If \( m^* (\mathcal{E} \otimes \mathcal{F}) = \mathcal{F} \otimes (\mathcal{E}^*) \) \( \phi: \mathcal{E} \rightarrow \mathcal{F} \) and by seesaw principle, now pull-back equation (a) by the morphism
\[
A \rightarrow A \times A \quad \alpha \mapsto (\alpha, -\alpha)
\]
we get \( \mathcal{E}^* \otimes (\mathcal{E}^*) = \mathcal{E}^* \).

\( X(\mathcal{K}) \) assume non-empty fix \( \mathcal{B} \) on X.

Let \( T \) be an integral variety. We say that \( \mathcal{E} \in \text{Pic}^0(\mathcal{X} \times T) \) is a subfamily of \( \text{Pic}^0(\mathcal{X}) \) parametrized by \( T \) if

(a) \( \mathcal{E} \in \text{Pic}^0(\mathcal{X} \times T) \) for all \( T \in T \).

(b) \( \mathcal{E}^T = \mathcal{E} \in \text{Pic}^0(\mathcal{X}) \).

So by seesaw, \( \mathcal{E} \) is uniquely determined by family \( (\mathcal{E}_t)_{t \in T} \) and by condition (b).

Theorem: There is a subfamily \( \mathcal{E} \) of \( \text{Pic}^0(\mathcal{X}) \), parametrized by an irreducible smooth complete variety \( \mathcal{B} \), with the following universal property:

For every subfamily \( \mathcal{E} \) of \( \text{Pic}^0(\mathcal{X}) \), parametrized by an irreducible variety \( T \), there is a unique morphism \( \mathcal{Y}: T \rightarrow \mathcal{B} \) with \( \text{Pic}(\mathcal{X} \times T) = \mathcal{E} \).

Conversely, let \( \mathcal{E} \) be an extension field of \( \mathcal{K} \).

(a) By base change, we have \( \text{Pic}(\mathcal{X}) \subset \text{Pic}(\mathcal{X}) \).

(b) \( \text{Pic}^0(\mathcal{X} \times T) = \mathcal{E} \subset \text{Pic}(\mathcal{X}) \) and its Picard class is obtained from \( \mathcal{E} \) by base change to \( T \).

(c) \( \text{Pic}^0(\mathcal{X} \times T) = \mathcal{E} \subset \text{Pic}(\mathcal{X}) \) by identifying \( T \) with \( \mathcal{E} \).

Remark: By seesaw principle, the Picard class \( \mathcal{E} \) is uniquely characterized by the conditions:

(a) \( \mathcal{E} \) is a Picard variety of \( X \) and
(b) \( \mathcal{E} \) is compact class. If \( (B, \mathcal{F}) \) is another such pair, then \( \exists \mathcal{Y}: B \rightarrow B \) such that \( \mathcal{F} = \text{Pic}(\mathcal{X} \times B) \) and \( \mathcal{E} = \text{Pic}(\mathcal{X} \times B) \).

(c) \( \mathcal{E} \) is uniquely determined by the condition
\( \forall t \in T, \mathcal{E} \subset \text{Pic}(\mathcal{X}) \).

(d) \( \mathcal{E} \subset \text{Pic}(\mathcal{X}) \) for all \( t \in T \).

(e) \( \mathcal{E} \subset \text{Pic}(\mathcal{X}) \) for all \( t \in T \).

(f) \( \mathcal{E} \subset \text{Pic}(\mathcal{X}) \) for all \( t \in T \).

(g) \( \mathcal{E} \subset \text{Pic}(\mathcal{X}) \) for all \( t \in T \).

(h) \( \mathcal{E} \subset \text{Pic}(\mathcal{X}) \) for all \( t \in T \).

