Seminar Notes: Λ-adic Forms of Half-Integral Weight

Advanced Mathematical Formulations of the Λ-adic Shimura Correspondence

Xiaorun Wu

Based on: Hida, H., *On Λ-adic forms of half integral weight for $SL(2)/\mathbb{Q}$*,
LMS Lecture Note Series 215 (1995), 139–166.
April 6, 2026

Abstract

This document presents an exhaustive, mathematically dense exposition of Haruzo Hida's 1995 construction of the Λ-adic Shimura correspondence. We provide explicit definitions of the adelic metaplectic cocycles, double coset decompositions for half-integral Hecke operators, the Katz–Igusa geometric framework for p-adic modular forms, the universal ordinary Hecke algebras, and the p-adic interpolation of the Weil representation via theta measures.

Contents

1 The Classical Shimura Correspondence

1.1 Classical Spaces and Automorphic Factors

Let $N, k \in \mathbb{Z}_{\geq 1}$ with $4 \mid N$, and let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Define the standard theta series:

$$\theta(z) = \sum_{n\in\mathbb{Z}} \exp(2\pi i n^2 z), \quad z \in \mathfrak{H} \tag{1}$$

The theta multiplier system for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ is defined as:

$$j(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)} = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz + d)^{1/2} \tag{2}$$

where $\left(\frac{c}{d}\right)$ is the extended Jacobi symbol, and ε_d is the root of unity:

$$\varepsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4 \\ i & d \equiv 3 \pmod 4 \end{cases} \tag{3}$$

Definition 1.1 (Classical Space $S_{k+1/2}(N, \chi)$)**.** *The space $S_{k+1/2}(N, \chi)$ consists of holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$, vanishing at the cusps of $\Gamma_0(N)$, satisfying the transformation law for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$:*

$$f\left(\frac{az + b}{cz + d}\right) = \chi(d) j(\gamma, z)^{2k+1} f(z) = \chi(d)\varepsilon_d^{-2k-1} \left(\frac{c}{d}\right)^{2k+1} (cz + d)^{k+1/2} f(z) \tag{4}$$

1.2 Shimura's Theorem and Dirichlet Series

Theorem 1.2 (Shimura, 1973)**.** *Let $t \in \mathbb{Z}_{\geq 1}$ be square-free. There exists a Hecke-equivariant linear map $\mathcal{S}_t : S_{k+1/2}(N, \chi) \to S_{2k}(N/2, \chi^2)$. If $f(z) = \sum_{n=1}^\infty a(n)q^n \in S_{k+1/2}(N, \chi)$ is a common eigenform for all $\tilde{T}_{l^2}$ $(l \nmid N)$, then $F(z) = \mathcal{S}_t(f)(z) = \sum_{n=1}^\infty A(n)q^n$ is an eigenform for T_l, determined by the formal identity of Dirichlet series:*

$$\sum_{n=1}^{\infty} A(n)n^{-s} = L(s - k + 1, \chi\chi_t\psi_k) \sum_{m=1}^{\infty} a(tm^2)m^{-s} \tag{5}$$

where the quadratic characters are defined as:

$$\chi_t(d) = \left(\frac{t}{d}\right), \quad \psi_k(d) = \left(\frac{-1}{d}\right)^k \tag{6}$$

2 Adelic Metaplectic Structures

2.1 Local and Global Cocycles

Let $S = \mathrm{SL}(2)_{/\mathbb{Q}}$. The metaplectic group $\tilde{S}(\mathbb{A})$ requires a rigorous definition via local 2-cocycles $\beta_v \in H^2(S(\mathbb{Q}_v), \{\pm 1\})$.

Definition 2.1 (Hilbert Symbol). *For $c, d \in \mathbb{Q}_v^\times$, $(c,d)_v \in \{\pm 1\}$. It is the unique non-degenerate bilinear pairing such that $(c,d)_v = 1 \iff \exists x, y, z \in \mathbb{Q}_v$ (not all zero) solving $z^2 - cx^2 - dy^2 = 0$.*

Define the rational section $x : S(\mathbb{Q}_v) \to \mathbb{Q}_v$ by:

$$x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0 \end{cases} \tag{7}$$

The Kubota cocycle $\beta_v(g_1, g_2)$ for $g_1, g_2 \in S(\mathbb{Q}_v)$ is defined as:

$$\beta_v(g_1, g_2) = \left(\frac{x(g_1 g_2)}{x(g_1)}, \frac{x(g_1 g_2)}{x(g_2)}\right)_v (x(g_1), x(g_2))_v (-1, w_v)_v \tag{8}$$

where $w_v = x(g_1 g_2) x(g_1)^{-1} x(g_2)^{-1}$.

