Modules of Congruence of Hecke Algebras and L-functions Associated with Cusp Forms (Hida 1988)

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Overview

This talk is about Hida's 1988 paper: *Modules of Congruence of Hecke Algebras and L-Functions Associated with Cusp Forms*, which builds crucially on his previous works, some of which we have discussed in the seminar (eg. *Galois representations into $GL_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms* and *Iwasawa modules attached to congruences of cusp forms*, both published in 1986).

The main result is of the flavor that we are now familiar with from various important conjectures in number theory: 'analytic invariant' = 'arithmetic invariant'.

The Hida-theoretic flavor of the result lies in the fact that we are not only producing this identity at one cusp form. Instead, we provide a p-adic interpolation of one variable of the canonical algebraic part of the special value of a certain L-function by varing the form along the spectrum of the corresponding irreducible component of the p-adic universal ordinary Hecke algebra. To be more precise, we have to make several definitions.

1 Preliminary Definition

Let $f = \sum_{n=1}^{\infty} a(n,f)q^n$ be a primitive cusp form of $S_k(\Gamma_0(M),\psi)$. One can define the L-function associated to f using the Fourier coefficients, and written in Euler product form using the eigenvalues of Frobenius as:

$$L(s,f) := \sum_{n=1}^{\infty} a(n,f)n^{-s} = \prod_{\ell} \left[(1-\alpha_\ell \ell^{-s})(1-\beta_\ell \ell^{-s})\right]^{-1}.$$

However, we are working with a different L-function (here ψ_0 is the primitive character that induces ψ)

$$\mathfrak{D}(s,f) := \prod_{\ell} \left[(1-\bar{\psi}_0(\ell)\alpha_\ell^2 \ell^{-s})(1-\bar{\psi}_0(\ell)\alpha_\ell\beta_\ell \ell^{-s})(1-\bar{\psi}_0(\ell)\beta_\ell^2 \ell^{-s})\right]^{-1}.$$

Conceptually, this L-function is closely related to the adjoint L-function, which is the L-function associated with the symmetric square representation of π_f.

On the arithmetic side, we need to introduce the p-adic universal ordinary Hecke algebra.

Fix $\mathcal{O}$ a DVR finite flat over $\mathbb{Z}_p$ inside $\bar{\mathbb{Q}}_p$ and K its fraction field. Fix a positive integer N prime to p and let $S_k(\Gamma_1(Np^r))$ be the space of holomorphic cusp forms of weight k for the respective congruence subgroup. Define Hecke operators $T(n)$ and actions of integers ℓ prime to Np^r as usual.

Let $h_k(\Gamma_1(Np^r);\mathbb{Z})$ be the subalgebra of the linear endomorphism algebra of $S_k(\Gamma_1(Np^r))$ generated over $\mathbb{Z}$ by the $T(n)$'s for all n. The action of the ℓ's are already contained inside and we denote them by the diamond operator $\langle \ell \rangle$.

If we let $S_k(\Gamma_1(Np^r);\mathbb{Z}) \subset S_k(\Gamma_1(Np^r))$ be the subspace of integral Fourier coefficients, then we can define a pairing (which is perfect over $\mathbb{Z}$)

$$\langle\, ,\, \rangle : h_k(\Gamma_1(Np^r);\mathbb{Z}) \times S_k(\Gamma_1(Np^r);\mathbb{Z}) \to \mathbb{Z} \text{ by } \langle h\, ,\, f\rangle = a(1, f|h).$$

Put $h_k(\Gamma_1(Np^r);\mathcal{O}) := h_k(\Gamma_1(Np^r);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}$. We know it is free of finite rank over $\mathcal{O}$ with rank equals to the dimension of $S_k(\Gamma_1(Np^r))$.

Let

$$Z_N = \varprojlim_r (\mathbb{Z}/Np^r\mathbb{Z})^\times \cong \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$$

. By viewing $\ell \in Z_N$, we can induce a continuous character

$$Z_N \to h_k(\Gamma_1(Np^r);\mathcal{O}) \text{ by } \ell \mapsto \langle \ell \rangle.$$

Therefore, we can consider $h_k(\Gamma_1(Np^r);\mathcal{O})$ as an $\mathcal{O}[[Z_N]]$-algebra as well as a Λ-algebra, with Λ the usual Iwasawa algebra $\mathcal{O}[\Gamma]]$, wtih $\Gamma = 1 + p\mathbb{Z}_p$.