(i) \( \mathcal{E} \subset \text{Pic}(\mathcal{X}) \) for all \( t \in T \).
Now the morphism is given by
\[ f(t) = t^2 = \frac{p^2}{\pi} + \text{const} \]

This is clear by the restriction of \((\mathbf{d} \times \mathbf{y})^* \phi = \phi\) to the fibre \(X \times \mathfrak{f}\) and then using the rule
\[ (f \circ g)^* = g^* \circ f^* \] to show that
\[ (\mathbf{c}_t = (\mathbf{c} \times \mathbf{y})^* \phi (\mathbf{h}^{-1} \times \mathbf{h}))^* = \mathbf{p}_{\mathfrak{q}}^* \mathbf{g}^* \mathbf{p}_{\mathfrak{q}} = \mathbf{p}_{\mathfrak{q}} \]

Together with the canonical group structures induced by tensor product of line bundles, \(\mathbf{p}_{\mathfrak{q}}^* \mathbf{c} (\mathbf{x})\)

is an abelian variety of \(K\).

**In Summary:**

Let \(X\) be an irreducible smooth complete variety over \(K\) and let \(P_0 \in \mathbf{X}(\text{Spec } K)\) be a base point of \(X\). Then the group \(\mathbf{p}_{\mathfrak{q}}^* \mathbf{c} (\mathbf{x})\) has a unique structure as an abelian variety over \(K\), called the Picard variety and denoted by \(\mathbf{p}_{\mathfrak{q}}^* \mathbf{c} (\mathbf{x})\), with the properties:

(a) There is \(\phi \in \mathbf{p}_{\mathfrak{q}}^* \mathbf{c} (\mathbf{x})\) such that \(\phi^2 = 0\) for \(\mathbf{c} \in \mathbf{p}_{\mathfrak{q}}^* \mathbf{c} (\mathbf{x})\) and \(P_0\), it is trivial.

(b) For any subfamily \(\mathbf{c}\) of \(\mathbf{p}_{\mathfrak{q}}^* \mathbf{c} (\mathbf{x})\) parametrized by an irreducible variety \(\text{Spec } K\), the set-theoretic map

\[ T \to \mathbf{p}_{\mathfrak{q}}^* \mathbf{c} (\mathbf{x}) \]

is actually a morphism over \(K\). The uniquely determined class \(\phi\) is called
The Poincaré class.

**Theorem of Square** (Mostly just state, without proof).

For details, check 8.5 of Beauville.

**Prop** Let \( \overline{\mathcal{C}} \in \text{Pic}(A) \) and \( a \in A \). Then \( \varphi_{\overline{\mathcal{C}}}(a) = T_a(\overline{\mathcal{C}}) - \overline{\mathcal{C}} \in \text{Pic}^0(A)_k \) and \( \varphi_{\overline{\mathcal{C}}} : A \to \text{Pic}^0(A) \) is a homomorphism of abelian varieties over \( k \).

**Thm** (Theorem of Square). For \( a, b \in A \), we have

\[
T_a^*(\overline{\mathcal{C}}) + \overline{\mathcal{C}} = T_b^*(\overline{\mathcal{C}}) + T_b^*(\overline{\mathcal{C}})
\]

**Prop** A class \( \overline{\mathcal{C}} \in \text{Pic}(A) \) is ample iff \( \ker(\varphi_{\overline{\mathcal{C}}}) \) is finite and in addition \( H^0(A, n\overline{\mathcal{C}}) \to 0 \) for some positive integer \( n \).

Most importantly, the following statements:

**Prop** For \( \overline{\mathcal{C}} \in \text{Pic}(A) \), the following statements are equivalent:

1. \( \overline{\mathcal{C}} \in \text{Pic}^0(A) \)
2. \( \ker(\varphi_{\overline{\mathcal{C}}}) = A \).
3. For every ample \( \overline{\mathcal{D}} \in \text{Pic}(A) \), there is \( a \in A \) such that \( \overline{\mathcal{D}} = T_a^*(\overline{\mathcal{C}}) - \overline{\mathcal{C}} \).
4. There is an ample \( \overline{\mathcal{D}} \in \text{Pic}(A) \) such that \( \overline{\mathcal{D}} = T_a^*(\overline{\mathcal{C}}) - \overline{\mathcal{C}} \) for some \( a \in A \).