Definition 2.2 (Topological Central Extension). *$\tilde{S}(\mathbb{A})$ is defined as the set $S(\mathbb{A}) \times \{\pm 1\}$ with the group law:*

$$(g_1, \zeta_1) \cdot (g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 \prod_v \beta_v(g_1, g_2)) \tag{9}$$

This gives the exact sequence:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \tilde{S}(\mathbb{A}) \xrightarrow{\pi} S(\mathbb{A}) \longrightarrow 1 \tag{10}$$

2.2 Genuine Representations

Because $\prod_v (c,d)_v = 1$ for all $c, d \in \mathbb{Q}^\times$, the cocycle is trivial on $S(\mathbb{Q}) \times S(\mathbb{Q})$.

Lemma 2.3. *There exists a unique group homomorphism $s : S(\mathbb{Q}) \to \tilde{S}(\mathbb{A})$ defined by $s(\gamma) = (\gamma, 1)$.*

The space of automorphic forms $\mathcal{A}_{k+1/2}(\tilde{S})$ consists of smooth, slowly increasing functions $\varphi : \tilde{S}(\mathbb{A}) \to \mathbb{C}$ satisfying:

$$\varphi(s(\gamma)g) = \varphi(g) \quad \forall \gamma \in S(\mathbb{Q}) \tag{11}$$

$$\varphi(\zeta g) = \zeta \varphi(g) \quad \forall \zeta \in \{\pm 1\} \text{ (Genuineness condition)} \tag{12}$$

$$\varphi(g k_\theta) = \exp(i(k + 1/2)\theta)\varphi(g) \quad \forall k_\theta \in \widetilde{SO}(2, \mathbb{R}) \tag{13}$$

3 Explicit Hecke Operators and Double Cosets

The representation theory of $\tilde{S}(\mathbb{Q}_l)$ diverges from $S(\mathbb{Q}_l)$ because the covering does not split over the maximal compact subgroup $K_l = \mathrm{SL}_2(\mathbb{Z}_l)$. It only splits over specific congruence subgroups.

3.1 Double Coset Decomposition

For a prime $l \nmid N$, the local Hecke algebra for the metaplectic group is generated not by double cosets of degree l, but by degree l^2. Define the element $\delta_{l^2} = \left(\begin{smallmatrix} l^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We require the decomposition of the double coset $\Gamma_0(N)\delta_{l^2}\Gamma_0(N)$ into left cosets.

Lemma 3.1. *The double coset decomposes as the disjoint union of $l^2 + l + 1$ left cosets:*

$$\Gamma_0(N)\delta_{l^2}\Gamma_0(N) = \coprod_{b \pmod{l^2}} \Gamma_0(N)\begin{pmatrix} 1 & b \\ 0 & l^2 \end{pmatrix} \coprod_{h \pmod{l}} \Gamma_0(N)\begin{pmatrix} l & h \\ 0 & l \end{pmatrix}\begin{pmatrix} a_h & b_h \\ N & 0 \end{pmatrix} \coprod \Gamma_0(N)\begin{pmatrix} l^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

where a_h, b_h are chosen such that $a_h \equiv 1 \pmod{N}$ and the determinant is 1.

3.2 Action on Fourier Coefficients

Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$. The action of $\tilde{T}_{l^2}$ is defined via this double coset decomposition, factoring in the automorphic factor $j(\gamma, z)^{2k+1}$.

Theorem 3.2. *The action on Fourier coefficients yields:*

$$a(n; f|\tilde{T}_{l^2}) = a(nl^2) + \chi(l)\left(\frac{(-1)^k n}{l}\right) l^{k-1}a(n) + \chi(l^2)l^{2k-1}a(n/l^2) \tag{15}$$

where $a(n/l^2) = 0$ if $l^2 \nmid n$.

3.3 The U_{p^2} Operator and Ordinarity

For a prime $p \mid N$, the double coset decomposition for δ_{p^2} simplifies because the lower-left entry is divisible by the level.

$$\Gamma_0(N)\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}\Gamma_0(N) = \coprod_{u \pmod{p^2}} \Gamma_0(N)\begin{pmatrix}1 & u\\ 0 & p^2\end{pmatrix} \tag{16}$$

This leads directly to the operator U_{p^2}:

$$(f|U_{p^2})(z) = \frac{1}{p^2}\sum_{u=0}^{p^2-1} f\left(\frac{z+u}{p^2}\right) = \sum_{n=1}^{\infty} a(np^2)q^n \tag{17}$$

Definition 3.3 (Hida Ordinary Projector)**.** *Fix $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathcal{O}$ be a p-adic valuation ring. The projector $e \in End_{\mathcal{O}}(S_{k+1/2}(N, \chi; \mathcal{O}))$ is defined as the p-adic limit in the strong operator topology:*

$$e = \lim_{m\to\infty} (U_{p^2})^{m!} \tag{18}$$

The ordinary subspace is defined as $\tilde{V}_k = eS_{k+1/2}(N, \chi; \mathcal{O})$.

