# DIOPHANTINE GEOMETRY WEEK 07 NOTES

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Abstract. In this week, we will briefly talk about Néron-Tate Heights and Fermat's descent Theorem. Again, for more details, please check Bianca's notes and Bombieri [1] chapter 9.1 - 9.4.

Contents



## 1. Introduction

The theorem of Néron-Tate height is the main recipe needed to extend Mordell-Weil theorem for Elliptic Curves to Abelian Varieties, namely the finite generation of the group of rational points of an abelian variety defined over a number field.

We shall see in next week's material for a historical overview. For this week, we will introduce tools that would allow us to extend this result to the general Mordell-Weil Theorem.

As in the case of the Elliptic Curve, we will split into two steps: in the first step, we will outline the proof for weak Mordell-Weil Theorem for general abelian varieties, and then we give a generalized version of Fermat's descent theorem, which will allow us to prove strong Mordell-Weil Theorem.

## 2. Néron Tate Height and Fermat's Descent Theorem

So ultimately, our goal is to prove the following lemma, which will be crucial in completing the proof of Mordell-Weil Theorem:

**Lemma 2.1.** *Let $K$ be a number field and let $c$ be ample and even. Then $\widehat{h}_c$ vanishes exactly on the torsion subgroup of $A(K)$. Moreover, there is a unique scalar product $\langle , \rangle$ on the abelian group $A(K) \otimes_{\mathbb{Z}} \mathbb{R}$ such that*

$$\widehat{h}_c(x) = \langle x \otimes 1, x \otimes 1 \rangle$$

*for every $x \in A(K)$,*

where we will soon define $\widehat{h}$, which is commonly known as Néron-Tate Height. Over the rest of the section, we will give a survey of important results in Néron-Tate heights.

Let $K$ be a field and let $A$ be an abelian variety over $K$. Let $X$ be a complete variety over $K$. Then we know that we have the height homomrphism

$$\mathbf{h} : \mathrm{Pic}(X) \to \mathbb{R}^{X(\overline{K})}/O(1),$$

which associates $\mathbf{c}$ with the equivalence class of heights $\mathbf{h_c}$.

But the problem with Weil Heights is, there do not exist a canonical height function associated to $\mathbf{c} \in \operatorname{Pic}(X)$, as they are only determined up to a bounded constant.

To solve this, we take our resolution to theorem of cube. for every $\mathbf{c} \in \operatorname{Pic}(A)$ we have a quadratic function

$$\operatorname{Mor}(X, A) \to \operatorname{Pic}(X), \quad \varphi^* \mapsto \varphi^*(c).$$

where we may decompose $\mathbf{c}$ into an even and odd part $\mathbf{c} = \mathbf{c}_+ + \mathbf{c}_-$, and we there fore have the associated decomposition of height homomorphism:

$$q : \operatorname{Mor}(X, A) \to \mathbb{R}^{X(\overline{K})}O(1), \quad \varphi \mapsto \mathbf{h}_{\varphi^*(c)}.$$

We conclude that $q = q_+ + q_-$ for the quadratic form $q_+(\varphi) := h_{\varphi*}(c_+)$ and the linear form $q_-(\varphi) := h_{\varphi*}(c_-)$. The most important fact is that, this decomposition is unique. Motivating by the observation, we have the following:

**Observation 2.1.** *Let $h_{c\pm}$ be an arbitrary height function in the class $\boldsymbol{h}_{c\pm}$ . For any integer $n$, we have $n^2\boldsymbol{h}_{c_+} = \boldsymbol{h}_{[n]*(c_+)}$ and $n\boldsymbol{h}_{c_-} = \boldsymbol{h}_{[n]*(c_-)}$. By theorem of height function, there is a constant $C(n)$ such that for every $a \in A$*

$$|h_{c+}(na) - n^2 h_{c+}(a)| \leq C(n)$$

*and*

$$|h_{c-}(na) - n h_{c-}(a)| \leq C(n).$$

**Definition 2.2** (Quasi-Homogeneous)**.** *Let $\mathcal{N}$ be a multiplicatively closed subset of $\mathbb{R}$ (resp. $\mathbb{R}_+$) acting on a set $S$ by means of a map such that $n(mx) = (nm)x$ for $x \in S$ . A function $h : S \to R$ is quasi-homogeneous of degree $d \in N$ (resp. $d \in R+$) for $N$ if for $n \in \mathbb{N}$ there is a positive constant $C(n)$ such that*

$$|h(nx) - n^d h(x)| \leq C(n)$$

*for every $x \in S$, and is homogeneous of degree $d$ for $N$ if h(nx) = ndh(x).*

We then have the following theorem:

**Theorem 2.3.** *Let $\mathcal{N}$ act on the set $S$ as before and let $h : S \to \mathbb{R}$ be quasihomogeneous of degree $d > 0$. If $\mathcal{N}$ has an element of absolute value $> 1$, then there is a unique homogeneous function $\widehat{h} : S \to R$ of degree $d$ for $\mathcal{N}$ such that $\widehat{h} - h$ is bounded.*

The proof of this is purely algebraic, which we will omit here. The readers are welcome to check [1], chapter 9.