**Def** The \( \overline{\mathcal{C}} \) Picard variety \( \text{Pic}^0(A) \) is called the...
dual abelian variety of $A$ and will be denoted by $\tilde{A}$.

Con: The dual abelian variety $\tilde{A}$ has the same dimension as $A$.

Theorem of cube

This section discusses elementary facts for involutions and quadratic functions on an abelian group then prove the Theorem of Cube.

Def: $M$ abelian group. With \( \tau \) involution, defined as a linear map $M \rightarrow M$, $x \mapsto x^\tau$, with $\tau(x^\tau) = x$, for $\forall x \in M$.

- $x \in M$ even if $x^\tau = x$
- $x \in M$ odd if $x^\tau = -x$

Lem: $x \in M$, then $2x$ has a decomposition into an even and an odd part. If the subgroup of odd elements is divisible by 2, then $x$ also has such a decomposition.

Pf: $2x = (x + x^\tau) + (x - x^\tau)$.

Divisibility by 2 $\Rightarrow \exists \ z$ odd, $2z = x - x^\tau$.

Let $x_+ := x - z$, $x_- := z$.

$(x_+)^\tau = x^\tau - z^\tau = (x - 2z - z) = x_+$.

So $x_+$ is even, done.
A abelian variety over $K$. Consider the involution

\[ \xi \mapsto [-1]^{\xi} \text{ on the abelian group } \text{Pic}(A) \]

Hence the bundle is even (odd) if

\[ [-1]^f \simeq L / [-1]^f \cdot L \otimes [\xi] \]

**Def:** $q : M \to N$ set map on abelian groups

If $b : M \times M \to N$ $(xy) \mapsto q(xy) - q(x) - q(y)$

is bilinear, the $q$ is quadratic function

with $b$ associated bilinear form

**Quadratic form:** quadratic function which is homogeneous of degree 2.

Involution: $q^\dagger(x) = q(-x)$

By Lemma 8.6.2, decompose $2q$ into $Q$ even,

$L$ odd, given by

$Q(x) = q(x) + q(-x)$ and $L(x) = q(x) - q(-x)$

$Q$: associated quadratic form

$L$: associated linear form

**Lem:** $q : M \to N$ be a quadratic function, n.d.

Then $q(x+y) = \frac{q^2(x+y)}{2} - \frac{q^2(x-y)}{2} - q(x)$

for all $x, y \in M$

**Cor:** A quadratic function is even if and only if it is a quadratic form.
Let $M = (\mathbb{Z}/2)^2$, $N = \mathbb{Z}/2$.

$q : M \to N \quad q(x) = 0 \quad \iff \exists x = 0$

then $q$ is called quadratic function not linear.

**Def**

$b \in \mathbb{N}^+$, $I \subseteq \{1, \ldots, k\}$, define $S_I : M^k \to M$

$S_I (x_1, \ldots, x_k) = \sum x_i$

set $S_\emptyset = 0$

**Lem**

Let $q : M \to N$ be a quadratic function and let $b$ be an integer. If $b \geq 3$, then we have

for $\bar{x} \in M^b$. 

\[
\sum_{I \subseteq \{1, \ldots, b\}} |I| q(S_I(\bar{x})) = 0.
\]

Apply the lemma to the abelian group $M = \text{Mor}(X, A)$ of morphisms from $X$ to $A$ and to the abelian group $N = \text{Pic}(X)$.

**Thm**

Let $X$ be a variety over the field $k$ and $A$ be an abelian variety over $k$ with $\mathfrak{c} \in \text{Pic}(A)$. Then the map of $\text{Mor}(X, A)$ into $\text{Pic}(X)$ given by $\varphi \mapsto \varphi^*(\mathfrak{c})$ is quadratic.