4 Geometric Framework: Katz and Igusa Curves

To interpolate forms of different weights, we rely on Katz's geometric definition of p-adic modular forms.

4.1 Metaplectic Sheaves over Moduli Stacks

Let $X_1(4N)$ be the modular curve over $\mathbb{Z}_p$ classifying elliptic curves E with a point of order $4N$. Let $\pi : \mathcal{E} \to X_1(4N)$ be the universal elliptic curve. Define the relative dualizing sheaf $\omega = \pi_*\Omega^1_{\mathcal{E}/X_1(4N)}$. For half-integral weights, we introduce a line bundle $\mathcal{M}$ equipped with an isomorphism:

$$\mathcal{M}^{\otimes 2} \xrightarrow{\sim} \omega \quad \text{over } X_1(4N) \tag{19}$$

A classical form of weight $k + 1/2$ is a global section $f \in H^0(X_1(4N), \mathcal{M} \otimes \omega^{\otimes k})$.

4.2 The Ordinary Locus and the Igusa Tower

Let $A \in H^0(X_1(4N)_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$ be the Hasse invariant. The ordinary locus $X_1(4N)^{ord}$ is the open affine subscheme where A is invertible. Over $X_1(4N)^{ord}$, the p-divisible group $E[p^\infty]$ sits in an exact sequence:

$$0 \longrightarrow \hat{E} \longrightarrow E[p^\infty] \longrightarrow E_{et} \longrightarrow 0 \tag{20}$$

where $\hat{E} \cong \mu_{p^\infty}$ is the connected component.

Definition 4.1 (Igusa Curve)**.** *The Igusa curve $Ig(p^m)$ over $X_1(4N)^{ord}$ classifies trivializations of the connected component:*

$$\phi_m : \mu_{p^m} \xrightarrow{\sim} E[p^m]^o \tag{21}$$

This induces an isomorphism $\phi_m^* : \omega_{E/Ig} \xrightarrow{\sim} \mathcal{O}_{Ig(p^m)}$.

4.3 p-adic Modular Forms

The Igusa tower is the inverse limit $Ig(p^\infty) = \varprojlim_m Ig(p^m)$. It carries a natural action of $\mathbb{Z}_p^\times$ via automorphisms of μ_{p^∞}.

Definition 4.2. *A p-adic modular form of half-integral weight is an element of the coordinate ring of the Igusa tower:*

$$V_{\infty,\mathcal{O}} = H^0(Ig(p^\infty)_\mathcal{O}, \mathcal{M}) \tag{22}$$

Because sections of ω are trivialized over $Ig(p^\infty)$, classical forms of any weight embed into $V_{\infty,\mathcal{O}}$. The action of $x \in \mathbb{Z}_p^\times$ on a classical form of weight $k+1/2$ is given by multiplication by $x^{k+1/2}$.

5 Λ-adic Moduli and Universal Hecke Algebras

5.1 The Iwasawa Algebra Λ

Let $\Gamma = 1+p\mathbb{Z}_p$. The Iwasawa algebra is the continuous group ring $\Lambda = \mathcal{O}[[\Gamma]]$. Fix a topological generator $u = 1+p$. The map $\gamma \mapsto 1+X$ gives an isomorphism $\Lambda \cong \mathcal{O}[[X]]$. Arithmetic primes $P_k \in \mathrm{Spec}(\Lambda)$ are kernels of continuous homomorphisms $\kappa_k : \Lambda \to \overline{\mathbb{Q}}_p$ given by $\kappa_k(u) = u^k$ for $k \in \mathbb{Z}_{\geq 2}$.

