Fourier transform and the global Gan–Gross–Prasad conjecture for unitary groups

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Abstract

We confirm the global Gan–Gross–Prasad conjecture for unitary groups under some local restrictions for the automorphic representations. We also obtain some result towards the Flicker–Rallis conjecture characterizing the image of weak base change from any unitary group via distinction by the general linear subgroup.

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1 Introduction to Main results

The study of periods and heights related to automorphic forms and Shimura varieties have recently received a lot of attention. One pioneering example is the work of Harder–Langlands–Rapoport ([24]) on the Tate conjecture for Hilbert-Blumenthal modular surfaces. Another example which motivates the current paper is the study of the Neron–Tate heights of Heegner points or CM points on the modular curve $X_0(N)$ by Gross and Zagier ([20]) in 1980s, on Shimura curves by S. Zhang in 1990’s, completed by Yuan–Zhang–Zhang ([54]) recently, and Kudla–Rapoport–Yang ([37]), Bruinier–Ono ([5]) in various prospectives. At almost the same time as the Gross–Zagier’s work, Waldspurger ([49]) discovered a formula that relates certain toric periods to some Rankin L-series, the same type L-function appeared in the Gross–Zagier formula. In 1990’s, Gross and Prasad formulated a conjectural generalization of Waldspurger’s work to higher rank orthogonal groups ([18],[19]) (later refined by Ichino–Ikeda ([28])). Recently, Gan, Gross and Prasad have generalized the conjectures to classical groups ([11]) including unitary groups and symplectic groups. The conjectures are on the relation between period integrals and certain L-values. The main result of this paper is to confirm their conjecture for unitary groups under some local restrictions.

In the following we want to describe the main results of the paper in more details.

Gan–Gross–Prasad conjecture for unitary groups. Let $E/F$ be a quadratic extension of number fields. Let $W$ be a (non-degenerate) hermitian space of dimension $n$ and denote by $U(W)$ the corresponding unitary group as an algebraic group over $F$. Let $G'_n := Res_{E/F}GL_n$ be the restriction of scalar of $GL_n$. We recall the local base change map when a place $v$ is split or the representation is unramified. If a place $v$ of $F$ is split in $E/F$, we may identify $G'_n(F_v)$ with $GL_n(F_v) \times GL_n(F_v)$ and identify $U(W)(F_v)$ with a subgroup consisting of elements of the form $(g,g^{-1})$, $g \in GL_n(F_v)$ and $g^t$ is the transpose of $g$. Let $p_1, p_2$ be the two isomorphisms between $U(W)(F_v)$ with $GL_n(F_v)$ induced by the two projections from $GL_n(F_v) \times GL_n(F_v)$ to $GL_n(F_v)$. If $\pi_v$ is an irreducible admissible representation of $U(W)(F_v)$, we define the local base change $BC(\pi_v)$ to be the representation $p_1^{*} \pi_v \otimes p_2^{*} \pi_v$ of $G'_n(F_v)$ where $p_i^{*} \pi_v$ is a representation of $GL_n(F_v)$ obtained by the isomorphism $p_i$. Note that for the split case, the local base change map is injective. When $v$ is non-split and $U(W)$ is unramified at $v$, there...
is a local base change map at least when \( \pi_v \) is an unramified representation of \( U(W)(F_v) \), cf. \([11]\). Now let \( \pi \) be a cuspidal automorphic representation of \( U(W)(A) \), an automorphic representation \( \Pi \) of \( G'_{n}(A) \) is called the weak base change of \( \pi \) if \( \Pi_v \) is the local base change of \( \pi_v \) for all but finitely many places \( v \) where \( \pi_v \) is unramified (\([22]\)). We will then denote it by \( BC(\pi) \). If \( \Pi = BC(\pi) \) is cuspidal, by strong multiplicity one for \( GL_n \), it is unique.

Throughout this article, we will assume the following hypothesis on the base change

**Hypothesis \((\ast)\):** For all \( W \) and all dimension \( n \), the weak base change exists and satisfies the following local–global compatibility at all split places \( v \): the \( v \)-component of \( BC(\pi) \) is the local base change of \( \pi_v \) for all but finitely many places \( v \) where \( \pi_v \) is unramified (\([22]\)). We will then denote it by \( BC(\pi) \).

**Remark 1.** By a result of Harris–Labesse (\([22, \text{Theorem 2.2.2}]\)), the hypothesis is valid if (1) \( \pi \) have supercuspidal components at at least two split places and (2) either \( n \) is odd or all archimedean places of \( F \) are complex.

Let \( W, W' \) be two hermitian spaces. Then for almost all \( v \), the unitary groups \( U(W)(F_v) \) and \( U(W')(F_v) \) are isomorphic. We say that two automorphic representations \( \pi, \pi' \) of \( U(W)(A) \) and \( U(W')(A) \) respectively are nearly equivalent if \( \pi_v \simeq \pi'_v \) for all places \( v \) of \( F \). Conjecturally, all automorphic representations in a Vogan's L-packet form precisely a single nearly equivalence class. By the strong multiplicity one theorem for \( GL_n \), if \( \pi, \pi' \) are nearly equivalent, their weak base changes are the same.

We recall the notion of (global) distinction following Jacquet. Let \( G \) be a reductive group over \( F \) and \( H \) a subgroup. Let \( \mathcal{A}_0(G) \) be the space of cuspidal automorphic forms on \( G(A) \). We define a period integral

\[
\ell_H : \mathcal{A}_0(G) \to \mathbb{C}
\]

\[
\phi \mapsto \int_{Z_G \cap H(A)H(F) \backslash H(A)} \phi(h)dh
\]

whenever the integral makes sense. Similarly, if \( \chi \) is a character of \( H(F) \backslash H(A) \), we define

\[
\ell_{H,\chi}(\phi) = \int_{Z_G \cap H(A)H(F) \backslash H(A)} \phi(h)\chi(h)dh.
\]

For a cuspidal automorphic representation \( \pi \) (viewed as a subrepresentation of \( \mathcal{A}_0(G) \)), we say that it is (\( \chi \)-, resp.) distinguished by \( H \) if the linear functional \( \ell_H \) (\( \ell_{H,\chi} \), resp.) is nonzero when restricted to \( \pi \). Even if the multiplicity one fails for \( G \), this definition still makes sense as our \( \pi \) comes with a fixed embedding into \( \mathcal{A}_0(G) \).

To state the main theorem of this paper on the global Gan–Gross–Prasad conjecture, we let \( W \subset V \) be a fixed pair of (non-degenerate) hermitian spaces of dimension \( n \) and \( n + 1 \) respectively. The embedding \( W \subset V \) induces an embedding of unitary groups \( \iota : U(V) \subset U(W) \). We will consider \( U(W) \) as a subgroup of \( U(V) \times U(W) \) by the diagonal embedding.

**Theorem 1.1.** Let \( \pi \) be a cuspidal automorphic representation of \( U(V) \times U(W) \). Suppose that
(1) All $v|\infty$ is split in $E/F$.

(2) For two distinct places $v \in \{v_1, v_2\}$ (non-archimedean) split in $E/F$, $\pi_v$ is supercuspidal.

Then the following are equivalent

(i) $L(\Pi, R, 1/2) \neq 0$ for the weak base change $\Pi$ of $\pi$. Here $L(\Pi, R, s) = L(\Pi_{n+1} \times \Pi_n, s)$ is the Rankin–Selberg $L$-function if we write $\Pi = \Pi_n \otimes \Pi_{n+1}$.

(ii) For some hermitian space $W' \subset V'$ of dimension $n$ and $n + 1$ respectively, and some automorphic representation $\pi'$ of $U(V') \times U(W')$ nearly equivalent to $\pi$, such that $\pi'$ is distinguished by $U(W')$.

Note that we don’t assume that the representation $\pi'$ occurs with multiplicity one in the space of cuspidal automorphic forms $\mathcal{A}_0(U(V') \times U(W'))$. By $\pi'$ we do mean a subspace of $\mathcal{A}_0(U(V') \times U(W'))$.

The theorem confirms the global conjecture of Gan–Gross–Prasad for unitary group under the local restrictions (1) and (2). The two conditions are due to some technical issue we now describe. Our approach is by a simple version of Jacquet–Rallis relative trace formulae. The first assumption is due to the fact that we only prove the existence of smooth transfer for a $p$-adic field (cf. Remark 2). The second assumption is due to the fact that we use a “cuspidal” test function at a split place and use a test function with nice support at another split place (cf. Remark 3).

We remark that previously, it is known by [15],[16] that $(ii) \implies (i)$ for both unitary and orthogonal groups.

Remark 2. In the archimedean case we have some partial result for the existence of smooth transfer (Theorem 3.18). If we assume the local–global compatibility of weak base change at a non-split archimedean place, we may replace the first assumption by the following: if $v|\infty$ is non-split, then $W, V$ are positive definite (hence $\pi_v$ is finite dimensional) and

$$\text{Hom}_{U(W)(F_v)}(\pi_v, \mathbb{C}) \neq 0.$$ 

Remark 3. In the theorem, we may weaken the second condition and only require that $\pi_{v_1}$ is supercuspidal and $\pi_{v_2}$ is tempered.

Remark 4. We recall some by-no-means complete history related to this conjecture. In the lower rank cases, a lot of works have been done on the global Gan–Gross–Prasad conjecture for orthogonal groups: the work of Waldspurger on $SO(2) \times SO(3)$ ([49]), the work of Garrett ([13]), Piatetski-Shapiro–Rallis, Garret–Harris, Harris–Kudla ([21]), Gross–Kudla ([17]), and Ichino ([27]) on the case of $SO(3) \times SO(4)$ or the so-called Jacquet’s conjecture. For the general case, Ginzburg–Jiang–Rallis’s work ([15],[16] etc.) proves one direction of the conjectures for both orthogonal and the unitary cases.
Remark 5. The original local Gross–Prasad conjecture ([18],[19], for the orthogonal case) has also been essentially resolved based on a recent series of work by Waldspurger and Moeglin ([52],[53] etc.). It is expected that similar technique should work for the local conjecture of Gan–Gross–Prasad in the unitary case ([11]). But in our paper we won’t need this. According to the local conjecture of Gan–Gross–Prasad for unitary groups and the multiplicity one of \( \pi' \), such relevant ([11]) pair \((W', V')\) and \( \pi' \) in the theorem should be unique.

Remark 6. Concerning the refined version of the Gan–Gross–Prasad conjecture of Ichino–Ikeda type ([28]) (explicitly stated by N. Harris [23] in the unitary case), we hope that the approach of trace formula may eventually establish the unitary case. One prototype is the case of \( SO(2) \times SO(3) \) by Chen and Jacquet ([6]) where they essentially reproved the Waldspurger formula using trace formula.

An application to non-vanishing of central L-values  We have an application to the existence of non-vanishing twist of Rankin–Selberg L-function.

**Theorem 1.2.** Let \( E/F \) be a quadratic extension of number fields such that all archimedean places are split. Let \( \sigma \) be a cuspidal automorphic representation of \( GL_{n+1}(\mathbb{A}_E) \). Assume that \( \sigma \) is a weak base change of an automorphic representation of some unitary group \( U(V) \) which is locally supercuspidal at two split places of \( F \). Then there exists a cuspidal automorphic representation \( \tau \) of \( GL_n(\mathbb{A}_E) \) such that the central value of the Rankin–Selberg L-function

\[
L(\sigma \times \tau, \frac{1}{2}) \neq 0
\]

**Flicker–Rallis conjecture.** Let \( \eta = \eta_{E/F} \) be the quadratic character of \( F^\times \backslash \mathbb{A}^\times \) associated to the quadratic extension \( E/F \) by class field theory. By abuse of notation, we will denote by \( \eta \) the quadratic character \( \eta \circ \det \) (\( \det \) being the determinant map) of \( GL_n(\mathbb{A}) \).

**Conjecture 1.3** (Flicker–Rallis, [9]). An automorphic cuspidal representation \( \Pi \) on \( \text{Res}_{E/F} GL_n(\mathbb{A}) \) is a weak base change from a cuspidal automorphic \( \pi \) on some unitary group of \( n \)-variables if and only if it is distinguished (\( \eta_{E/F} \)-distinguished, resp.) by \( GL_{n,F} \) if \( n \) is odd (even, resp.).

Another result of the paper is to confirm one direction of Flicker-Rallis conjecture under the same local restrictions as in the previous theorem. In fact, this result is used in the proof of our previous theorem on the global Gan–Gross–Prasad conjecture for unitary groups.

**Theorem 1.4.** Suppose that a cuspidal automorphic representation \( \pi \) of \( U(W)(\mathbb{A}) \) satisfies:

1. All \( v|\infty \) is split in \( E/F \).
2. For two distinct places \( v \in \{v_1, v_2\} \) (non-archimedean) split in \( E/F \), \( \pi_v \) is supercuspidal.

Then the weak base change \( BC(\pi) \) is (\( \eta_- \), resp.) distinguished by \( GL_{n,F} \) if \( n \) is odd (even, resp.).
Remark 7. If $\Pi$ is distinguished by $GL_{n,F}$, then $\Pi$ is conjugate self-dual ([9]). Moreover, the partial Asai L-function has a pole at $s = 1$ if and only if $\Pi$ is distinguished by $GL_{n,F}$ ([8], [10]). In [14], it is further proved that if the central character of $\Pi$ is distinguished, then $\Pi$ is conjugate self-dual if and only if $\Pi$ is distinguished (resp., distinguished or $\eta$-distinguished) if $n$ is odd (resp., even).

We briefly describe the contents of each section. In section 2, we prove the main theorems based on the existence of transfer. In section 3 we reduce the existence of transfer to the existence of local transfer around zero of the “Lie algebras” (an infinitesimal version). In section 4, we show the existence of smooth transfer on Lie algebras for a $p$-adic field.