Let $b \geq 3$ and $X = A^k$ with $i$th projection $p_i$ onto $A$. For $I \subseteq \{1, \ldots, k\}$, we have

$S_I (p_1, \ldots, p_k) = \sum_{i \in I} p_i$. 

$\|I\|$
Lemma 8.110 shows that the theorem implies
\[ \sum (-1)^{i} \text{dim } H_{i}(\mathcal{M}_{1}, \ldots, \mathcal{M}_{n}, \mathcal{A}(\mathcal{C})) = 0 \]

For \( b=3 \), the equation
\[ \sum (-1)^{i} \text{dim } H_{i}(\mathcal{M}_{1}, \ldots, \mathcal{M}_{n}, \mathcal{A}(\mathcal{C})) = 0 \]
\[ \text{for } i \leq 1 \]

is called the theorem of cubes.

For more information about the theorem of cubes for varieties, and a version of Riemann-Roch for abelian varieties,

\[ \dim \left( T(A, L) \right) = \sqrt{\text{det } E(x, y)} \left| \right. \]

The isogeny multiplication by \( n \)

A abelian variety over \( K \) field.

This section: study \( \text{End}: A \rightarrow A \) given by multiplication by \( n \in \mathbb{Z} \) of the abelian variety \( A \).

Significance: Construction of Néron-Tate Height, needed for proof of Mordell-Weil.

Over \( \mathbb{C} \): \( k = \mathbb{C} \) we have
\[ \text{dim } A(\mathbb{C}) = 2 \dim (A) \]
\[ A(\mathbb{C}) \cong (\mathbb{C}/n \mathbb{Z}) \]
Prop. Let \( C \in \text{Pic}(\mathbb{A}) \) and \( n \in \mathbb{Z} \), then
\[
\text{Inj}_n^\ast(C) = \frac{n^2}{2} \cdot \mathbb{Z} + \frac{n^2 - n}{2} \cdot \text{Inj}^\ast_C.
\]

In particular, we have \( \text{Inj}_n^\ast(C) = n^2 \cdot C \) if \( C \) is even and \( \text{Inj}_n^\ast(C) = -\frac{n^2}{2} \cdot C \) if \( C \) is odd.

Prop. 8.7.2 Let \( n \in \mathbb{Z} \setminus \{0\} \). Then \( \text{Inj}_n \) is a finite flat surjective morphism of degree \( n \), \( \text{Inj}_n \) is an étale morphism and \( \text{Inj}_n \sim (\mathbb{Z} / n \mathbb{Z}) \).

If \( p = \text{char} (k) \mid n \), then \( \text{Inj}_n \) is not separable.

Finally, we include some material about curves & Jacobians.

Def. The Picard Variety of \( C \) is called the Jacobian Variety of \( C \).

where \( C \) is an irreducible smooth projective curve over a field \( k \) of genus \( g \geq 1 \) with base point \( P_0 \in C(k) \).

Cor. The Jacobian Variety of \( C \) has dimension \( g \).
Lemma: Assume that the ground field $K$ is algebraically closed. Let $L$ be a line bundle on $C$. Then for every $P \in C$, we have
\[
\dim P(C, L(-[P])) \geq \dim P(C, L) - 1.
\]
Equality holds iff $P$ is not a base-point of $L$.

Lemma: Let $L$ be a line bundle on $C$ and let $r \geq 1$. Then \[
\dim P(C, L(-\sum_{j=1}^{r} [P_j])) = \dim P(C, L) - r
\]
for all $(P_1, \ldots, P_r) \in U$.

Lemma: Let $r \geq 1, \ldots, g$. Then \[
U_r := \{ (P_1, \ldots, P_r) \in C^r \mid \forall i \neq j : P_i \neq P_j \}
\]
is an open dense subset of $C^r$.