5.2 The Module of Λ-adic Forms

Definition 5.1 (Ordinary Λ-adic Space)**.** *The space $\tilde{\mathbb{S}}(N, \chi; \Lambda)$ consists of formal power series:*

$$\mathcal{F} = \sum_{n=1}^{\infty} A_n(\mathcal{F}) q^n \in \Lambda[[q]] \tag{23}$$

satisfying the specialization property: $\forall k \geq 2$,

$$\mathcal{F} \pmod{P_k} = \sum_{n=1}^{\infty} \kappa_k(A_n(\mathcal{F})) q^n \in eS_{k+1/2}(\Gamma_0(4Np^r), \chi\omega^{-k}) \tag{24}$$

where ω is the Teichmüller character.

Theorem 5.2 (Geometric Freeness)**.** *Using the exactness of the ordinary projector e on the cohomology of the Igusa tower, the space $\tilde{\mathbb{S}}(N, \chi; \Lambda)$ is a finite, free Λ-module of rank d.*

5.3 Universal Ordinary Hecke Algebras

Definition 5.3 (Hecke Algebras)**.** *We define the integral and half-integral universal algebras as Λ-subalgebras of endomorphisms:*

$$\mathbf{h}(N, \chi^2; \Lambda) = \Lambda[\{T_l\}_{l \nmid Np}, U_p] \subset End_\Lambda(\mathbb{S}(N, \chi^2; \Lambda)) \tag{25}$$

$$\tilde{\mathbf{h}}(4N, \chi; \Lambda) = \Lambda[\{\tilde{T}_{l^2}\}_{l \nmid 4Np}, U_{p^2}] \subset End_\Lambda(\tilde{\mathbb{S}}(N, \chi; \Lambda)) \tag{26}$$

Theorem 5.4 (Vertical Control Theorem)**.** *There exist canonical isomorphisms of finite free $\mathcal{O}$-algebras:*

$$\mathbf{h}(N, \chi^2; \Lambda)/P_k\mathbf{h} \xrightarrow{\sim} \mathbf{h}_{2k}^{ord}(Np^r, \chi^2\omega^{-2k}; \mathcal{O}) \tag{27}$$

$$\tilde{\mathbf{h}}(4N, \chi; \Lambda)/P_k\tilde{\mathbf{h}} \xrightarrow{\sim} \tilde{\mathbf{h}}_{k+1/2}^{ord}(4Np^r, \chi\omega^{-k}; \mathcal{O}) \tag{28}$$

6 The Weil Representation and Theta Interpolation

The core of Hida's construction of the morphism between these algebras relies on interpolating the theta distribution associated to the Weil representation.

6.1 The Dual Reductive Pair

Let V be the quadratic space over $\mathbb{Q}$ defined by $V = \{x \in M_2(\mathbb{Q}) \mid \mathrm{Tr}(x) = 0\}$ equipped with the quadratic form $Q(x) = -\det(x)$. The orthogonal group is $O(V) \cong \mathrm{PGL}(2)$. The groups $\tilde{S}$ and $O(V)$ form a reductive dual pair inside the symplectic group $Sp(W)$, where $W = V \otimes \mathbb{Q}^2$.

6.2 The Weil Representation ω_ψ

Let $\psi = \prod \psi_v$ be the standard additive character of $\mathbb{A}/\mathbb{Q}$. The Weil representation ω_ψ of $\tilde{S}(\mathbb{A}) \times O(V)(\mathbb{A})$ acts on the space of Schwartz-Bruhat functions $\mathcal{S}(V(\mathbb{A}))$. For $g \in \tilde{S}(\mathbb{A})$, the action on $\Phi \in \mathcal{S}(V(\mathbb{A}))$ is explicitly generated by:

$$\omega_\psi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\Phi(x) = \psi(bQ(x))\Phi(x) \tag{29}$$

$$\omega_\psi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)\Phi(x) = |a|_{\mathbb{A}}^{3/2}\Phi(ax) \tag{30}$$

$$\omega_\psi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\Phi(x) = \gamma \int_{V(\mathbb{A})} \Phi(y)\psi(\langle x, y\rangle)\, dy \tag{31}$$

where $\langle x, y\rangle = Q(x+y) - Q(x) - Q(y)$ is the bilinear form.

6.3 Theta Kernels and the Integral Lift

Define the theta kernel:

$$\Theta(g, h; \Phi) = \sum_{x \in V(\mathbb{Q})} \omega_\psi(g)\Phi(h^{-1}x) \tag{32}$$

The classical Shimura lift is the integral operator:

$$\mathcal{S}(f)(h) = \int_{\tilde{S}(\mathbb{Q})\backslash\tilde{S}(\mathbb{A})} f(g)\Theta(g, h; \Phi)\, dg \tag{33}$$

6.4 p-adic Interpolation of Theta Measures

Hida defines a sequence of local Schwartz functions $\Phi_{m,p} \in \mathcal{S}(V(\mathbb{Q}_p))$ that vary analytically. Specifically, he constructs a bounded measure $d\mu_\Theta$ on $\Gamma \cong 1 + p\mathbb{Z}_p$ with values in the space of p-adic modular forms on $X_1(N/2)$.

