, V(ad(\phi))^*) \to H^1(I_p, V(ad(\phi))^*/V_+^*)\right)$$
where X^* denotes the Pontryagin dual.

If $A = \mathcal{O}$, a naive expectation is that the order of $Sel(ad(\phi))$ is governed by the p-part of $L(1, ad(\phi))$.

2 The Classical Case: Elliptic Curves over $\mathbb{Q}$

Assume ϕ_0 is the Galois representation attached to a modular elliptic curve $E/\mathbb{Q}$ situated inside the Jacobian $J = J_0(p)$. E has multiplicative reduction at p. Consider the dual projection $\pi : J \to E$. We obtain the decomposition $J = E + A$ where $A = \ker(\pi)$, and the intersection $E \cap A$ is finite. Let $f_0 \in S_2(\Gamma_0(p))$ be the associated primitive form. Using the canonical period $U(f_0) = C^{-1}(2\pi i)\Omega_+\Omega_-$, the intersection number formula is given by:

$$\frac{L(1, ad(\phi_0))}{U(f_0)} = \sqrt{|E \cap A|}$$

Let $\mathbb{H}$ be the Hecke algebra in $\mathrm{End}(J)$. The projection $\lambda : \mathbb{H} \to \mathbb{Z} \subset \mathrm{End}(E)$ and $\lambda' : \mathbb{H} \to \mathrm{End}(A)$ allow us to define two critical finite modules:

$$C_0 = \mathrm{Im}(\lambda) \otimes_{\mathbb{H}} \mathrm{Im}(\lambda')$$

$$C_1 = \Omega_{\mathbb{H}/\mathbb{Z}} \otimes_{\mathbb{H}} \mathrm{Im}(\lambda) = \ker(\lambda)/\ker(\lambda)^2$$

The scheme-theoretic intersection $C_{0,p}$ mirrors the physical intersection $E \cap A$. Crucially, Wiles and Taylor proved that $|C_{0,p}|[cite_start] = |C_{1,p}|$. Furthermore, Wiles established the isomorphism $C_{1,p} \cong Sel(ad(\phi_0))^*$. This yields the celebrated order formula:

$$p\text{-part of } \frac{L(1, ad(\phi_0))}{U(f_0)} = |Sel(ad(\phi_0))|$$

3 One-Variable Iwasawa Theory

We lift f_0 to a p-adic family of p-ordinary eigenforms $f_k \in S^*_{k+2}(\Gamma_0(p))$. This family produces a Galois representation $\phi : G \to GL_2(\Lambda)$ where $\Lambda = \mathcal{O}[[T]]$. The Pontryagin dual $Sel^*(ad(\phi))$ is a torsion Λ-module of finite type.Let S_Λ be the space of p-ordinary Λ-adic cusp forms, which is free of finite rank over Λ. Let $\mathbf{f}$ be the unique Λ-adic form interpolating f_k. We construct the Λ-adic congruence modules analogously:

$$C_{0,\Lambda} = \mathrm{Im}(\lambda) \otimes_{\mathbb{H}} \mathrm{Im}(\lambda')$$

$$C_{1,\Lambda} = \Omega_{\mathbb{H}/\Lambda} \otimes_{\mathbb{H}} \mathrm{Im}(\lambda) = \ker(\lambda)/\ker(\lambda)^2$$

We have $C_{0,\Lambda} \cong \Lambda/(\eta(T))$ for a characteristic power series $\eta(T) \in \Lambda$. By the Wiles-Taylor machinery, we deduce:

$$(\eta(T)) = char_\Lambda(C_{1,\Lambda}) \quad \text{and} \quad C_{1,\Lambda} \cong Sel^*(ad(\phi))$$

Evaluating at classical weights $u^k - 1$, we recover the interpolation of L-values:

$$\eta(u^k - 1) = \frac{L(1, ad(\phi_k))}{U(f_k)} \quad \text{(up to } p\text{-adic units)}$$

We view $\eta(T)$ as the algebraic avatar of the canonical p-adic L-function.

4 Two-Variable Case and the Main Conjecture

We introduce the universal cyclotomic character $\nu : G \to \mathbb{Z}_p[[S]]^\times$, where $S = \gamma - 1$ for a topological generator $\gamma \in \Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. We study the enlarged Selmer group $Sel^*(ad(\phi) \otimes \nu^{-1})$ over the two-variable Iwasawa algebra $\mathcal{O}[[T, S]]$.

Theorem 4.1 (Greenberg, Tilouine, Urban). *The module $Sel^*(ad(\phi) \otimes \nu^{-1})$ is a torsion $\mathcal{O}[[T, S]]$-module of finite type. Its characteristic power series takes the form $S\Psi(T, S)$, satisfying the Control Theorem:*

$$\Psi(T, 0) \Big| \eta(T) \frac{da}{dT}(T)$$

where $a(T)$ is the eigenvalue of $T(p)$.

Analytically, Hida constructed a two-variable p-adic L-function $L(T, S) \in \eta(T)^{-1} S\mathcal{O}[[T, S]]$ interpolating critical values. Writing $\eta(T)L(T, S) = S\Phi(T, S)$, the central objective of the theory is to establish the following:

Conjecture 4.2 (Main Conjecture). *Up to a unit in $\mathcal{O}[[T, S]]$, we have:*

$$(\Phi) = (\Psi)$$

5 Strategy of Proof via Siegel Modular Forms

To prove the divisibility $\Phi \mid \Psi$, Urban employs the arithmetic of p-adic families of Siegel modular forms on $GSp(4)$. Let f be a nearly p-ordinary cohomological Hecke eigenform on $GSp(4)$. Weissauer attaches a Galois representation $\rho_f \to GL_4$ whose characteristic polynomials match the Hecke eigenvalues outside p.

Assumption 5.1 (Ordinarity Conjecture). *For a nearly p-ordinary f, the image of the decomposition group at p under ρ_f is contained in a Borel subgroup of $GSp(4)$.*

We construct a Klingen-style Eisenstein series $\mathcal{E}$ over $\mathcal{O}[[T, S]]$ induced from the Λ-adic form $\mathbf{f}$. The Galois representation $\rho_{\mathcal{E}}$ maps into the standard maximal parabolic subgroup:

$$\rho_{\mathcal{E}} \cong \begin{pmatrix} \phi & * \\ 0 & \nu \det(\phi)\, {}^t\phi^{-1} \end{pmatrix} \subset GSp_4(\mathcal{O}[[T, S]])$$

The constant term calculation shows that the Eisenstein ideal Eis, which measures congruences between $\mathcal{E}$ and cusp forms g, is generated by $\eta(T)L(T, S)$. For an Eisenstein prime $P \mid \Phi(T, S)$, suppose $g \equiv \mathcal{E} \pmod{P}$. Under the Ordinarity Conjecture, ρ_g takes values in $GSp(4)$ and is absolutely irreducible, even though it is residually reducible.The adjoint action of $\rho_{\mathcal{E}}$ on the unipotent radical of the parabolic subgroup is isomorphic to $ad(\phi) \otimes \nu^{-1}$. The irreducibility of ρ_g forces the extension modulo P to be non-split. This yields a non-trivial cohomology class in $Sel(ad(\phi) \otimes \nu^{-1})$. By applying this to each height-one prime dividing the Eisenstein ideal, we bound the Selmer group from below, establishing the divisibility:

$$\mathrm{Eis} \mid \Psi$$

Consequently, the precise structure of the Klingen Eisenstein congruences fundamentally implies the two-variable Main Conjecture.