Proof: For any $r \geq 0$, we have a map
\[
\sigma_r : C^r \to \bigoplus_{i=1}^{r} (\bigoplus_{j=1}^{r} \mathcal{O}(-[P_j]))
\]
since $j = 3$, and addition on $J$ is the morphisms we easily deduce that $0$. $s_r$.
is a morphism. Note that its image is closed because $C^r$ is complete.

Let $\alpha := \pi(P_1, \ldots, P_r)$, then the fiber over $\alpha$ is

$$\mathcal{F}_{\pi^{-1}(\alpha)} = \{ (\alpha, \ldots, \alpha) \} \subseteq C^r \times \cdots \times C^r$$

Let $r \in \mathbb{N}$ and $(P_1, \ldots, P_r) \in \mathcal{U}_r$.

Then the fiber over $\alpha$ is obtained by permuting the entries, namely

$$\mathcal{F}_{\pi^{-1}(\alpha)}(P_1, \ldots, P_r) = \{ (P_{\pi(1)}, \ldots, P_{\pi(r)}) \} \subseteq C^r \times \cdots \times C^r$$

By the dimension theorem, we conclude that $\dim j_\ast(C^r) = r$. In particular, Corollary 8.10.7 of [Bor] imply that morphism $j_\ast$ is surjective.

Moreover, $\Theta = j_\ast(C^{g-1})$ is indeed an divisor.

Proof: The map $j: C \to J$, $P \mapsto \text{cl}(C(P) - C(P_0))$

is a closed embedding.

Aside: An important role Jacobian Variety played is the theta divisor.

$$\Theta := \underbrace{j(c) + \cdots + j(c)}_{g-1 \text{ times}}$$
The three Lemmas presented just now, in particular verified that \( \Theta \) is indeed a divisor.

\[ \Theta := \left[-1 \right]^* \Theta = \mathcal{J}(c) \quad \mathcal{J}(c) \]

As a divisor on \( \mathcal{J} \), we also consider

\[ \Theta := \left[-1 \right]^* \Theta = \mathcal{J}(c) \quad \mathcal{J}(c) \]

In \( \mathcal{P}(\mathcal{J}) \), we use \( \Theta := \mathcal{J}(c)(\Theta) \) and \( \Theta := \left[-1 \right]^* \Theta \). For a \( \mathcal{G} \mathcal{J} \), we set \( \mathfrak{f}_a := \prod_{a} \mathfrak{f}_a \).

\( \mathfrak{f}_a \) is the map \( C \rightarrow \mathcal{J} \) given by \( \mathfrak{f}_a(P) := a \).

**Prop.** Assume \( k \) is algebraically closed. For all \( \mathfrak{f}_1, \ldots, \mathfrak{f}_g \in \mathcal{J} \), we have the rational equivalence relation

\[ \sum_{i=1}^{g} \mathfrak{f}_i(P) \sim \mathfrak{f}_a(\Theta) \]

of divisors on \( C \).

where \( \alpha := \mathfrak{f}_a(\mathfrak{f}_1, \ldots, \mathfrak{f}_g) \).

If \( k \) is not algebraically closed, then Proposition 8.4.0 of [Ban] shows that rational equivalence holds over any field of definition for \( \mathfrak{f}_a \).

**Rmk.** the idea is to show that the intersection of \( \mathfrak{f}(c) \) and \( \Theta + \alpha \) is transverse for generic \( \mathfrak{f}_1, \ldots, \mathfrak{f}_g \). Then the proposition...
will follow in the generic case from our first step below. An application of the theorem of square will lead to the general case.

**Proof Outline**

**Step I:** For $(P_1, \ldots, P_g) \in C^g$ with $\dim P_2 C, O(\mathcal{L}_C, P_1) = 1$, we have

$$\Theta - \alpha \cap \beta \in \mathcal{S}(P_1, \ldots, P_g)$$

**Step II:** For $1 \leq r \leq g$, and $(P_1, \ldots, P_r) \in U_{1r}$, the differential $\tilde{d}_{P_1} : T_{\tilde{F}}(P_1, \ldots, P_r) \to T_{\tilde{F}_{3r}}(P_1, \ldots, P_r)$ is injective.