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2 Relative trace formulae of Jacquet–Rallis

Let $F$ be a field of character zero. For a reductive group $H$ acting on an affine variety $X$, we say that a point $x \in X(F)$ is

- $H$-semisimple if $Hx$ is Zariski closed in $X$ (when $F$ is a local field, equivalently, $H(F)x$ is closed in $X(F)$ for the analytic topology, cf. [2]).

- $H$-regular if the stabilizer $H_x$ of $x$ has the minimal dimension.

If no confusion, we will simply use the words “semisimple” and “regular”. And we say that $x$ is regular semisimple if it is regular and semisimple. In this paper, we will be interested in the following two cases

- $X = G$ is a reductive group and $H = H_1 \times H_2$ is a product of two (reductive) subgroups of $G$ where $H_1$ ($H_2$, resp.) acts by left (right, resp.) multiplication.

- $X = V$ is a vector space (considered as an affine variety) with an action by a reductive group $H$.

For later use, we also recall that the categorical quotient of $X$ by $H$ (cf.[2],[39]) consists of a pair $(Y, \pi)$ where $Y$ is an algebraic variety and $\pi : X \to Y$ is an $H$-morphism such that for any pair $(Y', \pi')$ with $H$-morphism $\pi' : X \to Y'$, there exists a unique morphism $\phi : Y \to Y'$ such that $\pi' = \phi \circ \pi$. If such a pair exists, then it is unique up to a canonical isomorphism. When $X$ is affine (in all our cases), the categorical quotient always exists. Indeed we may construct as follows. Consider the affine variety

$$X/H := Spec\mathcal{O}(X)^H$$
together with the obvious quotient morphism

$$\pi = \pi_{X,G} : X \to \text{Spec } O(X)^H.$$  

Then \((X//H, \pi)\) is a categorical quotient of \(X\) by \(H\). By abuse of notation, we will also let \(\pi\) denote the induced map \(X(F) \to (X//H)(F)\) if no confusion arises.

Now we fix some notations. Let \(F\) be a number field or a local field, and let \(E\) be a quadratic extension.

- **The general linear case.** We will consider the \(F\)-algebraic group \(G' = \text{Res}_{E/F}(GL_{n+1} \times GL_n)\) and two subgroups: \(H'_1\) is the diagonal embedding of \(\text{Res}_{E/F}GL_n\) (where \(GL_n\) is embedded into \(GL_{n+1}\) by \(g \mapsto \text{diag}[g, 1]\)) and \(H'_2 = GL_{n+1,F} \times GL_{n,F}\) embedded into \(G'\) in the obvious way. In this paper for an \(F\)-algebraic group \(H\), we will denote by \(Z_H\) the center of \(H\). We note that \(Z_{G'} \cap Z_{H'_1}\) is trivial.

- **The unitary case.** We will also consider a pair of hermitian spaces over the quadratic extension \(E\) of \(F\): \(V\) and a codimension one subspace \(W\). Suppose that \(W\) is of dimension \(n\). Without loss of generality, we may always assume \(V = W \oplus Eu\) where \(u\) has norm one: \((u, u) = 1\). In particular, the isometric class of \(V\) is determined by \(W\). We have an obvious embedding of unitary groups \(U(W) \hookrightarrow U(V)\). Let \(G = G^W = U(V) \times U(W)\) and let \(\Delta : U(W) \to G\) be the diagonal embedding with image \(H\) (or \(H_W\) to emphasize the dependence on \(W\)) as a subgroup of \(G\).

For a number field \(F\), let \(\eta = \eta_{E/F} : F^\times \backslash \mathbb{A}^\times \to \{\pm 1\}\) be the quadratic character associated to \(E/F\) by class field theory. By abuse of notation we will also denote by \(\eta\) the character of \(H'_2(\mathbb{A})\) defined by \(\eta(h) := \eta(det(h_1)) (\eta(det(h_2)),\text{ resp.})\) if \(h = (h_1, h_2) \in GL_{n-1}(\mathbb{A}) \times GL_n(\mathbb{A})\) and \(n\) odd (even, resp.). Fix a character \(\eta' : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times\) (not necessarily quadratic) such that its restriction \(\eta'|_{\mathbb{A}^\times} = \eta\). We similarly define the local analogue \(\eta_v, \eta'_v\).

### 2.1 Orbital integrals

We first introduce the local orbital integrals appearing in the relative trace formulae of Jacquet–Rallis. We refer [57, sec. 2] on important properties of orbits (namely, double cosets). Later on in sec. 3 we will also recall some of them. We now let \(F\) be a local field of characteristic zero. And let \(E\) be a quadratic semisimple \(F\)-algebra. Namely, \(E\) is either a quadratic extension or \(F \times F\).

We start with the **general linear case.** If an element \(\gamma \in G'(F)\) is \(H'_1 \times H'_2\)-regular semisimple, for simplicity we will say that it is regular semisimple. For such a \(\gamma\) and \(f' \in C_c^\infty(G'(F))\), we define its orbital integral as:

\[
O(\gamma, f') := \int_{H'_1(F)} \int_{H'_2(F)} f'(h_1^{-1}\gamma h_2)\eta(h_2)dh_1dh_2.
\]
The integral converges absolutely. This depends on the choice of Haar measure. But in this paper, the choice of measure is not crucial since we will only concern non-vanishing problem. In the following, we always pre-assume that we have made a choice of a Haar measure on each group. The integral is then absolutely convergent. Up to ±1 (due to the character \( \eta \)), the integral depends only on the orbit of \( \gamma \).

We may simplify the orbital integral as follows. Identify \( H_1' \backslash G' \) with \( \text{Res}_{E/F}GL_{n+1} \). Then we consider the morphism between \( F \)-varieties:

\[
\nu : \text{Res}_{E/F}GL_{n+1} \to S_{n+1}, \quad g \mapsto g\bar{g}^{-1}
\]

where \( S_n \) is the subvariety of \( \text{Res}_{E/F}GL_n \) defined by the equation \( ss' = 1 \). By Hilbert Satz-90, this defines an isomorphism of two affine varieties

\[
\text{Res}_{E/F}GL_{n+1}/GL_{n+1,F} \simeq S_{n+1},
\]

and in the level of \( F \)-points:

\[
GL_{n+1}(E)/GL_{n+1}(F) \simeq S_{n+1}(F).
\]

We may integrate \( f' \) over \( H_1'(F) \) to get a function on \( \text{Res}_{E/F}GL_{n+1}(F) \):

\[
\tilde{f}'(x) := \int_{H_1'(F)} f'((x, 1)h_1)dh_1, \quad x \in \text{Res}_{E/F}GL_{n+1}(F).
\]

Now assuming that \( n \) is odd, so the character \( \eta \) on \( H_2' \) is indeed only nontrivial on the component \( GL_{n+1,F} \). Then we may introduce a function on \( S_{n+1}(F) \) as follows: when \( \nu(x) = s \in S_{n+1}(F) \), we define

\[
\tilde{f}'(x) := \int_{GL_{n+1}(F)} \tilde{f}'(xg)\eta'(xg)dg.
\]

Then \( \tilde{f}' \in \mathcal{C}_c^\infty(S_{n+1}(F)) \) and all functions in \( \mathcal{C}_c^\infty(S_{n+1}(F)) \) arise this way. Now it is easy to see that for \( \gamma = (\gamma_1, \gamma_2) \):

\[
O(\gamma, f') = \eta'(\det(\gamma_1\gamma_2^{-1})) \int_{GL_n(F)} \tilde{f}'(h^{-1}sh)\eta(h)dh, \quad s = \nu(\gamma_1\gamma_2^{-1}).
\]

If \( n \) is even, we simply define in the above

\[
\tilde{f}'(x) := \int_{GL_{n+1}(F)} \tilde{f}'(xg)dg,
\]

and then

\[
O(\gamma, f') = \int_{GL_n(F)} \tilde{f}'(h^{-1}sh)\eta(h)dh, \quad s = \nu(\gamma_1\gamma_2^{-1}).
\]
Suppose \((z, h) \circ s = s\). As \(tr(A) \neq 0, d \neq 0\), up to modification by elements in \(Z_0\), we may assume that \(z = 1\). Then the first assertion follows from the fact that the stabilizer of \(s\) is trivial for the \(GL_n\)-action on \(S_{n+1}\). When \(tr(A) \neq 0, d \neq 0\), besides \(N_{E/F}(tr(A))\), \(N_{E/F}d\), the following are also \(Z_{G'} \times GL_{n,F}\)-invariant:

\[
\frac{tr \wedge^i A}{(tr(A))^i}, \quad \frac{cA^j b}{(tr(A))^{j+1}d}, \quad 1 < i \leq n, 0 \leq j \leq n - 1.
\]
Then we claim that two $Z$-regular semisimple $s, s'$ are in the same $Z_{G'} \times GL_{n,F}$-orbit if and only if they have the same invariants (listed above). One direction is obvious. We now assume that $s, s'$ are $Z$-regular semisimple and have the same invariants. In particular, the values of $N_{E/F}(tr(A)), N_{E/F}d$ are the same. Replacing $s'$ by $zs'$ for a suitable $z \in Z_{G'}$, we may assume that $s'$ and $s$ have the same $tr(A)$ and $d$. Then $s, s'$ have the same value for $tr \land i A, 1 \leq i \leq n$ and $cA^j b, 0 \leq j \leq n - 1$. Then by [57, sec.2], $s$ and $s'$ are conjugate by $GL_{n,F}$ since they are also $GL_{n,F}$-regular semisimple. Therefore, the $Z_{G'} \times GL_{n,F}$-orbit of $s$ consists of $s \in S_{n+1}$ such that for a fixed tuple $(\alpha, \beta, \alpha_i, \beta_j)$

\[ N_{E/F}(tr(A)) = \alpha, N_{E/F} = \beta, \]

and

\[ \alpha_i = \frac{tr \land i A}{(tr(A))^i}, \beta_j = \frac{cA^j b}{(tr(A))^{j+1}d} 1 < i \leq n, 0 \leq j \leq n - 1. \]

The second set of conditions can be rewritten as

\[ tr \land i A - \alpha_i (tr(A))^i = 0, cA^j b - \beta_j (tr(A))^{j+1}d = 0, \]

for $1 \leq i \leq n, 0 \leq j \leq n - 1$. This shows that the $Z_{G'} \times GL_{n,F}$-orbit of $s$ is Zariski closed.

Let $\chi'$ be a character of the center $Z_{G'}(F)$ that is trivial on $Z_{H_2'}(F)$. If an element $\gamma \in G'(F)$ is $Z$-regular semisimple, we define a $\chi'$-orbital integral:

\[ (2.4) \quad O_{\chi'}(\gamma, f') := \int_{H_2'(F)} \int_{Z_{H_2'}(F) \setminus H_2'(F)} \int_{Z_{G'}(F)} f'(h_1^{-1} z^{-1} \gamma h_2) \chi'(z) \eta(h_2) dz dh_1 dh_2. \]

The integral is then absolutely convergent.

We now move to the unitary case. Similarly, we will simply use “regular semisimple” for the action of $H \times H$ on $G$. For a regular semisimple $\delta \in G(F)$ and $f \in C_c^\infty(G(F))$, we define its orbital integral

\[ O(\delta, f) = \int_{H(F) \times H(F)} f(x^{-1} \delta y) dx dy. \]

It converges absolutely. Similar to the linear case, we may simplify the orbital integral $O(\delta, f)$. Introduce a new function on $U(V)(F)$:

\[ (2.5) \quad \bar{f}(g) = \int_{U(W)(F)} f(g, 1) h) dh, \quad g \in U(V)(F). \]

Then for $\delta = (\delta_{n+1}, \delta_n) \in G(F)$, we may rewrite the previous integral as

\[ (2.6) \quad O(\delta, f) = \int_{H(F)} \bar{f}(y^{-1} (\delta_{n+1} \delta_n^{-1}) y) dy. \]
We thus have the action of $U(W)$ on $U(V)$ by conjugation. Then the element $\delta = (\delta_{n+1}, \delta_n) \in G(F)$ is $H \times H$-regular semisimple if and only if $\delta_{n+1} \delta_n^{-1} \in U(V)(F)$ is $U(W)$-regular semisimple for the conjugation action. An element $\delta \in U(V)(F)$ is $U(W)$-regular semisimple if and only if $\delta^i u$, $i = 0, 1, \ldots, n$, form an $E$-basis of $V$. We also need to consider the action of the center $Z_G$. We similarly define the notion of $Z$-regular semisimple. Then Lemma 2.1 easily extends to the unitary case. Let $\chi$ be a character of the center $Z_G(F)$. If an element $\delta \in G(F)$ is $Z$-regular semisimple, we define a $\chi$-orbital integral:

\[(2.7) \quad O_\chi(\delta, f) := \int_{H(F) \times H(F)} \int_{Z_G(F)} f(x^{-1} z \delta y) \chi(z) dz dx dy.\]

The integral is then absolutely convergent.

### 2.2 RTF on the general linear group

We will use “RTF” to stand for “relative trace formula”. Now we recall the construction of Jacquet–Rallis’ RTF on the general linear side ([35]). Fix a Haar measure on $Z_{G'}(\mathbb{A})$, $H'_i(\mathbb{A})$ ($i = 1, 2$) etc., and the counting measure on $Z_{G'}(F) H'_i(F)$ ($i = 1, 2$) etc.