Define the ordinary part $h_k^0(\Gamma_1(Np^r);\mathcal{O})$ by the product of all local rings of $h_k(\Gamma_1(Np^r);\mathcal{O})$ on which $T(p)$ acts as a unit. By the duality from above, one can establish bijections between the set of $\mathcal{O}$-algebra morphisms $h_k(\Gamma_1(Np^r);\mathcal{O}) \to \bar{\mathbb{Q}}_p$ and the set of normalized eigenforms. A similar statement can be made for the ordinary part and normalized eigenforms with its p-th Fourier coefficient being a p-adic unit.

We have previously seen the definition of the p-adic universal ordinary Hecke algebra

$$h^0(N;\mathcal{O}) := \varprojlim_r h_k^0(\Gamma_1(Np^r);\mathcal{O}),$$

and we have seen as a crucial result that $h^0(N;\mathcal{O})$ is independent of the choice of $k \geq 2$ and free of finite rank over Λ.

Fix a topological generator u of Γ and identify Λ with $\mathcal{O}[[X]]$ as usual. Let $\epsilon : \Gamma \to \mathcal{O}^\times$ be a character of order p^{r-1}.

We know

$$\Lambda/\omega_{r,k} \cong \mathcal{O}[\Gamma/\Gamma_r], \quad \omega_{r,k} = (1+X)^{p^{r-1}} - u^{kp^{r-1}},$$

and

$$\Lambda/P_{k,\epsilon}\Lambda \cong \mathcal{O}, \quad P_{k,\epsilon} = (1+X) - \epsilon(u)u^k.$$

From the discussion of control theorems, we have seen that

$$h^0(N;\mathcal{O})/\omega_{r,k}h^0(N;\mathcal{O}) \xrightarrow{\sim} h_k^0(\Gamma_1(Np^r);\mathcal{O}).$$

2 Big Picture and Statement of Theorem 0.1 (Section 0)

Now let $\mathfrak{L} = Frac(\Lambda)$, $\mathfrak{K}/\mathfrak{L}$ a finite extension, and $\mathfrak{J}$ the integral closure of Λ in $\mathfrak{K}$. Let $X(\mathfrak{J};\mathcal{O})$ denote the p-adic topological space of all $\mathcal{O}$-valued points of Spec$(\mathfrak{J})$ and $X_{alg}(\mathfrak{J};\mathcal{O})$ the subset consisting of points lying over $X(\Lambda;\mathcal{O})$ of the form $P_{k,\epsilon}$. That is to say, for $P \in X_{alg}(\mathfrak{J};\mathcal{O})$, we have $P \cap \Lambda = P_{k,\epsilon}$, where we write $k(P) = k$ and $\epsilon_P = \epsilon(P) = \epsilon$.

Let $\lambda : h^0(N;\mathcal{O}) \otimes_\Lambda \mathfrak{J} \to \mathfrak{J}$ (conceptually this correspondes to picking one irreducible branch of the big Hecke algebra). One can associate with λ a Dirichlet character $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathcal{O}^\times$ of λ by precomposing with $Z_N \to h^0(N;\mathcal{O})$ and restrict to $(\mathbb{Z}/Np\mathbb{Z})^\times$. One further can define $\psi_p := \psi|_{(\mathbb{Z}/p\mathbb{Z})^\times} = \omega^a$.

For each $P \in X(\mathfrak{J};\mathcal{O})$, we know

$$h^0(N;\mathcal{O}) \otimes_\Lambda \mathfrak{J} \otimes_{\mathfrak{J}} \mathfrak{J}/P \cong h^0(N;\mathcal{O}) \otimes_\Lambda (\Lambda/(P \cap \Lambda)),$$

Thus, for $P \in X_{alg}(\mathfrak{J};\mathcal{O})$ with $k(P) \geq 2$, we can define via the control theorem

$$\lambda_P : h^0_k(\Gamma_1(Np^r);\mathcal{O}) \to \mathcal{O}$$

associated to λ, which agrees with λ after mod P.