For a Λ-adic half-integral weight form $\mathcal{F} \in \tilde{\mathbb{S}}$, the convolution integral:

$$\mathcal{G} = \mathcal{F} * d\mu_\Theta := \int_{\mathbb{Z}_p^\times} \mathcal{F}(g)\, d\mu_\Theta(g, h) \tag{34}$$

converges p-adically and defines a map $\tilde{\mathbb{S}} \to \mathbb{S}$. Crucially, because the Weil representation commutes with the Hecke action, this induces an equivariant map on the Hecke algebras.

7 Main Theorem: The Λ-adic Shimura Lift

Theorem 7.1 (Λ-adic Shimura Correspondence)**.** *The convolution pairing induces a canonical, surjective Λ-algebra homomorphism Sh rendering the following diagram commutative for all* $k \geq 2$*:*

$$\begin{array}{ccc} \mathbf{h}(N, \chi^2; \Lambda) & \xrightarrow{\quad Sh \quad} & \tilde{\mathbf{h}}(4N, \chi; \Lambda) \\ {\scriptstyle mod\ P_k}\downarrow & & \downarrow{\scriptstyle mod\ P_k} \\ \mathbf{h}_{2k}^{ord}(Np^r, \chi^2\omega^{-2k}; \mathcal{O}) & \xrightarrow{\mathcal{S}_t^{class}} & \tilde{\mathbf{h}}_{k+1/2}^{ord}(4Np^r, \chi\omega^{-k}; \mathcal{O}) \end{array} \tag{35}$$

The map Sh is uniquely determined by its action on the dense set of Hecke generators:

$$Sh(T_l) = \tilde{T}_{l^2} \quad \textit{for all primes } l \nmid 4Np \tag{36}$$

$$Sh(U_p) = U_{p^2} \tag{37}$$

8 Galois Representations and Congruence Ideals

The existence of Sh allows the transfer of arithmetic data from GL(2) to the metaplectic cover.

8.1 Pullback of Galois Representations

Let $\lambda : \tilde{\mathbf{h}}(4N, \chi; \Lambda) \to \overline{\mathbb{Q}}_p$ be an $\mathcal{O}$-algebra homomorphism (corresponding to a Λ-adic eigenform in $\tilde{\mathbb{S}}$). By Theorem 7.1, λ factors as:

$$\begin{array}{ccc} \mathbf{h}(N, \chi^2; \Lambda) & \xrightarrow{\quad \text{Sh} \quad} & \tilde{\mathbf{h}}(4N, \chi; \Lambda) \\ & {\scriptstyle \lambda_{int}}\searrow \quad \swarrow{\scriptstyle \lambda} & \\ & \overline{\mathbb{Q}}_p & \end{array} \tag{38}$$

Theorem 8.1 (Galois Representation)**.** *To the integral weight homomorphism* λ_{int}*, Hida theory associates a continuous, ordinary Galois representation:*

$$\rho_\lambda : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(Frac(\Lambda)) \tag{39}$$

unramified outside Np*. For* $l \nmid Np$*, the characteristic polynomial of* $\rho_\lambda(Frob_l)$ *is given directly by the half-integral Hecke eigenvalues:*

$$X^2 - \lambda(\tilde{T}_{l^2})X + \chi^2(l)\langle l \rangle l^{-1} = 0 \tag{40}$$

Thus, $Tr(\rho_\lambda(Frob_l)) = \lambda(\tilde{T}_{l^2})$*.*

8.2 Congruence Ideals

Definition 8.2 (Congruence Ideal)**.** *Let* $\mathcal{I} = \ker(Sh)$*. We have the exact sequence of Λ-modules:*

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathbf{h}(N, \chi^2; \Lambda) \xrightarrow{Sh} \tilde{\mathbf{h}}(4N, \chi; \Lambda) \longrightarrow 0 \tag{41}$$

The structure of $\mathcal{I}$ governs congruences between the image of the Shimura lift and the complementary subspace of forms. By the Iwasawa Main Conjecture, the characteristic ideal of the Selmer group associated to ρ_λ is intimately tied to the p-adic L-function $\mathcal{L}_p(s, \text{Sym}^2(\rho_\lambda))$, which itself interpolates the critical values governed by the Fourier coefficients of the half-integral weight family.