For $f' \in \mathcal{C}_c^\infty(G'(\mathbb{A}))$, we define a kernel function

$$K_{f'}(x, y) = \sum_{\gamma \in G'(F)} f'(x^{-1} y \gamma).$$

For a character $\chi'$ of $Z_{G'}(F) \backslash Z_{G'}(\mathbb{A})$, we define the $\chi'$-part of the kernel function

$$K_{f', \chi'}(x, y) = \int_{Z_{G'}(F) \backslash Z_{G'}(\mathbb{A})} \sum_{\gamma \in G(F)} f'(x^{-1} y z^{-1} \gamma) \chi(z) dz.$$

We then consider a distribution on $G'(\mathbb{A})$:

$$I(f') = \int_{H'_1(F) \backslash H'_1(\mathbb{A})} \int_{H'_2(F) \backslash H'_2(\mathbb{A})} K_{f'}(h_1, h_2) \eta(h_2) dh_1 dh_2.$$

Similarly, for a character $\chi'$ of $Z_{G'}(F) \backslash Z_{G'}(\mathbb{A})$ that is trivial on $Z_{H'_2(\mathbb{A})}$, we define the $\chi'$-part of the distribution

$$I_{\chi'}(f') = \int_{H'_1(F) \backslash H'_1(\mathbb{A})} \int_{Z_{H'_2(\mathbb{A})} H'_2(F) \backslash H'_2(\mathbb{A})} K_{f', \chi'}(h_1, h_2) \eta(h_2) dh_1 dh_2.$$

Note that in general these integrals may diverge. But for our purpose, it suffices to consider test functions $f'$ satisfying some local conditions. We say that a function $f' \in \mathcal{C}_c^\infty(G'(\mathbb{A}))$ is nice with respect to $\chi'$ if it is decomposable $f' = \otimes_v f'_v$ and satisfies:

- For at least one place $v_1$, $f'_{v_1} \in \mathcal{C}_c^\infty(G'(F_{v_1}))$ is a essentially a matrix coefficient of a supercuspidal representation with respect to $\chi'_{v_1}$. This means that $\tilde{f}'(g) = \int_{Z_{G(F_{v_1})}} f'(gz) \chi'_{v_1}(z) dz$ is a matrix coefficient of a supercuspidal representation.
• For at least one place \( v_2 \), \( f'_{v_2} \) is supported on the locus of \( Z \)-regular semisimple elements.

• \( f'_\infty \) is \( K_\infty \)-finite where \( K_\infty \) is the maximal compact subgroup of \( G'(F_\infty) \).

**Lemma 2.2.** Suppose that \( f' = \otimes_v f'_v \) is nice with respect to \( \chi' \).

• As a function on \( H'_1(\mathbb{A}) \times H'_2(\mathbb{A}) \), \( K_{f'}(h_1, h_2) \) is compactly supported modulo \( H'_1(F) \times H'_2(F) \). In particular, the integral \( I(f') \) converges absolutely.

• As a function on \( H'_1(\mathbb{A}) \times H'_2(\mathbb{A}) \), \( K_{f',\chi'}(h_1, h_2) \) is compactly supported modulo \( H'_1(F) \times H'_2(F)Z_{H'_2}(\mathbb{A}) \). In particular, the integral \( I_{\chi'}(f') \) converges absolutely.

**Proof.** The kernel function \( K_{f'} \) can be written as

\[
\sum_{\gamma \in H'_1(F) \setminus G'(F)/H'_2(F)} \sum_{(\gamma_1, \gamma_2) \in H'_1(F) \times H'_2(F)} f'(h_1^{-1}\gamma_1^{-1}\gamma_2 h_2),
\]

where the outer sum is over regular semisimple \( \gamma \). First we claim that the outer sum has only finite many non-zero terms. Let \( \Omega \) be the support of \( f' \). Note that the invariants of \( G'(\mathbb{A}) \) defines a continuous map from \( G'(\mathbb{A}) \) to \( X(\mathbb{A}) \) where \( X \) is the categorical quotient of \( G' \) by \( H'_1 \times H'_2 \). So the image of \( \Omega \) will be a compact set in \( X(\mathbb{A}) \). On the other hand the image of \( h_1^{-1}\gamma_1^{-1}\gamma_2 h_2 \) is in the discrete set \( X(F) \). Moreover for a fixed \( x \in X(F) \) there is at most one \( H'_1(F) \times H'_2(F) \) double coset with given invariants. This shows the outer sum has only finite many non-zero terms.

It remains to show that for a fixed \( \gamma_0 \in G'(F) \), the function on \( H'_1(\mathbb{A}) \times H'_2(\mathbb{A}) \) defined by \( (h_1, h_2) \mapsto f'(h_1^{-1}\gamma_0 h_2) \) has compact support. Consider the continuous map \( H'_1(\mathbb{A}) \times H'_2(\mathbb{A}) \to G'(\mathbb{A}) \) given by \( (h_1, h_2) \mapsto h_1^{-1}\gamma_0 h_2 \). When \( \gamma \) is regular semisimple, this defines an homeomorphism onto a closed subset of \( G'(\mathbb{A}) \). This implies the desired compactness and completes the proof the first assertion. The second one is similarly proved using the \( Z \)-regular semi-simplicity. \( \square \)

The last lemma allows us to decomposes the distribution into a finite sum of orbital integrals

\[
I(f') = \sum_{\gamma} O(\gamma, f'),
\]

where the sum is over regular semisimple \( \gamma \in H'_1(F) \setminus G'(F)/H'_2(F) \). If \( f' = \otimes_v f'_v \) is decomposable, we may decompose the orbital integral as a product of local orbital integrals:

\[
O(\gamma, f') = \prod_v O(\gamma, f'_v),
\]

where \( O(\gamma, f'_v) \) is defined as in the previous subsection. Similarly, we have

\[
I_{\chi'}(f') = \sum_{\gamma} O_{\chi'}(\gamma, f'),
\]

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where the sum is over regular semisimple \( \gamma \in Z_{G'}(F)H_1'(F) \backslash G'(F)/H_2'(F) \).

For a cuspidal automorphic representation \( \Pi \) of \( G'(\mathbb{A}) \) whose central character is trivial on \( Z_{H_2'}(\mathbb{A}) \), we define a spherical character (a relative version of character):

\[
I_\Pi(f') = \sum_{\phi \in \mathcal{B}(\Pi)} \left( \int_{H_1'(F) \backslash H_1'(\mathbb{A})} \Pi(f') \phi(x) dx \right) \left( \int_{Z_{H_2'}(\mathbb{A})H_2'(F) \backslash H_2'(\mathbb{A})} \phi(x) dx \right)
\]

where the sum is over an orthonormal basis \( \mathcal{B}(\Pi) \) of \( \Pi \). For nice \( f' \), we may choose the basis such that the sum is finite.

We then have a simple RTF:

**Theorem 2.3.** If \( f' \) is nice with respect to \( \chi' \), then \( I_{\chi'}(f') \) is equal to

\[
\sum_{\{\gamma\}} O_{\chi'}(f') = \sum_{\Pi} I_\Pi(f'),
\]

where the sum on the LHS runs over all regular semisimple

\[
\gamma \in H_1'(F) \backslash G'(F)/Z_{G}(F)H_2'(F)
\]

and the sum on the RHS runs over all cuspidal automorphic representations \( \Pi \) with central character \( \chi' \).

**Proof.** It suffices to treat the spectral side. Since \( f'_{v_3} \) is essentially a matrix coefficient of a super-cuspidal representation, an argument for the simple version of Arthur–Selberg trace formula also applies to the current case and we obtain that the kernel \( I_{\chi'}(f') \) only has only cuspidal part. This yields an absolutely convergent sum

\[
I_{\chi'}(f') = \sum_{\Pi} I_\Pi(f'),
\]

where \( \Pi \) runs over automorphic cuspidal representations of \( G'(\mathbb{A}) \) with central character \( \chi' \).

\[\square\]

### 2.3 RTF on unitary groups

We now recall the RTF of Jacquet–Rallis on the unitary case. For \( f \in C_c^\infty(G(\mathbb{A})) \) we consider a kernel function

\[
K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y),
\]

and a distribution

\[
J(f) := \int_{H(F) \backslash H(\mathbb{A})} \int_{H(F) \backslash H(\mathbb{A})} K_f(x, y) dx dy.
\]
For a fixed central character \( \chi = (\chi_n+1, \chi_n) : Z_G(\AA) \simeq U(1)(\AA)^2 \to \mathbb{C}^\times \) we introduce the kernel
\[
K_{f,\chi}(x, y) = \int_{Z_G(F)\backslash Z_G(\AA)} K_f(zx, y)\chi(z)dz = \int_{Z_G(F)\backslash Z_G(\AA)} K_f(x, z^{-1}y)\chi(z)dz,
\]
and a distribution
\[
J_\chi(f) := \int_{H(F)\backslash H(\AA)} \int_{H(F)\backslash H(\AA)} K_{f,\chi}(x, y)dxdy.
\]
Note that the center \( Z_G \) is anisotropic and its intersection with \( H \) is trivial.

The integrals \( J(f) \) and \( J_\chi(f) \) are usually not convergent. Similar to the general linear case, we will consider a simplified trace formula and for our purpose we only discuss the convergence issue for the \( \chi \)-part for a fixed \( \chi \). We say that such a function \( f \in C_c^\infty(G(\AA)) \) is nice with respect to \( \chi \) if \( f = \otimes_v f_v \) satisfies

- For at least one place \( v_1 \), \( f_{v_1} \) is a essentially a matrix coefficient of a supercuspidal representation with respect to \( \chi_{v_1} \). This means that \( \tilde{f}_{v_1}(g) = \int_{Z_G(F_{v_1})} f(gz)\chi_{v_1}(z)dz \) is a matrix coefficient of a supercuspidal representation.

- For at least one place \( v_2 \), \( f_{v_2} \) is supported on the locus of \( Z \)-regular semisimple elements.

- \( f_\infty \) is \( K_\infty \)-finite where \( K_\infty \) is the maximal compact subgroup of \( G(F_\infty) \).

For a cuspidal automorphic representation \( \pi \), we define a spherical character as a distribution on \( G(\AA) \):
\[
J_\pi(f) = \sum_{\phi \in B(\pi)} \left( \int_{H(F)\backslash H(\AA)} \pi(f)\phi(x)dx \right) \left( \int_{H(F)\backslash H(\AA)} \phi(x)dx \right),
\]
where the sum is over an orthonormal basis \( B(\pi) \) of \( \pi \). For nice \( f \), we may choose the basis such that the sum is finite.

For nice test functions, we have a simple RTF.

**Theorem 2.4.** If \( f \) is nice with respect to \( \chi \), then \( J_\chi(f) \) is equal to
\[
\sum_{\delta} O_\chi(\delta, f) = \sum_\pi J_\pi(f),
\]
where the LHS runs over all regular simisimple orbits
\[
\delta \in H(F)\backslash G(F)/Z_G(F)H(F),
\]
and the RHS runs over all cuspidal automorphic representations \( \pi \) with central character \( \chi \).

Here in the RHS, by a \( \pi \) we mean a sub-representation of the space of cuspidal automorphic forms. So, a prior, two such representations may be isomorphic (as we don’t know yet the multiplicity one for such a \( \pi \)).

**Proof.** The proof is the same as for the RTF on the linear side.
2.4 Comparison: fundamental lemma and transfer

Some preparation. We first recall the matching of orbits without proof. The proof can be found [44] and [57]. Now the field $F$ is either a number field or a local field of characteristic zero. Note that we may embed $S_{n+1}(F)$ to $GL_{n+1}(E)$. Also we may realize $U(V)$ as a subgroup of $GL_{n+1}$. Even though such realization is not unique, the following notion is independent of the realization: we say that $\delta \in U(V)(F)$ and $s \in S_{n+1}(F)$ match if $s$ and $\delta$ (both considered as elements in $GL_{n+1}(E)$) are conjugate by an element in $GL_n(E)$. Then it is proved in [57] that this defines a natural bijection between the set of regular semisimple orbits of $S_{n+1}(F)$ and the disjoint union of regular semisimple orbits of $U(V)$ where $V = W \oplus Eu$ (with $(u, u) = 1$) and $W$ runs over all (isometric classes of) hermitian spaces over $E$.

To state the matching of test functions, we need to introduce a “transfer factor”: it is a compatible family of functions $\{\Omega_v\}_v$ indexed by all places $v$ of $F$, where $\Omega_v$ is defined on the regular semisimple locus of $S_{n+1}(F_v)$, and they satisfy:

- If $s \in S_{n+1}(F)$ is regular semisimple, then we have a product formula
\[ \prod_v \Omega_v(s) = 1. \]

- For any $h \in GL_n(F_v)$ and $s \in S_n(F_v)$, we have $\Omega(h^{-1}sh) = \eta(h)\Omega_v(s)$.

We may construct it as follows. We have fixed a character $\eta' : E^\times \backslash A_E^\times \rightarrow \mathbb{C}^\times$ (not necessarily quadratic) such that its restriction $\eta'|_{A^\times} = \eta$. Then we may define
\[ \Omega_v(s) := \eta'_v((\det(s)^{-(n+1)/2}\det(es, ..., es^n))). \]

It is easy to verify that the family $\{\Omega_v\}_v$ defines a transfer factor.

We also extend this to a transfer factor on $G'$: it is a compatible family of functions (to abuse notation) $\{\Omega_v\}_v$ on the regular semisimple locus of $G'(F_v)$, indexed by all places $v$ of $F$, such that

- If $\gamma \in G'(F)$ is regular semisimple, then we have a product formula $\prod_v \Omega_v(\gamma) = 1$.

- For any $h_i \in H'_i(F_v)$ and $s \in S_n(F_v)$, we have $\Omega(h_1\gamma h_2) = \eta(h_2)\Omega_v(\gamma)$.