Therefore, λ_P corresponds to a normalized eigenform $f_P \in S_{k(P)}(\Gamma_1(Np^{r(P)})$. In particular, $f_P \in S_{k(P)}(\Gamma_0(Np^{r(P)}, \epsilon_P\psi\omega^{-k(P)})$. This form f_P is called the ordinary form belonging to λ at P.

We say λ is primitive if f_P is primitive. If λ is primitive, one can decompose

$$h^0(N;\mathcal{O}) \otimes_\Lambda \mathfrak{K} = \mathfrak{K} \oplus \mathfrak{Q}$$

such that λ coincides with the projection of $h^0(N;\mathcal{O}) \otimes_\Lambda \mathfrak{J}$ to the first factor $\mathfrak{K}$. (conceptually, this is the branch that corresponds to λ)

Let $h(\mathfrak{K})$ and $h(\mathfrak{Q})$ resp. to be the projection of $h^0(N;\mathcal{O}) \otimes_\Lambda \mathfrak{J}$ into the first and second factor resp. Then we can define the main object of the arithmetic side, i.e. the module of congruence:

$$\mathfrak{C}(\lambda) = h(\mathfrak{K}) \oplus h(\mathfrak{Q})/\delta(h^0(N;\mathcal{O}) \otimes_\Lambda \mathfrak{J}),$$

where δ is the diagonal map.

Now we are ready to state Theorem 0.1. Refer to paper.

3 Minimal Forms and Primitive L-Functions (Section 7)

The purpose of this section is to state the main result Theorem 0.1 in terms of the L-function, associated with the Gelbart-Jacquet lift, which will be defined below.

Conceptually, since this L-function is defined using primitive forms, it is invariant under twists of f. Therefore, we have to eliminate the ambiguities introduced by twists on the arithmetic side too.

3.1 Definition 7.1: Primitive L-function

Let f be a primitive form in $S_k(\Gamma_1(Np^r))$ and π be the corresponding automorphic represenation of $GL_2(\mathbb{A})$. Let $\hat{\pi}$ be the base change lift of π to GL(3) given by Gelbart-Jacquet. Let $L(s, \hat{\pi})$ be the L-function associated with $\hat{\pi}$ and define

$$D(s, f) = L(s - k + 1, \hat{\pi}).$$

Note that $L(s, \hat{\pi})$ coincides with the symmetric square L-function.

3.2 Definition 7.2: Minimal form

For each Dirichlet character $\chi : \mathbb{Z} \to \mathbb{C}$ modulo M, we can define the twist of f by

$$f|[\chi] = \sum_{n=1}^{\infty} \chi(n) a(n, f) q^n \in S_k(\Gamma_1(NM^2p^r)).$$

A primitive form f is said to be *minimal* if the conductor $C(f) \leq C(f|[\chi])$ for all Dirichlet characters χ. In the class of twists of f, there exists at least one minimal form.

3.3 Major steps to modify objects appeared in Theorem 0.1

Below we outline the roles of the respective propositions/theorems in eliminating the ambiguity of twisting in the arithmetic side.

Proposition 7.3 compute the discrepancy between the $D(s, f)$ and $\mathfrak{D}(s, f)$. It particular, it differs by finitely many Euler factors at primes where the local representation π_ℓ is supercuspidal and $\pi_\ell \otimes \eta_\ell \cong \pi_\ell$.

Proposition 7.7 and the lemmas above provides theoretical basis to define primitive homomorphism λ associated with an algebraic homomorphism λ_0 and the conductor of λ_0

Theorem 7.9 explains the effect at the level of forms (eg. the form of λ at P) when we twist a $\mathfrak{J}$-algebraic morphism λ. One then defines what it means for a primitive homomorphism λ to be *minimal*: if the ordinary form f_P associated to λ is minimal for some $P \in X_{alg}$.

Theorem 7.11 says you can find a set of places (the set relevant in proposition 7.3) that is uniform across different specializations P of the family (up to finitely many P) if λ is minimal.

Corollary 7.12 is the end goal that reformulates Theorem 0.1 in terms of $D(s, f)$ when λ is minimal. The discrepancy on the arithmetic side is adjusted by multiplying finitely many extra factors to the H in Theorem 0.1 at places corresponding exactly to the extra factors of $D(s, f)$ in Proposition 7.3. We can do this because we have found a set uniform to the whole family λ instead of only at certain P according to Theorem 7.11.