**Step III:** For generic $(P_1, \ldots, P_g) \in C^g$, the intersection of $\Theta - \alpha$ and $\beta$ is transverse.

**Step IV:** Proof of proposition for generic $(P_1, \ldots, P_g) \in C^g$.

**Step V:** Proof of proposition for all $(P_1, \ldots, P_g) \in C^g$.

**Conclusion:** For all $(P_1, \ldots, P_g) \in U_{1g}$ and $\alpha = \beta$, we have $\sum_{i=1}^{g} [P_i] = \gamma^* (\Theta)$ as an identity.
of divisors

For \( \Delta \in J = \text{Pic}^0(C) \), we have

\[
\bar{\alpha} \cdot \bar{\Theta}^{-1} - \tilde{3} \cdot \Theta^{-1} = \bar{\Delta}.
\]

Finally, note there are two Poincare classes in the context of Jacobians.

\( \bar{\pi} \in \text{Pic}^1(C) \)

\( \bar{\pi} \in \text{Pic}^1(J, \bar{J}) \), where \( J \) is the dual abelian variety of \( J \).

Prop. Let \( \Delta \) denote the diagonal in \( C \times C \) then

\[
\text{red}(C \times J)^x(\bar{\pi} \circ \Delta) = 0 \circ (0 - C \times \text{Pic}^0(C))
\]

Prop. Let \( m : J \times J \to J \) be addition and let \( \pi_1, \pi_2 \) be the projections of \( J \times J \) onto the corresponding factors. For \( \bar{\alpha} = m^* \bar{\Theta} = \pi_1^* \bar{\Theta} = \pi_2^* \bar{\Theta} \in \text{Pic}^1(J) \), we have

\[
(j \times \text{id}_J)^x(C) = \bar{\pi} \circ \Delta
\]

Prop. Let \( \phi : J \to J \) be the morphism \( J \to J \) introduced in 3.5-1. Let \( \bar{\alpha} = m^* \bar{\Theta} = \pi_1^* \bar{\Theta} = \pi_2^* \bar{\Theta} \in \text{Pic}^1(J) \) as in the previous proposition, then

\[
(\text{id}_J \times \phi)^x(\bar{\pi} \circ \Delta) = (\text{id}_J \times \phi)^x(\bar{\pi} \circ \Delta) = \bar{\alpha}.
\]
We summarize here our findings.

Given a curve $C$, $g \geq 1$; generic base point $P_0 \in C(K)$. If natural embedding $j$ of $C$ into the Jacobian variety $J$, by $\delta J$, we have a dual homomorphism $\tilde{j} : J \to \delta J$. The theta divisor is defined by

$$\Theta := \tilde{j}(c_1 + \cdots + c_{g-1})$$

and the corresponding class in $Pic(J)$ is denoted by $\theta$. Let $\Theta := \tilde{j}^* \Theta$ and $\psi_0 : J \to J$ be the natural morphism introduced in 8.5.1 of [Buiu].

There are 3 canonical morphisms from $J(x)$ to $J$, namely:

1. Addition in $x$ first projection $p_1$.
2. Second projection $p_2$.

The pull-back of the Poincaré class $\frac{1}{y} \in Pic(J)$ by $\tilde{j} \times \tilde{j}$ is equal to the class

$$c := \delta^* \Theta - p_1^* \Theta - p_2^* \Theta$$

and it follows from previous
Proposition that
\[ c = m^0 \cdot \gamma^0 \cdot p^0. \]

Finally, we note that:
- The map \( \varphi_0 \) is an isomorphism of \( J \rightarrow J \) whose inverse is \( \gamma \).
- Moreover, \( \Theta \) is ample.