We may construct it as follows: write $\gamma = (\gamma_1, \gamma_2)$ and $s = \nu(\gamma_1\gamma_2^{-1})$, if $n$ is odd, we set:
\[ \Omega_v(\gamma) := \eta'(\det(\gamma_1\gamma_2^{-1}))\eta'_v((\det(s)^{-(n+1)/2}\det(es, ..., es^n))), \]
and if $n$ is even, we set:
\[ \Omega_v(\gamma) := \eta'_v((\det(s)^{-n/2}\det(es, ..., es^n))). \]

For a place $v$ of $F$, we consider $f' \in C(S_{n+1}(F_v))$ and the tuple $(f_W)_V, f_W \in C^\infty_c(U(V)(F_v))$ indexed by the set of all (isometric classes of) hermitian spaces $W$ over $E_v = E \otimes F_v$ and we
set \( V = W \oplus E_v u \). We say that \( f' \in \mathcal{C}(S_n(F_v)) \) and the tuple \((f_W)_W\) are (smooth) transfer of each other if
\[
\Omega_v(s)O(s, f') = O(\delta, f_W),
\]
whenever a regular semisimple \( s \in S_{n+1}(F_v) \) matches \( \delta \in U(V)(F_v) \).

Similarly we extend the definition to (smooth) transfer between \( C^\infty_c(G'(F_v)) \) and \( C^\infty_c(G^W(F_v)) \) where \( G^W = U(V) \times U(W) \). It is then obvious that the the existence of the two transfers are equivalent.

For a split place \( v \), the existence of transfer is almost trivial. To see this, we may directly work with smooth transfer on the original groups. We may identify \( GL_n(E \otimes F_v) = GL_n(F_v) \times GL_n(F_v) \) and suppose the function \( f'_n = f'_{n,1} \otimes f'_{n,2} \). We identify the unitary group \( U(W)(F_v) \simeq GL_n(F_v) \) and let \( f_n \in C^\infty_c(GL_n(F_v)) \). Similarly we have \( f'_{n+1} \) for \( GL_{n+1}(F_v) \) etc..

**Proposition 2.5.** If \( v \) is split in \( E \), then the smooth transfer exists. In fact we may take the convolution \( f_i = f'_{i,1} \ast f_{i,2}^\vee \) where \( i = n, n + 1 \) and \( f_{i,2}^\vee(g) = f'_{i,2}(g^{-1}) \)

**Proof.** In this case the quadratic character \( \eta_v \) is trivial. For \( f' = f'_{n+1} \otimes f'_n \), the orbital integral \( O(\gamma, f') \) can be computed in two steps: first we integrate over \( H_2(F_v) \) then over the rest. Define
\[
\bar{f'}_i(x) = \int_{GL_i(F_v)} f'_{i,1}(xy)f'_{i,2}(y)dy = f'_{i,1} \ast \bar{f'}_{i,2}(x), \quad i = n, n + 1.
\]

Then obviously we have the orbital integral for \( \gamma = (\gamma_{n+1}, \gamma_n) \in G'(F_v) \) and \( \gamma_i = (\gamma_{i,1}, \gamma_{i,2}) \in GL_i(F_v) \times GL_i(F_v) \), \( i = n, n + 1 \):
\[
O(\gamma, f') = \int_{GL_n(F_v)} \int_{GL_n(F_v)} \bar{f'}_{n+1}(x\gamma_{n+1,1}\gamma_{n+1,2}y)\bar{f'}_{n}(x\gamma_{n,1}\gamma_{n,2}y)dxdy.
\]

Now the lemma follows easily. \( \square \)

Now we use \( E/F \) to denote a local (genuine) quadratic field extension.

**Theorem 2.6.** If \( E/F \) is non-archimedean, then the smooth transfer exists.

The proof will occupy section 3 and 4.

Let \( \chi \) be a character of \( Z_G(F) \) and define the character \( \chi' \) of \( Z_{G'}(F) \) to be the base change of \( \chi \).

**Corollary 2.7.** If \( f' \) and \( f_W \) match, then the \( \chi \)-orbital integrals also match:
\[
O_{\chi}(\delta, f_W) = \Omega(\gamma)O_{\chi'}(\gamma, f')
\]
if \( \gamma \) and \( \delta \) match.
Proof. It suffices to verify that the orbital integrals are compatible with multiplication by central elements in the following sense: consider $z \in E^\times \times E^\times$ identified with the center of $G'(F)$ in the obvious way. We denote by $\bar{z}$ the Galois conjugate coordinate-wise. Then $z/\bar{z} \in E^1 \times E^1$ which can be identified with the center of $G(F)$ in the obvious way. Assume that $\delta$ and $\gamma$ match. Then so do $z\gamma$ and $z/\bar{z}\delta$. We have by assumption that $f'$ and $f_W$ match:

$$\Omega(z\gamma)O(z\gamma,f') = O(z/\bar{z}\delta,f_W)$$

for all $z$. It is an easy computation to show that our definition of transfer factors satisfy

$$\Omega(z\gamma) = \Omega(\gamma).$$

We need two more theorems to prove the trace formula identity.

The first one is the fundamental lemma for units in the spherical Hecke algebras. Let $E/F$ be an unramified quadratic extension (non-archimedean). There are precisely two isometric classes of hermitian space $W$: one with a self-dual lattice is denoted by $W_0$ and the other $W_1$.

For $W_0$ we consider the stabilizer $K$ of a self-dual lattice.

**Theorem 2.8** (Yun,[55]). There is a constant $c(n)$ depending only on $n$ such that the fundamental lemma of Jacquet–Rallis holds for all quadratic extension $E/F$ with residue character larger than $c(n)$; namely, the function $1_K \in \mathcal{C}_c^\infty(G'(F))$ and the pair $f_{W_0} = 1_K$, $f_{W_1} = 0$ are transfer of each other.

The second result is a theorem of automorphic-Cebotarev-density type. It will allows to separate (cuspidal) spectrums without using the fundamental lemma for the full spherical Hecke algebras at non-split places. It is stronger than the strong multiplicity one theorem for $GL_n$.

**Theorem 2.9** (Ramakrishnan). Let $E/F$ be a quadratic extension. Two cuspidal automorphic representations $\Pi_1, \Pi_2$ of $\text{Res}_{E/F}GL_n(\mathbb{A})$ are isomorphic if and only if $\Pi_{1,v} \simeq \Pi_{2,v}$ for almost all places $v$ of $F$ that are split in $E/F$.

The proof can be found in [45].

The trace formula identity. We first have the following coarse form.

**Proposition 2.10.** Fix a character $\chi$ of $Z_G(F)\backslash Z_G(\mathbb{A})$ and let $\chi'$ be its base change. Fix a split place $v_0$ and a supercuspidal representation $\pi_{v_0}$ of $G(F_v)$ with central character $\chi_{v_0}$. Suppose that

- $f'$ and $(f_W)_W$ are all nice functions and are smooth transfer of each other.
Let \( \Pi_{v_0} \) be the local base change of \( \pi_{v_0} \). Then \( f'_{v_0} \) is essentially a matrix coefficient of \( \Pi_{v_0} \) and is related to \( f_{W,v_0} \) as prescribed in Prop. 2.5 (In particular, \( f_{W,v_0} \) is essentially a matrix coefficient of \( \pi_{v_0} \)).

Fix \( \otimes_v \pi_v^0 \) where the product is over almost all split places \( v \) and each \( \pi_v^0 \) is irreducible unramified. Then we have

\[
\sum_{\Pi} I_{\Pi}(f') = \sum_{W} \sum_{\pi_W} I_{\pi_W}(f_W),
\]

where the sums run over all automorphic representations \( \Pi \) of \( G'(\mathbb{A}) \) and \( \pi_W \) of \( G(\mathbb{A}) \) with central characters \( \chi', \chi \) respectively such that

- \( \pi_{W,v} \simeq \pi_v^0 \) for almost all split \( v \).
- \( \pi_{W,v_0} \) is the fixed supercuspidal representation \( \pi_{v_0} \).
- \( \Pi = BC(\pi_W) \) is a weak base change of \( \pi_W \), and \( \Pi_{v_0} \) is the local base change of \( \pi_{v_0} \). In particular \( \Pi \) is cuspidal and the LHS contains at most one term.

**Proof.** We may assume that all test functions are decomposable. Let \( S \) be a finite set of places such that for any \( v \) outside \( S \), \( f'_v \) and \( f_{W,v} \) are units of the spherical Hecke algebras (in particular, \( v \) is non-archimedean and unramified in \( E/F \)) and for all Hermitian spaces \( W \) such that \( f_W \neq 0 \), they are the same outside \( S \) and are all unramified. So we may identify \( G(\mathbb{A}^S) \) for all such \( W \) appearing here. Now we enlarge \( S \) so that for all non-split \( v \) outside \( S \), the fundamental lemma for unit holds. The fundamental lemma for the spherical Hecke algebra holds at a split place. Therefore for any \( f',S \) in the spherical Hecke algebra \( \mathcal{H}(G'(\mathbb{A}^S)//K'S) \) \((K'S = \prod_{v \notin S}) \) is the usual maximal compact subgroup) and \( f^S \in \mathcal{H}(G(\mathbb{A}^S)//K^S) \) such that at a non-split \( v \), \( f'_v, f_v \) are the units, we have a trace formula identity

\[
I(f'_S \otimes f'^S) = \sum_{W} J(f_{W,S} \otimes f^S).
\]

Again all these test functions are nice so we may apply the simple trace formulae:

\[
\sum_{\Pi} I_{\Pi}(f'_S \otimes f'^S) = \sum_{W} \sum_{\pi_W} J_{\pi_W}(f_{W,S} \otimes f^S).
\]

Here all \( \Pi, \pi_W \) are cuspidal automorphic representations whose component at \( v_0 \) are the given ones. Let \( \lambda_{\Pi^S} \) \( (\lambda_{\pi_W^S}, \text{ resp.}) \) be the linear functional of the spherical Hecke algebras \( \mathcal{H}(G'(\mathbb{A}^S)//K'S) \) \( (\mathcal{H}(G(\mathbb{A}^S)//K^S), \text{ resp.}) \). Then we observe that

\[
I_{\Pi}(f'_S \otimes f'^S) = \lambda_{\Pi^S}(f'^S)I_{\Pi}(f'_S \otimes 1_{K'S})
\]

and similarly for \( J_{\pi_W}(f_{W,S} \otimes f^S) \). Note that we are only allowed to take the unit elements in the spherical Hecke algebras. Therefore we can view both sides as linear functionals on the
spherical Hecke algebra \( \mathcal{H}(G'(A^{S, \text{split}})//K'^{S, \text{split}}) \) where “split” indicate we only consider the product over all split places outside \( S \). These linear functionals are linearly independent. In particular, for the fixed \( \otimes_v \pi_v^0 \), we may have an equality as claimed in the theorem. Since such \( \Pi \)'s are cuspidal, there exists at most one \( \Pi \) by the automorphic-Cebotarev-density theorem of Ramakrishnan.

Now we come to the trace formula identity which will allow us to deduce the main theorems in Introduction.

**Proposition 2.11.** Let \( E/F \) be a quadratic extension such that all archimedean places \( v|\infty \) are split. Fix a hermitian space \( W_0 \) and define \( V_0 \), the group \( G \) etc. as before. Let \( \pi \) be a cuspidal automorphic representation of \( G \) such that for a split place \( v_0 \), \( \pi_{v_0} \) is supercuspidal. Consider decomposable nice functions \( f' \) and \( (f'_W)_W \) satisfying the same conditions as in the previous proposition. Then we have a trace formula identity:

\[
I_{\Pi}(f') = \sum_W \sum_{\pi_W} J_{\pi_W}(f_W),
\]

where \( \Pi = BC(\pi) \) and the sum in the RHS runs over all \( W \) and all \( \pi_W \) nearly equivalent to \( \pi \).

**Proof.** Apply the previous proposition to \( \pi_v^0 = \pi_v \) for almost all split \( v \). Then in the sum of the RHS there, all \( \pi_W \) have the same weak base change \( \Pi \). Note that the local base change map are injective for split places and for unramified representations at non-split unramified places. This implies that all \( \pi_W \) are in the same nearly equivalence class.

**A non-vanishing result.** Finally, to see that the second condition on the niceness of a test function does not lose generality in some sense, at least for supercuspidal representations, we will need some “regularity” result for the distribution \( J_\pi \). By the multiplicity one result [3] and [48], we may fix an appropriate choice of generator \( \ell_{H_v} \in \text{Hom}_{H_v}(\pi_v, \mathbb{C}) \) (\( \ell_{H_v} = 0 \) if the space is zero) and decompose

\[
\ell_H = c_\pi \prod_v \ell_{H_v},
\]

where \( c_\pi \) is a constant depending on the cuspidal automorphic representation \( \pi \) (and its realization in \( \mathcal{A}_0(G) \)). This gives a decomposition of the spherical character as a product of local spherical characters

\[
J_\pi(f) = |c_\pi|^2 \prod_v J_{\pi_v}(f_v),
\]

where the spherical character is defined as

\[
J_{\pi_v}(f_v) = \sum_{\phi_v \in \mathcal{B}(\pi_v)} \ell_{H_v}(\pi_v(f_v)\phi_v)\overline{\ell_{H_v}(\phi_v)}.
\]
Note that $J_{\pi_v}$ is a distribution of positive type, namely, for all $f_v \in \mathcal{C}_c^\infty(G(F_v))$,

$$J_{\pi_v}(f_v \ast f_v^*) \geq 0, \quad f_v^*(g) := \overline{f(g^{-1})}.$$ 

We will also say that a function of the form $f_v \ast f_v^*$ is of positive type.

**Proposition 2.12.** Let $\pi_v$ be a tempered representation of $G(F_v)$. Then there exists function $f_v \in \mathcal{C}_c^\infty(G(F_v))$ supported in the $Z$-regular semisimple locus such that

$$J_{\pi_v}(f_v) \neq 0.$$

The proof is given in the Appendix (Theorem A.2). Equivalently, the result can be stated as follows: the support of the spherical character $J_{\pi_v}$ (as a distribution on $G(F_v)$) is not contained in the complement of regular semisimple locus.