We then introduce the Tate's limit argument:

**Theorem 2.4.** *Let $\boldsymbol{c} \in Pic(A)$ and let $\boldsymbol{c} = \boldsymbol{c}_+ + \boldsymbol{c}_-$ be a decomposition into an even part $\boldsymbol{c}+$ and an odd part $\boldsymbol{c}_-$. Then the classes $\boldsymbol{h}_{c_\pm}$ are independent of the choice of the decomposition. There is a unique homogeneous height function $\widehat{\boldsymbol{h}}_{c_\pm}$ in the class $\boldsymbol{h}_{c_\pm}$ , of degree 2 in the + case and degree 1 in the − case.*

This theorem allows the definition of Néron-Tate height:

**Definition 2.5** (Néron-Tate height)**.** *The height function $\widehat{\boldsymbol{h}}_c := \widehat{\boldsymbol{h}}_{c_+} + \widehat{\boldsymbol{h}}_{c_-}$ is called the Néron–Tate height associated to c.*

To complete this section, we associate our bilinear form with Néron Tate Height, which would allow us to prove the Fermat Descent Theorem.

Let $M$ be an abelian group and let $b$ be a real-valued symmetric bilinear form on $M$. We have in mind the example $M = A(K)$ and a certain bilinear form associated to a Néron–Tate height. The kernel of $b$ is the abelian group

$$N := \{x \in M | b(x, y) = 0 \text{ for every } y \in M\}.$$

Then $b$ induces a symmetric bilinear form $b$ on $\overline{M} := M/N$ and the kernel of $b$ is zero. Since $b$ is real valued, $\overline{M}$ is torsion free and all torsion elements of $M$ are contained in $N$. We conclude that

$$\overline{M} \to \overline{M}_{\mathbb{R}} := \overline{M} \otimes_{\mathbb{Z}} \mathbb{R}, mm \otimes 1$$

is injective. Let $\overline{M}'$ be a finitely generated subgroup of $\overline{M}$. The restriction of $b$ to the free abelian group $\overline{M}'$ extends uniquely to a bilinear form $\overline{b}'$ on $\overline{M}'_{\mathbb{R}}$ . Let $\overline{M}'_{\mathbb{Q}} = \overline{M}' \otimes_{\mathbb{Z}} \mathbb{Q}$. An easy argument shows that

$$\overline{M}'_{\mathbb{Q}} \subset \overline{M}_{\mathbb{Q}}$$

and so $\overline{M}'_{\mathbb{R}} \subset \overline{M}_{\mathbb{R}}$. Since $\overline{M}_{\mathbb{R}}$ is the union of all $\overline{M}'$ and the bilinear forms $\overline{b}'$ coincide on overlappings by uniqueness, we have a unique extension of $b$ to a bilinear form $b_{\mathbb{R}}$ on $\overline{M}_{\mathbb{R}}$.

Thus we would like that bilinear form on $b_{\mathbb{R}}(x, y)$ determines a scalar product and an associated norm $\|x\|^2 = b_{\mathbb{R}}(x, x)$ on $M_{\mathbb{R}}$. In fact, we have the following lemma:

**Lemma 2.6.** *With the notation and assumptions above, the bilinear form $b_{\mathbb{R}}$ is positive definite if and only if for every finitely generated subgroup $\overline{M}'$ of $\overline{M}$ and for every $C > 0$ the set*

$$\{x \leq \overline{M} \mid b_{\mathbb{R}}(x, x) \leq C\}$$

*is finite.*

Finally, our goal is to offer a explicit formula that would allow us to calculate Néron-Tate heights.

**Theorem 2.7.** *Let $K$ be a number field and let $c$ be ample and even. Then $\widehat{h}_c$ vanishes exactly on the torsion subgroup of $A(K)$. Moreover, there is a unique scalar product $\langle , \rangle$ on the abelian group $A(K) \otimes_{\mathbb{Z}} \mathbb{R}$ such that*

$$\widehat{h}_c(x) = \langle x \otimes 1, x \otimes 1\rangle$$

*for every $x \in A(K)$.*

For a complete proof of this would require more algebraic geometry input, which we shall omit here.

With this, we are now ready to prove the second part-Fermat Descent Theorem.

## References


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