### 2.5 Proof of Theorem 1.1: $(ii) \implies (i)$

**Proposition 2.13.** Let $E/F$ be a quadratic extension such that all archimedean places $v|\infty$ are split. Let $W$ be a codimension one subspace of a hermitian space $V$ of dimension $n + 1$. Let $\pi$ be a cuspidal automorphic representation of $U(V) \times U(W)$ such that for a split places $v_1$, $\pi_{v_1}$ is supercuspidal and for a split $v_2 \neq v_1$, $\pi_{v_2}$ is tempered. Denote by $\Pi$ its weak base change. If $\pi$ is distinguished by the diagonal embedding of $U(W)$, then $L(\Pi, R, 1/2) \neq 0$ and $\Pi$ is $\eta$-distinguished by $H'_2 = GL_{n+1,F} \times GL_{n,F}$. In particular, in Theorem 1.1, $(ii)$ implies $(i)$.

**Proof.** We apply Prop. 2.11. It suffices to show that there exist $f'$ as in Prop. 2.11 such that

$$I_{\Pi}(f') \neq 0.$$

We will first choose an appropriate $f := f_W$ and then choose $f'$ to be a transfer of the tuple $(f_W, 0, \ldots, 0)$ where for all hermitian space other than $W$ we choose the zero functions. We choose $f'$ satisfying the conditions of Prop. 2.11. Then the trace formula identity from Prop. 2.11 is reduced to

$$I_{\Pi}(f') = \sum_{\pi_W} J_{\pi_W}(f_W).$$

Note that for all $\pi_W$, they have the same local component at $v_1, v_2$ by the Hypothesis on the local–global compatibility for weak base change at split places.

We choose $f = f_W = \otimes_v f_v$ as follows. By the assumption on the distinction of $\pi$, we may assume that for some function $g$ of positive type $J_\pi(f) > 0$. We may assume that at $v_1$, $g_{v_1}$ is essentially a matrix coefficient of $\pi_{v_1}$. This is clearly possible. Then we have

$$J_{\pi}(g) = |c_\pi|^2 \prod_v J_\pi(g_v) > 0$$

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and for all $\pi_W$ nearly equivalent to $\pi$:

$$J_{\pi_W}(g) \geq 0.$$  

Now we choose $f_v = g_v$ for every place $v$ other than $v_2$. We choose $f_{v_2}$ to be supported in the $Z$-regular semisimple locus. By Prop. 2.12, we may choose an $f_{v_2}$ such that

$$J_{\pi_{v_2}}(f_{v_2}) \neq 0.$$  

For this choice of $f$, the trace identity is reduced to

$$I_\Pi(f') = J_{\pi_{v_2}}(f_{v_2}) \left( \sum_{\pi_W} |c_{\pi_W}|^2 J_{\pi_{v_2}}(f^{(v_2)}) \right)$$

where the superscript means the away from $v_2$-part. In the sum, every term is non-negative as we choose $f^{(v_2)}$ of positive type. And at least one of these terms is non-zero. Therefore we conclude that for this choice the RHS above is non-zero. This shows that $I_\Pi(f') \neq 0$ and completes the proof.

\[\square\]

### 2.6 Proof of Theorem 1.4

A key ingredient is the following Burger–Sarnark type principle following a terminology of Prasad [41].

**Proposition 2.14.** Let $V$ be a hermitian space of dimension $n$ and $W$ a nondegenerate subspace of codimension one. Let $\pi$ be a cuspidal automorphic representation of $U(V)(\mathbb{A})$. Fix a finite (non-empty) set $S$ of places and an irreducible representation $\sigma_v$ of $U(W)(F_v)$ for each $v \in S$ such that

- If $v \in S$ is archimedean, both $W$ and $V$ are positive definite at $v$.
- If $v \in S$ is non-archimedean and split, $\sigma_v^0$ is induced from a representation of $Z_v K_v$ where $K_v$ is a compact open subgroup and $Z_v$ is the center of $U(W)(F_v)$.
- If $v \in S$ is either archimedean or non-split, the contragredient of $\sigma_v^0$ appears as a quotient of $\pi_v$ restricted to $U(W)(F_v)$.

Then there exists a cuspidal automorphic representation $\sigma$ of $U(W)(\mathbb{A})$ such that

- $\sigma_v = \sigma_v^0$ for all $v \in S$.
- the linear form $\ell_W$ on $\pi \otimes \sigma$ is non-zero.

We first show the following variant of [41, Lemma 1].
Lemma 2.15. Suppose that we are in the following situations

- $F$ is a number field.
- $G$ is a reductive algebraic group defined over $F$, and $H$ is a reductive subgroup of $G$.
- $S$ is a finite set of places of $F$ such that if $v \in S$ is archimedean, then $G(F_v)$ is compact. Denote $G_S = \prod_{v \in S} G(F_v)$ and $H_S = \prod_{v \in S} H(F_v)$.
- the center $Z$ of $H$ is assumed to be anisotropic over $F$.

Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A})$. Let $\otimes_{v \in S} \mu_v$ be an irreducible representation of $H_S$ such that

1. Assume that $\mu_v$ appears as a quotient of $\pi_v$ restricted to $H(F_v)$ for all $v \in S$.
2. For each non-archimedean $v \in S$, $\mu_v$ is supercuspidal representations of $H(F_v)$, and it is an induced representation $\mu_v = \text{Ind}_{Z_vK_v}^{H_v} \nu_v$ from a representation $\mu_v$ of a subgroup $Z_vK_v$ where $K_v$ are certain open compact subgroups of $H_v$.

Then there is an automorphic representation $\mu' = \prod_v \mu'_v$ of $H(\mathbb{A})$ and functions $f_1 \in \pi, f_2 \in \mu'$ such that

(i) $\int_{H(F) \backslash H(\mathbb{A})} f_1(h) \bar{f}_2(h) dh \neq 0$,

and

(ii) If $v \in S$ is archimedean $\mu'_v = \mu_v$. If $v \in S$ is non-archimedean, $\mu'_v = \text{Ind}_{Z_vK_v}^{H_v} \nu'_v$ is induced from $\nu'_v$ where $\nu'_v|_{K_v} = \nu_v|_{K_v}$.

Proof. The proof is a variant of [41, Lemma 1]. If $v \in S$ is archimedean, let $K_v = H(F_v)$ and $\nu_v = \mu_v$. It is compact by assumption. We consider the restriction of $\pi_v$ to $K_v$ for each $v \in S$. By the assumption and Frobenius reciprocity, $\nu_v|_{K_v}$ is a quotient representation of $\pi_v|_{K_v}$. Since $K_v$ is compact, $\nu_v|_{K_v}$ is also a sub-representation for $v \in S$. This means that we may find a function $f$ on $G(\mathbb{A})$ whose $K_S = \prod_{v \in S} K_v$ translates span a space which is isomorphic to $\otimes_{v \in S} \nu_v|_{K_v}$ as $K_S$-modules. By the same argument as in [41, Lemma 1] (using weak approximation), we may assume that such $f$ has non-zero restriction (denoted by $\tilde{f}$) to $H(F) \backslash H(\mathbb{A})$. Now note that $Z(F) \backslash Z(\mathbb{A})$ is compact. For a character $\chi$ of $Z(F) \backslash Z(\mathbb{A})$ we may define

$$\tilde{f} \mapsto \tilde{f}_\chi(h) := \int_{Z(F) \backslash Z(\mathbb{A})} f(zh)\chi^{-1}(z)dz, h \in H(F) \backslash H(\mathbb{A}).$$

As $Z_S$ and $K_S$ commute, each of $\tilde{f}$ and $\tilde{f}_\chi$ generates a space of functions on $H(F) \backslash H(\mathbb{A})$ which is isomorphic to $\otimes_{v \in S} \nu_v|_{K_v}$ as $K_S$-modules. There must exist some $\chi$ such that it is
non-zero. For such a \( \chi \), it is necessarily true that \( \chi_v|_{Z_v \cap K_v} = \omega_{\mu_v}|_{Z_v \cap K_v} \) where \( \omega_{\mu_v} \) is the central character of \( \nu_v \). In particular, we may replace \( \mu_v = \text{Ind}_{\nu_v}^{H_v} \nu_v \) by \( \mu'_v := \text{Ind}_{\nu_v}^{H_v} \nu'_v \) where \( \nu_v \) is an irreducible representation of \( Z_v K_v \) with central character \( \chi_v \) and \( \nu'_v|_{K_v} = \nu_v|_{K_v} \). Certainly such \( \mu'_v \) is still supercuspidal if \( v \in S \) is non-archimedean and if \( v \in S \) is archimedean, \( \mu'_v = \mu_v \).

Now we consider the space generated by \( \tilde{f}_\chi \) under \( Z_S K_S \) translations. This space is certainly isomorphic to \( \prod_{v \in S} \nu'_v \) as \( Z_S K_S \)-modules. The rest of the proof is the same as in [41, Lemma 1], namely applying [41, Lemma 2] to the space of \( H_S \)-translations of \( \tilde{f}_\chi \) which is isomorphic to \( \otimes_{v \in S} \text{Ind}_{\nu_v}^{H_v} \nu'_v \) as \( H_S \)-modules.

We now prove Prop. 2.14.

**Proof.** We apply the lemma above to the case \( H = U(W), G = U(V) \). Then the center is anisotropic. If \( v \in S \) is split non-archimedean, it is always true that \( \mu_v \) appears as a quotient of \( \pi_v \) (the local conjecture in [11] for the general linear group is known to hold for generic representations). The proposition then follows immediately.

**Remark 8.** Noting that any supercuspidal representation of \( GL_n(F_v) \) for a non-archimedean \( v \) is induced from an irreducible representation of an open subgroup which is compact modulo center. But an open subgroup of \( GL_n(F) \) compact modulo center is not necessarily of the form \( K_v Z_v \). As pointed out by Prasad to the author, it may be possible to choose an arbitrary supercuspidal \( \mu_v \) if one suitably extends the result in Lemma 2.15.

Now we come to the proof of Theorem 1.4.

**Proof.** We show this by induction on the dimension of \( W \). If \( n := \text{dim} W \) is equal to one, then the theorem is obvious. Now assume that for all dimension at most \( n \) hermitian spaces, the statement holds. Let \( V \) be a \( n+1 \)-dimensional hermitian space and \( W \) is a codimension one subspace. And let \( \pi \) be a cuspidal automorphic representation such that \( \pi_{v_1} \) is supercuspidal for a split place \( v_1 \). By [4], there exists a supercuspidal representation \( \sigma^0_{v_1} \) that verifies the assumptions of Prop. 2.14. Then we apply Prop. 2.14 to \( S = \{ v_1 \} \) to choose a cuspidal automorphic representation \( \sigma \) of \( U(W) \) such that \( \pi \otimes \sigma \) is distinguished by \( U(W) \) (for the diagonal embedding into \( U(V) \times U(W) \)). Then by Prop. 2.13, the weak base change of \( \pi \otimes \sigma \) is \( \eta \)-distinguished by \( H'_2 = GL_{n+1,F} \times GL_{n,F} \). By induction hypothesis, the weak base change of \( \sigma \) is \( \eta \)-distinguished by \( GL_{n,F} \) if \( n \) is odd (even, resp.). Together we conclude that the weak base change of \( \pi \) is \( \eta \)-distinguished by \( GL_{n+1,F} \) if \( n+1 \) is odd (even, resp.). This completes the proof.

**Remark 9.** In the most ideal situation, (namely, if we have the trace formula identity for all test functions \( f \), not just the nice ones), then we may use the proof of Prop. 2.13 to show first the existence of weak base change, then use the proof of Theorem 1.4 to show the distinction of the weak base change as predicted by the conjecture of Flicker–Rallis. But it seems impossible to characterize the image of the weak base change using the Jacquet–Rallis trace formulae.
2.7 Proof of Theorem 1.1: $(i) \implies (ii)$

Now we finish the proof of the other direction of Theorem 1.1: $(i) \implies (ii)$. It suffices to replace the condition (2) by that $\pi_{v_1}$ is supercuspidal at a split $v_1$, $\pi_{v_2}$ is tempered for $v_2 \neq v_1$ split.

By Theorem 1.4, the weak base change $\Pi = BC(\pi)$ is $\eta$-distinguished by $H'_2$. By the assumption on nonvanishing of $L(\Pi, R, 1/2)$, we know that $\Pi$ is also distinguished by $Res_{E/F}GL_n$. Therefore, $I_\Pi$ is non-zero distribution on $G'(\mathbb{A})$. Therefore some some decomposable $f' = \otimes_v f'_v$, $I_\Pi(f') \neq 0$. Note that the multiplicity one also holds in this case:

$$\dim Hom_{H'_i(F_v)}(\Pi_v, \mathbb{C}) \leq 1.$$ 

Similar to the decomposition of the distribution $J_\pi$, we may fix a decomposition

$$I_\Pi = c_\Pi \prod_v I_{\Pi_v}.$$ 

In particular, $c_\Pi \neq 0$ and for the $f'$ above, $I_{\Pi_v}(f'_v) \neq 0$ for all $v$. We want to modify $f'$ at the two places $v_1, v_2$ to apply Prop. 2.11.

It is easy to see that we may replace $f'_{v_1}$ by a function essentially a matrix coefficient of $\pi_{v_1}$. Now note that the distribution at the split place $v_2$

$$I_{\Pi_v}(f'_{v_2}) = J_{\pi_{v_2}}(f_{v_2})$$

if $\Pi_{v_2}$ is the local base change of $\pi_{v_2}$ and $f_{v_2}$ is the transfer of $f'_{v_2}$ as prescribed by Prop. 2.5. By Prop. 2.12, $J_{\pi_{v_2}}(f_{v_2}) \neq 0$ for some $f_{v_2}$ supported in $Z$-regular semisimple locus. Therefore, we may choose $f'_{v_2}$ supported in $Z$-regular semisimple locus such that $I_{\Pi_{v_2}}(f'_{v_2}) \neq 0$.

Now we replace $f'_{v_i}, i = 1, 2$ by the new choices. Then we let the tuple $(f_W)$ be a transfer of $f'$ satisfying the conditions in Prop. 2.11. By the trace formula identity of Prop. 2.11,

$$I_\Pi(f') = \sum W' J_{\pi_{W'}}(f_{W'})$$

where the sum in RHS runs over all $W'$, and all $\pi_{W'}$ nearly equivalent to $\pi$. There must be at least one term $J_{\pi_{W'}}(f_{W'}) \neq 0$ for some $W'$. This completes the proof.

Remark 10. The study of heights of some algebraic cycles on certain Shimura varieties constitutes a natural arithmetic version of the global Gan–Gross–Prasad conjecture ([11],[56]). The two themes—periods and heights—can be unified in an approach using relative trace formulae [57], at least for the unitary case in the current moment.

3 Reduction steps

In this and the next section, we will prove the existence of smooth transfer at a non-archimedean non-split place as well as a partial result at an archimedean non-split place.
In this section, we reduce the question to an analogue on “Lie algebras” (an infinitesimal version) and then to a local question around zero. Let $F$ be a local field of characteristic zero. In this section, both archimedean and non-archimedean local fields are allowed. Let $E = F[\sqrt{\tau}]$ be a quadratic extension where $\tau \in F^\times$. We remind the reader that, even though our interest is in the genuine quadratic extension $E/F$, we may actually allow $E$ to be split, namely, $\tau \in (F^\times)^2$.

### 3.1 Reduction to Lie algebra

We first reduce the smooth transfer to an infinitesimal version.

We consider the action of $H' := GL_{n,F}$ on the tangent space of the symmetric space $S_{n+1}$ at the identity matrix $1_{n+1}$:

\begin{equation}
S_{n+1} := \{ x \in M_{n+1}(E) \mid x + \bar{x} = 0 \}
\end{equation}

which will be called the “Lie” algebra of $S_{n+1}$. When no confusion arises, we will drop the subscript and write it $\mathfrak{S}$ for simplicity.

Let $W$ be a hermitian space of dimension $n$ and let $V = W \oplus Eu$ with $(u, u) = 1$. We consider the restriction to $H = H_W = U(W)$ of the adjoint action of $U(V)$ on $U(V)$ and its Lie algebra (as an $F$-vector space):

\begin{equation}
\mathfrak{U}(V) = \{ x \in End_E(V) \mid x + x^* = 0 \}
\end{equation}

where $x^*$ is the adjoint with respect to the hermitian form on $V$:

$$(xa, b) = (a, x^*b), \quad a, b \in V.$$ 

It will be more convenient to consider the action of $H'$ on the Lie algebra of $GL_{n+1}$:

$$\{ x \in M_n(E) \mid x = \bar{x} \} \simeq \mathfrak{gl}_{n+1}.$$

Relative to the $H$-action, we have notions of regular semisimple elements. Similarly for the $H'$-action. Analogous to the group case, regular semisimple elements have trivial stabilizers. We also define an analogous matching of orbits as follows. We may identify $End_E(V)$ with $M_{n+1}(E)$ by choosing a basis of $V$. Then for regular semisimple $x \in \mathfrak{S}(F)$ and $y \in \mathfrak{U}(V)(F)$, we say that they (and their orbits) match each other if $x$ and $y$ when considered as elements in $M_{n+1}(E)$ are conjugate by an element in $GL_n(E)$. We will also say that $x$ and $y$ are transfer of each other and denote the relation by $x \leftrightarrow y$.

Then, analogous to the case for groups, the notion of transfer defines a bijection between regular semisimple orbits

\begin{equation}
\mathfrak{S}(F)_{rs}/H'(F) \simeq \coprod_w \mathfrak{U}(V)(F)_{rs}/H(F)
\end{equation}
where the disjoint union runs over all isomorphism classes of \(n\)-dimensional hermitian space \(W\). We recall some results from [44], [57]. For the natural map \(\pi'_F : \mathcal{S}(F) \to (\mathcal{S}/\!/H')(F)\) and \(\pi_{W,F} : \mathfrak{U}(V)(F) \to (\mathfrak{U}(V)/\!/H)(F)\), the fiber of a regular semisimple element consists of precisely one orbit with trivial stabilizer. Moreover, \(\pi'_F\) is surjective. In particular, \(\pi'_F\) induces a bijection:

\[
\mathcal{S}(F)_{rs}/\!/H'(F) \simeq (\mathcal{S}/\!/H')(F)_{rs}.
\]

And \(\pi_{W,F}\) induces a bijection between \(\mathfrak{U}(V)(F)_{rs}/\!/H(F)\) and its image in \((\mathfrak{U}(V)/\!/H)(F)\).

A more intrinsic way is to establish an isomorphism between the categorical quotients between \(\mathcal{S}/\!/H'\) and \(\mathfrak{U}(V)/\!/H\). To state this more precisely, let us consider the invariants on them. We may choose a set of invariants on \(\mathcal{S}_{n+1}\)

\[
\text{tr} \wedge^i x, x \cdot x^j, e, \quad i = 1, 2, \ldots, n + 1, j = 1, 2, \ldots, n;
\]

and on \(\mathfrak{U}(V)\) for \(V = W \oplus Eu\):

\[
\text{tr} \wedge^i y, (y^j u, u), \quad i = 1, 2, \ldots, n + 1, j = 1, 2, \ldots, n.
\]

where \(x \in \mathcal{S}_{n+1}\) and \(y \in \mathfrak{U}(V)\). If we write \(\mathcal{S}_{n+1} \ni x = \begin{pmatrix} A & b \\ c & d \end{pmatrix}, A \in \mathcal{S}_n\), an equivalent set of invariants on \(\mathcal{S}_{n+1}\) are

\[
\text{tr} \wedge^i A, c \cdot A^j \cdot b, d \quad i = 1, 2, \ldots, n, j = 0, 1, 2, \ldots, n - 1.
\]

Similarly for the unitary case.

Denote by \(Q = \mathbb{A}^{2n+1}\) the \(2n+1\)-dimensional affine space (in this and the next section we are always in the local situation and \(\mathbb{A}\) will denote the affine line). Then the invariants above define a morphism

\[
\pi_{\mathcal{S}} : \mathcal{S}_{n+1} \to Q \\
x \mapsto (\text{tr} \wedge^i x, x \cdot x^j, e), \quad i = 1, 2, \ldots, n + 1, j = 1, 2, \ldots, n.
\]

To abuse notation, we will also denote by \(\pi_{\mathcal{S}}\) the morphism defined by the second set of invariants above. Similarly we have morphism denoted by \(\pi_{\mathfrak{U}}\) for the unitary case. We usually simply write \(\pi\) if there is no confusion.

**Lemma 3.1.** For each case \(V = \mathcal{S}\) or \(\mathfrak{U}(V)\), the pair \((Q, \pi_V)\) defines a categorical quotient of \(V\).

Equivalently, the set of invariants above is a set of generators of the ring of invariant polynomials for \(\mathcal{S}_{n+1}\) and \(\mathfrak{U}(V)\).

**Proof.** As this is a geometric statement, it suffices to treat the case \(V = \mathcal{S}\) or equivalently \(\mathfrak{g}l_{n+1}\). We will use Igusa’s criterion ([29, Lemma 4], or [42, Theorem 4.13]): Let a reductive group \(H\) act on an irreducible affine variety \(V\). Let \(Q\) be a normal irreducible affine variety, and \(\pi : V \to Q\) be a morphism which is constant on the orbits of \(H\) such that
(1) $Q - \pi(V)$ has codimension at least two;

(2) There exists a non-empty open subset $Q_0$ of $Q$ such that the fiber $\pi^{-1}(q)$ of $q \in Q_0$ contains exactly one closed orbit.

Then $(Q, \pi)$ is a categorical quotient for the $H$ action on $V$.

For $\mathfrak{gl}_{n+1}$, the morphism $\pi$ is clearly constant on the orbits of $H$. Now we define a section of $\pi$ close to the classical companion matrices. Consider

$$
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_n & b_{n-1} & \ldots & b_1 & 0 \\
a_n & a_{n-1} & \ldots & a_1 & d
\end{pmatrix}.
$$

Then its invariants are

$$
\text{tr } A \wedge^i A = (-1)^{i-1}b_i, c \cdot A^j \cdot b = a_{j+1}, d \quad i = 1, 2, \ldots, n, j = 0, 1, 2, \ldots, n - 1.
$$

This gives us an explicit choice of section of $\pi$ and it shows that $\pi$ is surjective. This verifies (1). By [44], for all regular semisimple $q \in Q$, the fiber of $q$ consists of at most one closed orbit. It follows by the explicit construction above that the fiber contains precisely one closed orbit. The regular semisimple elements form the complement of a principle divisor and hence we have verified condition (2). This completes the proof.

By this result, we have a natural isomorphism between the categorical quotient $\mathcal{G}//H'$ and $\mathfrak{U}(V)//H$. And this enables us to define the transfer of orbits. In the bijection (3.3) the appearance of disjoint union is due to the fact that the map between $\mathcal{F}$-points induced by $\pi_{\mathcal{G}}$ is surjective but the one by $\pi_{\mathfrak{U}(V)}$ is not.

By this lemma, more generally, we say that two semisimple elements $x \in \mathcal{G}(\mathcal{F})$ and $y \in \mathfrak{U}(V)(\mathcal{F})$ match each other if they have the same invariants, or equivalently their image in the quotients correspond to each other under the isomorphism between the categorical quotients. Given a semisimple $x \in \mathcal{G}(\mathcal{F})$ (not necessary regular), in general there may be more than one matching semisimple orbits in $\mathfrak{U}(V)(\mathcal{F})$.

We need also to define a transfer factor in the level of Lie algebra.

**Definition 3.2.** Consider the action of $H'$ on $X = \mathfrak{gl}_{n+1}$ or $\mathcal{G}$. A transfer factor is a smooth function $\omega : X(\mathcal{F})_{rs} \to \mathbb{C}^\times$ such that $\omega(x^h) = \eta(h)\omega(x)$.

Obviously, two transfer factors $\omega, \omega'$ differer by a $H'(\mathcal{F})$-invariant smooth function $\xi : X(\mathcal{F})_{rs} \to \mathbb{C}^\times$. If $\xi$ extends to a smooth function on $X(\mathcal{F})$ (and with moderate grows towards infinity for a norm on $X(\mathcal{F})$ if $\mathcal{F}$ is archimedean), we say that $\omega, \omega'$ are equivalent and denote by $\omega \sim \omega'$. 

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We have defined a transfer factor $\Omega$ earlier on the groups. We now define a transfer factor on the Lie algebra and we will only consider its equivalence class. If $\sqrt{\tau}x \in \mathfrak{S}(F)$ is regular semisimple, we define
\begin{align}
(3.4) \quad \omega(\sqrt{\tau}x) := \eta(\det(e, ex, ex^2, ..., ex^n))
\end{align}
where $e$ is the row vector $(0, 0, ..., 0, 1)$ and $(e, ex, ex^2, ..., ex^n)$ is the matrix consisting of the row vectors $eX^i$.

Now we may similarly define the notion of transfer of test functions on $\mathfrak{S}(F)$ and $\mathfrak{U}(V)(F)$.

For an $f' \in C^\infty_c(S_n(F))$, and a tuple $(f_W)_W$ where $f_W \in C^\infty_c(U(V)(F))$, they are called a (smooth) transfer of each other if for all matching regular semisimple $S_n(F) \ni x \leftrightarrow y \in \mathfrak{u}(V)(F)$, $V = W \oplus E_u$, we have
\[ \omega(x)O(x, f') = O(y, f_W). \]

For $n \in \mathbb{Z}_{\geq 1}$, we rewrite the “smooth transfer conjecture” of Jacquet-Rallis for the symmetric space $S$ and the unitary group $U(V)$:

**Conjecture $S_{n+1}$:** For any $f' \in C^\infty_c(S_{n+1})$, its transfer $(f_W)_W$ $(f_W \in C^\infty_c(U(V)))$ exists. And the other direction also holds, namely, given any a tuple $(f_W)_W$, there exists its transfer $f'$.

The corresponding statement for Lie algebra can be stated as

**Conjecture $\mathfrak{S}_{n+1}$:** For any $f' \in C^\infty_c(\mathfrak{S}_{n+1}(F))$, its transfer $(f_W)_W$ $(f_W \in C^\infty_c(\mathfrak{U}(V)(F)))$ exists. And the other direction also holds.

Note that the statement depends on the choice of a transfer factor. But it is obvious that the truth of the conjecture does not depend on the choice of the transfer factor within an equivalence class.

We now reduce the group version to the the Lie algebra version.

**Theorem 3.3.** Conjecture $\mathfrak{S}_{n+1}$ implies Conjecture $S_{n+1}$.

To prove this theorem, we need some preparation. Let us define a set for a scalar $\nu \in E$
\[ D_\nu = \{x | \det(\nu - x) \neq 0\}. \]
We will choose a basis of $V$ to realize the unitary group $U(V)$ as a subgroup of $GL_{n+1,E}$.

**Lemma 3.4.** Let $\xi \in E^1$. The map (“Cayley” transform)
\[ \alpha_\xi : M_{n+1}(E) - D_1 \rightarrow GL_{n+1}(E) \]
\[ x \mapsto -\xi(1 + x)(1 - x)^{-1}. \]
induces an $H$-equivariant isomorphism between $\mathfrak{S}_{n+1}(F) - D_1$ and $S_{n+1}(F) - D_\xi$. In particular, if we choose a sequence of distinct $\xi_1, \xi_2, ..., \xi_{n+2} \in E^1$, the images of $\mathfrak{S}_{n+1}(F) - D_1$ under $\alpha_{\xi_i}$ form a finite cover by open subset of $S_{n+1}(F)$.

Similarly, $\alpha_\xi$ induces an $U(W)$-equivariant isomorphism between $\mathfrak{U}(V)(F) - D_1$ and $U(V)(F) - D_\xi$. 28
Proof. First we verify that the image of $\alpha_\xi$ lies in $S = S_{n+1}$. We need to show
\[ \alpha(x)\overline{\alpha}(x) = (1 + x)(1 - x)^{-1}(1 + \bar{x})(1 - \bar{x})^{-1} = 1. \]
This is equivalent to
\[ (1 + x)(1 - x)^{-1}(1 + \bar{x}) = 1 - \bar{x} \]
\[ \iff (1 - x)(1 + x)(1 - x)^{-1}(1 + \bar{x}) = (1 - x)(1 - \bar{x}). \]
Note that $1 - x$ and $1 + x$ commutes:
\[ (1 + x)(1 + \bar{x}) = (1 - x)(1 - \bar{x}). \]
This reduces to (if $\text{char } F \neq 2$)
\[ x + \bar{x} = 0. \]
Note that if $\det(\xi - s) \neq 0$, $\alpha_\xi$ has an inverse defined by
\[ s \mapsto -(\xi + s)(\xi - s)^{-1}. \]
This shows that the image of $\alpha_\xi$ is $S - D_\xi$ and $\alpha_\xi$ defines an isomorphism between two affine varieties.

The same argument proves the assertion for unitary groups.

**Lemma 3.5.** The transfer factors are compatible under the Cayley map $\alpha_\mu$.

**Proof.** It suffices to consider the case $\mu = 1$ as the argument is the same for a general $\mu$. Note that $(1 + x)$ and $(1 - x)^{\pm 1}$ commute:
\[ \Omega((1 + x)(1 - x)^{-1}) = \eta'(\det((1 + 2(1 - x)^{-1})e_{i=0}^{n-1}). \]
Let $T = 2(1 - x)^{-1}$ Then it is easy to see that the determinant is equal to
\[ \det((1 + T)^i e_{i=0}^{n-1}) = \det(T^i e_{i=0}^{n-1}) \]
by elementary operations on a matrix. This is equal to
\[ 2^n(-1)^{(n-1)/2}\det((1 - x)^i e_{i=0}^{n-1}) = 2^n\det(x^i e_{i=0}^{n-1}). \]
Therefore we have proved that
\[ \Omega((1 + x)(1 - x)^{-1}) = \eta'(2^n\det(x^i e_{i=0}^{n-1}) = c \cdot \omega(x/\sqrt{7}). \]
for a non-zero constant $c$. 

\[ \Box \]
Now for more flexibility, we consider the statement following $P_\mu$ for $\mu \in F^\times$:

$P_\mu$: For $f \in C^\infty_c(\mathcal{G}_{n+1} - D_\mu)$, its transfer $(f_W)$ exists and can be chosen such that $f_J \in C^\infty_c(\mathcal{U}(V) - D_\mu)$. And the other direction also holds.

Then it is clear that $P_\nu$ holds for all $\mu \in F^\times$ implies Conjecture $S_{n+1}$ by partition of unity applying to the open cover of $S$ and $U(V)$ for distinct $\mu_0, \ldots, \mu_{n+1}$.

To prove Theorem 3.3, it remains to show the following:

**Lemma 3.6.** Conjecture $S_{n+1}$ implies $P_\mu$ for all $\mu \in F^\times$.

**Proof.** Fix such a $\mu$. Assume that $(f_W)$ is a transfer of $f \in C^\infty_c(\mathcal{G} - D_\mu)$. Let $Y = \text{supp}(f) \subseteq \mathcal{G}(F)$ and $Z = \pi'(Y)$. It suffices to show that for each $W$ there exists a function $\alpha_W \in C^\infty(\mathcal{U}(V)(F))$ (this means smooth when $F$ is archimedean; locally constant when $F$ is non-archimedean) satisfying

1. $\alpha_W$ is $H(F)$-invariant,
2. $\alpha_W|_{\pi^{-1}(Z)} \equiv 1$,
3. $\alpha_W|_{D_\mu} \equiv 0$.

Then we may replace $f_W$ by $f_W\alpha_W$ which will still be in $C^\infty(\mathcal{U}(V)(F))$ and has the same orbital integral as $f_W$.

Now note that $Z = \pi'(Y) \subseteq (\mathcal{G}/H')(F)$ is compact. And $D_\mu$ is the preimage under $\tilde{\pi}$ of a hypersurface denoted by $C$ in $(\mathcal{G}/H')(F)$. Then we may find a $C^\infty$ function $\beta$ on $(\mathcal{G}/H')(F)$ satisfying

1. $\beta|_Z \equiv 1$.
2. $\beta|_C \equiv 0$.

When $F$ is archimedean, one may construct such a $\beta$ using bump function. When $F$ is non-archimedean function, we may cover $Z$ by an open compact subsets $\tilde{Z}$ which is disjoint from $C$. Then we take $\beta$ to be the characteristic function of $\tilde{Z}$.

Then we may take $\alpha_W$ to be the pull-back of $\beta$ under $\pi_{W,F}$.

The other direction can be proved similarly. □

### 3.2 Reduction to local transfer around zero

From now on we will denote by $Q_n = \mathbb{A}^{2n+1}$ or simply $Q$ the common base $\mathcal{G}_{n+1}/H' \simeq \mathcal{U}(V)/H$ as an affine variety. The aim of this section is to reduce the existence of transfer to the existence of a local transfer near zero.
Localization. We fix a transfer factor \( \omega \) and let \( \pi' : \mathbb{G}(F) \rightarrow Q(F) \) and \( \pi : \mathcal{U}(V) \rightarrow Q \) be the induced maps on the rational points.

**Definition 3.7.** Let \( \Phi \) be a function on \( Q(F)_{rs} \) which vanishes outside a compact set of \( Q(F) \).

1. For \( x \in Q(F) \), we say that \( \Phi \) is a local orbital integral for \( \mathcal{G} \) around \( x \in Q(F) \) if there exists a neighborhood \( U \) of \( x \) and a function \( f \in C_c^\infty(\mathcal{G}(F)) \) such that for all \( y \in U_{rs} \), and \( z \) with \( \pi'(z) = y \) we have
   \[
   \Phi(y) = \omega(z)O(z, f).
   \]

2. Similarly we can define a local orbital integral for \( \mathcal{U}(V) \).

Note that if \( \Phi \) is a local orbital integral for a transfer factor \( \omega' \sim \omega \), it is also a local orbital integral for any other equivalent transfer factor \( \omega' \sim \omega \).

Then we have the following localization principle for orbital integrals.

**Proposition 3.8.** Let \( \Phi \) be a function on \( Q(F)_{rs} \) which vanishes outside a compact set \( \xi \) of \( Q(F) \). If \( \Phi \) is a local orbital integral for \( \mathcal{G} \) at \( x \) for all \( x \in Q(F) \), then it is an orbital integral, namely there exists \( f \in C_c^\infty(\mathcal{G}(F)) \) such that for all \( Y \in Q(F)_{rs} \), and \( z \) with \( \pi'(z) = y \) we have
   \[
   \Phi(y) = \omega(z)O(z, f).
   \]

Similar result holds for \( \mathcal{U}(V) \).

**Proof.** By assumption, for each \( x \in \xi \) we have an open neighborhood \( U_x \) and \( f_x \in C_c^\infty(\mathcal{G}(F)) \).
By the compactness we may find finitely many of them, say \( x_1, \ldots, x_m \), such that \( U_{x_i} \) cover \( \xi \). Then we apply “partition of unity” to the cover of \( Q(F) \) by \( U_{x_i} \) \( (i = 1, 2, \ldots, m) \) and \( Q(F) - \xi \) to obtain smooth functions \( \beta_i, \beta \) on \( Q(F) \) such that \( \text{supp}(\beta_i) \subseteq U_{x_i} \) and \( \text{supp}(\beta) \subseteq Q(F) - \xi \) and \( \beta + \sum_i \beta_i \) is the identity function on \( Q(F) \). Since \( \Phi \beta \equiv 0 \), we may write \( \Phi = \sum_{i=1}^m \Phi_i \) where \( \Phi_i = \Phi \beta_i \) is a function on \( Q(F)_{rs} \) which vanishes outside \( U_{x_i} \). Then \( \alpha_i = \beta_i \circ \pi' \) is a smooth \( H'(F) \)-invariant function on \( \mathcal{G}(F) \) and \( f_{x_i} \alpha_i \in C_c^\infty(\mathcal{G}(F)) \). We claim that for every \( y \in Q(F)_{rs} \) and \( z \in \pi'^{-1}(y) \), we have \( \omega(z)O(z, f_{x_i} \alpha_i) = \Phi_i(y) \). In deed, the left hand side is equal to \( \omega(z)O(z, f_{x_i}) \beta(y) \). If \( Y \) is outside \( U_{x_i} \), then both sides vanish. If \( y \in U_{x_i} \), then by the choice of \( f_{x_i} \), we have \( \omega(z)O(z, f_{x_i}) = \Phi(y) \). By the claim, we may take \( f = \sum_i f_{x_i} \alpha_i \) to complete the proof.

For \( f \in C_c^\infty(\mathcal{G}(F)) \), we define a “direct image” \( \pi_{*, \omega}(f) \) as the function on \( Q(F)_{rs} \):
   \[
   \pi_{*, \omega}(f)(x) := \omega(y)O(y, f)
   \]
where \( x \in Q(F)_{rs}, y \in (\pi')^{-1}(x) \). It clearly does not depend on the choice of \( y \). Similarly, for \( f_W \in C_c^\infty(\mathcal{U}(V)) \), we define a function \( \pi_{W,*}(f_W) \) on \( Q(F)_{rs} \) (extend by zero to those \( x \in Q(F)_{rs} \) such that \( \pi_W^{-1}(x) \) is empty). We will also write it as \( \pi_{*, \omega}(f_W) \) with the trivial transfer factor \( \omega = 1 \).
Definition 3.9. For $x \in Q(F)$, we say that the local transfer around $x$ exists, if for all $f \in C_c^\infty(G(F))$, there exist $(f_W)_W$ ($f_W \in C_c^\infty(\mathcal{U}(V))$) such that in a neighborhood of $x$, the following equality holds

$$\pi'_{*,\omega}(f) = \sum_W \pi_*(f_W),$$

and conversely for any tuple $(f_W)_W$, we may find $f$ satisfying the equality.

Descent of orbital integrals We recall some results of [2]. Let $V$ be a representation of a reductive group $H$. Let $\pi : V \to V//H$ be the categorical quotient. An open subset $U \subset V(F)$ is called saturated if it is the preimage of an open subset of $(V//H)(F)$.

Let $x \in V(F)$ be a semisimple element. Let $N^V_{H,x}$ be the normal space of $Hx$ at $x$. Then the stabilizer $H_x$ acts naturally on the vector space $N^V_{H,x}$. We call $(H_x, N^V_{H,x})$ the sliced representation at $x$.

An étale Luna slice (for short, a Luna slice) at $x$ is by definition ([2]) a locally closed smooth $H_x$-invariant subvariety $Z \ni x$ together a strongly étale morphism $\iota : Z \to N^V_{H,x}$ such that the $H$-morphism $\phi : H \times_{H_x} Z \to V$ is strongly étale. Here, an $H$-morphism between two affine varieties $\phi : X \to Y$ is called strongly étale if $\phi//H : X//H \to Y//H$ is étale and the induced diagram is Cartesian:

$$
\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
\downarrow & & \downarrow \\
X//H & \xrightarrow{\phi//H} & Y//H.
\end{array}
$$

When there is no confusion, we will simply say that $Z$ is an étale Luna slice.

We then have the Luna’s étale slice theorem: Let a reductive group $H$ acts on a smooth affine varieties $X$ and let $x \in X$ be semisimple. Then there exists a Luna slice at $x$. We will describe an explicit Luna slice in the next subsection for our case. We may indeed assume the morphism $\iota$ is essentially an open immersion in our case.

As an application, we have an analogue of Harish-Chandra’s compactness lemma ([25, Lemma 25]).

Lemma 3.10. Let $x \in V(F)$ be semisimple. Let $Z$ be an étale Luna slice at $x$. Then for any $H_x(F)$-invariant neighborhood $\xi$ of $x$ in $Z(F)$ whose image in $(Z//H_x)(F)$ is (relative) compact, and any compact subset $\Xi$ of $V(F)$, the set

$$\{h \in H_x(F)\setminus H(F) : \xi^h \cap \Xi \neq \emptyset\}$$

is relatively compact in $H_x(F)\setminus H(F)$.

Proof. We consider the étale Luna slice:

$$\phi : H \times_{H_x} Z \to V.$$
Consider the composition:
\[ H \times_{H_x} Z \simeq \mathcal{V} \times_{\mathcal{V}/H} Z/\!\!/H_x \hookrightarrow \mathcal{V} \times Z/\!\!/H_x. \]
The composition is a closed immersion. Shrinking \( Z \) if necessary, we may take the \( F \)-points to get a closed embedding
\[ i : (H \times_{H_x} Z)(F) \hookrightarrow \mathcal{V}(F) \times (Z/\!\!/H_x)(F). \]
We also have the projection
\[ H \times_{H_x} Z = (H \times Z)/\!\!/H_x \rightarrow H_x \setminus H. \]
We denote
\[ j : (H \times_{H_x} Z)(F) \rightarrow (H_x \setminus H)(F). \]
Note that \( H_x(F) \setminus H(F) \) sits inside \( (H_x \setminus H)(F) \) as an open and closed subset. Let \( \xi' \) be the image of \( \xi \) in \( (Z/H_x)(F) \). Then we see that the set
\[ \{ h \in H_x(F) \setminus H(F) : \xi^h \cap \Xi \neq \emptyset \} \]
is contained in
\[ ji^{-1}(\Xi \cap \xi') \]
which is obviously compact.

We also need the analytic Luna slice theorem ([2, Theorem 2.7]): there exists

1. an open \( H(F) \)-invariant neighborhood \( U \) of \( H(F)x \) in \( \mathcal{V}(F) \) with an \( H \)-equivariant retract \( p : U \rightarrow H(F)x \);
2. a \( H_x \)-equivariant embedding \( \psi : p^{-1}(x) \hookrightarrow N_{H_x,x}^Y(F) \) with an open saturated image such that \( \psi(x) = 0 \).

Denote \( S = p^{-1}(x) \) and \( N = N_{H_x,x}^Y(F) \). The quintuple \( (U, p, \psi, S, N) \) is then called an analytic Luna slice at \( x \).

\[ \begin{array}{ccc}
N_{H_x,x}^Y(F) & \xrightarrow{\psi} & p^{-1}(x) \\
\uparrow & & \downarrow \quad p \\
0 & \xrightarrow{x} & H(F)x
\end{array} \]

From an étale slice we may construct an analytic Luna slice from (cf. the proof of [2, Coro. A.2.4]). In our case, the existence of analytic Luna slice is self-evident once we describe the étale Luna slices in next subsection.

We recall some useful properties of analytic Luna slice ([2, Coro. 2.3.19]). If \( y \in p^{-1}(x) \), and \( z := \psi(y) \). Then we have
(1) \((H(F)_x)_z = H(F)_y\).

(2) \(N^\mathcal{V}_{H(F)_y,y} = N^\mathcal{N}_{H_x(F)_z,z}\) as \(H(F)_y\)-spaces.

(3) \(y\) is \(H\)-semisimple if and only if \(z\) is \(H_x\)-semisimple.

As an application, we state the Harish-Chandra (semisimple-) descent for orbital integrals.

**Proposition 3.11.** Let \(x \in \mathcal{V}(F)\) be semisimple and let \((U, p, \psi, S, N)\) be an analytic Luna slice at \(x\). Then there exists an neighborhood \(\xi \subset \psi(S)\) of \(0\) in \(N^\mathcal{V}_{H_x,x}(F)\) with the following properties

- For every \(f \in C_\infty^c(\mathcal{V}(F))\), we may associated to \(f_x \in C_\infty^c(N^\mathcal{V}_{H_x,x}(F))\) such that for all semisimple \(z \in \xi\) (with \(z = \psi(y)\)) such that \(\eta|_{H_y(F)} = 1\), we have

\[
\int_{H_y(F) \setminus H(F)} f(y^h)\eta(h)dh = \int_{H_y(F) \setminus H_x(F)} f_x(z^h)\eta(h)dh.
\]

- And conversely, given \(f_x \in C_\infty^c(N^\mathcal{V}_{H_x,x}(F))\), we may find \(f \in C_\infty^c(\mathcal{V}(F))\) such that 3.5 holds for all semisimple \(z \in \xi\) such that \(\eta|_{H_y(F)} = 1\).

**Proof.** Let \(\xi'\) be a relative compact neighborhood of \(x\) in \(S\) and let \(\xi = \psi(\xi')\). By Harish-Chandra’s compactness lemma 3.10, we may find a compact set \(C\) of \(H_x(F) \setminus H(F)\) that contains the set

\[\{h \in H_x(F) \setminus H(F) : \xi'^h \cap \text{supp}(f) \neq \emptyset\}\].

In the non-archimedean case, we may assume that \(C\) is compact open. Choose any function \(\alpha \in C_\infty^c(\mathcal{V}(F))\) such that the function

\[H_x(F) \setminus H(F) \ni h \mapsto \int_{H_x(F)} \alpha(gh)dg\]

is the characteristic function \(1_C\) in the non-archimedean case, and a bump function that takes value one on \(C\) and zero outside some larger compact subset \(C_1 \supset C\). We define a function on \(S\) by:

\[f_x(y) := \int_{H_y(F)} f(y^h)\alpha(h)\eta(h)dh\].

In the non-archimedean case, we may assume that \(S\) is a closed subset of \(\mathcal{V}\) and in the archimedean case, we may assume that \(S\) contains a closed neighborhood of \(x\) in \(\mathcal{V}\) whose image in \(N^\mathcal{V}_{H_x,x}(F)\) is the pre image of a closed neighborhood in the categorical quotient. Then possibly using a bump function in the archimedean case to modify \(f_x\), we may assume that \(f_x \in C_\infty^c(S)\). The map \(f \mapsto f_x\) depends on \(\xi\). We may also view \(f_x \in C_\infty^c(N^\mathcal{V}_{H_x,x}(F))\) via the embedding \(\psi : S \hookrightarrow N^\mathcal{V}_{H_x,x}(F)\).
Now the RHS of 3.5 is equal to
\[ \int_{H_y(F) \backslash H_x(F)} \int_{H(F)} f(y^g) \alpha(g) \eta(g) d\eta(h) dh \]
\[ = \int_{H_y(F) \backslash H_x(F)} \int_{H(F)} f(y^g) \alpha(h^{-1}g) \eta(g) d\eta(g) dh \]
\[ = \int_{H_y(F) \backslash H_x(F)} \int_{H_y(F) \backslash H(F)} \int_{H(F)} f(y^g) \alpha(h^{-1}pg) d\eta(g) d\eta(g) dh. \]
Interchange the order of the first two integrals and notice that when \( g \in C \)
\[ \int_{H_y(F) \backslash H_x(F)} \int_{H_y(F)} \alpha(h^{-1}pg) d\eta(g) dh = \int_{H_x(F)} \alpha(h^{-1}g) dh = 1. \]
By the Harish-Chandra compactness lemma 3.10, the value of the above integral outside \( C \)
does not matter when \( y \in \xi' \). We thus obtain
\[ \int_{H_y(F) \backslash H(F)} f(y^g) 1_{C}(g) \eta(g) dg. \]
This is equal to LHS when \( y \in \xi' \) (or equivalently, \( \psi(y) = z \in \xi \)).
To show the converse, we note that \( \psi(S) \) is saturated in \( N_{H_x,x}^y(F) \). Replacing \( f_x \) by \( f_x \cdot 1_S \)
in the non-archimedean case, and by \( f_x \cdot \alpha_S \) for some bump function \( \alpha \) in the archimedean
case, we may assume that \( \text{supp}(f_x) \subset \psi(S) \). Then we choose a function \( \beta \in \mathcal{C}_c^\infty(H(F)) \) such that
\[ (3.6) \quad \int_{H(F)} \beta(h) \eta(h) dh = 1. \]
Consider the surjective map
\[ H(F) \times S \to U \]
under which \( H(F) \times S \) is a \( H_x(F) \)-principal homogenous space over \( U \) (in the category of \( F \)-manifolds). It is obviously an submersion. We define \( f \in \mathcal{C}_c^\infty(U) \) by integrating \( \beta \otimes f_x \) over the fiber
\[ f(y^h) := \int_{H_x(F)} f_x(\psi(y^g)) \beta(g^{-1}h) dg, \quad y \in S, h \in H(F). \]
Then \( f \in \mathcal{C}_c^\infty(U) \) can be also viewed as in \( \mathcal{C}_c^\infty(V(F)) \). The LHS of 3.5 is then equal to
\[ \int_{H_y(F) \backslash H(F)} \int_{H_x(F)} f_x(\psi(y^g)) \beta(g^{-1}h) d\eta(h) dh \]
\[ = \int_{H_y(F) \backslash H(F)} \int_{H_y(F) \backslash H_x(F)} \int_{H_x(F)} f_x(\psi(y^g)) \beta(g^{-1}p^{-1}h) d\eta(h) dh \]
\[ = \int_{H_y(F) \backslash H_x(F)} \left( \int_{H(F)} \beta(g^{-1}h) \eta(h) dh \right) f_x(\psi(y^g)) dg. \]
By (3.6), this is equal to
\[
\int_{H_y(F) \setminus H_x(F)} f_x(\psi(y^g))\eta(g)dg.
\]

This completes the proof.

**Explicit étale Luna slices** Back to our case, we will first describe the sliced representations at a semisimple element of $\mathfrak{g}$ or $\mathfrak{U}(V)$. Then we exhibit an étale Luna slice at $x$. The key construction is already in [44]. But we need to show that their construction actually gives étale Luna slices. The steps are close to the Harish-Chandra’s descent method (cf. [36]).

We start with $\mathfrak{U}(V)$. It suffices to consider $\mathfrak{U}(W) \times W$. We may write an element in $\mathfrak{U}(W) \times W$ as $(X, w)$, $X \in \mathfrak{U}(W)$, $w \in W$. Denote by $W_2$ the subspace of $W$ generated by $X'w$, $i \geq 0$. By [44, Theorem 17.2], for $(X, w)$ to be semisimple, it is necessary that $W_2$ is a non-degenerate subspace. In this case, we have an orthogonal decomposition $W = W_1 \oplus W_2$. Then $X$ stabilizes both subspace. So we may write $X = diag[X_1, X_2]$ for $X_i \in \mathfrak{U}(W_i)$. Then $(X, w)$ is semisimple if and only if $X_2$ is semisimple (in the usual sense) in $\mathfrak{U}(W_2)$. And it is also self-evident that $(X_2, w)$ defines a regular semisimple element relative to the action of $\mathfrak{U}(W_2)$ on $\mathfrak{U}(W_2) \times W_2$. Then the stabilizer of $(X, w)$ is isomorphic to $\mathfrak{U}(W_1)_{X_1}$, the stabilizer of $X_1$ under the action of $\mathfrak{U}(W_1)$. It is a product of the restriction of scaler of unitary group of lower dimension over an extension of $F$ (including the general linear group). Let $\mathfrak{U}(W_1)_{X_1}$ be the respective Lie algebra. Then $\mathfrak{U}(W_1)_{X_1}$ acts on $\mathfrak{U}(W_1)_{X_1} \times W_1$. This representation of $\mathfrak{U}(W_1)_{X_1}$ is a product of representations of the same type (including the general linear case). The sliced representation at $x$ is then isomorphic to the product of the above representation of $\mathfrak{U}(W_1)_{X_1}$ on $\mathfrak{U}(W_1)_{X_1} \times W_1$ and the representation of the trivial group on the normal space at $(X_2, w)$ of $\mathfrak{U}(W_2)$-orbit of $(X_2, w)$ in $\mathfrak{U}(W_2) \times W_2$.

A similar description for $\mathfrak{G}_n$ also holds (cf. [44], [3]). And we have an equivalent version for the restriction of the adjoint action of $GL_{n+1,F}$ to $GL_{n,F}$. We describe the general form a semisimple element in the Lie algebra $\mathfrak{g}l_{n+1,F}$. Let

\[
x = \begin{pmatrix} X & u \\ v & d \end{pmatrix}, \quad X \in \mathfrak{g}l_{n,F},
\]

where $u \in F^n, v \in F_{\cdot}$ and we use $F^n$ ($F_\cdot$, resp.) to denote the $n$-dimensional space of column (row, resp.) vectors. There is an obvious pairing between $F^n$ and $F_\cdot$. Let $U_2$ be the subspace of $F^n$ spanned by $u, Xu, \ldots, X^n u$ and similarly $V_2$ the subspace of $F_\cdot$ spanned by $v, vX, \ldots, vX^n$. And let $V_1 := U_2^\perp \subset F_\cdot$ ($U_1 := V_2^\perp$, resp.) be the orthogonal complement of $U_2$ ($V_2$, resp.). Then for $x$ to be semisimple, it is necessary that $F_{\cdot}$ ($F^n$, resp.) is the direct sum of $U_2^\perp$ and $V_2$ ($V_2^\perp$ and $U_2$, resp.). Assuming this, according to the decomposition $F^n = U_1 \oplus U_2$, we may write $X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}$. Then by [44, sec. 8], for such $x$ to be semisimple, it is necessary and sufficient that $X_{11} \in End(U_1)$ is semisimple in the usual sense and $X_{12} = 0$. 

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Now we construct étale Luna slices for semisimple elements. As we shall see as follows, a general case are basically a composition of two extreme cases: 1) the case \( w = 0 \) (namely, \( r = 0 \) minimal), 2) the case \( x = (X, w) \) is regular semisimple (namely, \( r = 0 \) maximal). The first case is essentially the same as the classical case (the Luna slice for the adjoint representation, cf. [36, sec. 14]). For the second case we have to resort to the existence theorem of Luna. The general case can be reduced to those two basic cases.

We first describe a locally closed subvariety of \( x = (X, w) \) following [44, 18]. The case for \( \mathfrak{S} \) is similar following [44, sec. 7]. Let \( (X, w) \) be as above. Then we have an orthogonal decomposition \( W_1 \oplus W_2 \). Denote \( r = \text{dim} W_2 \geq 1 \). We define a closed subvariety \( \Xi \) of \( \mathfrak{U}(W) \times W \) consisting of \( (Y, u) \) such that \( u, Y u, ..., Y^{r-1} u \) spans \( W_2 \). In particular, \( x = (X, w) \in \Xi \). Now in [44] Rallis and Schiffmann define an isomorphism of varieties

\[
i_1 : \Xi \to \left( \mathfrak{U}(W_1) \times W_1 \right) / \left( \mathfrak{U}(W_2) \times W_2 \right)_{rs}
\]

whose inverse is defined as follows. According to the decomposition \( W_1 \oplus W_2 \) we may write \( Y \) as

\[
\begin{pmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{pmatrix}
\]

Then \( Y_{12} \in \text{Hom}(W_2, W_1) \). Define \( u' = Y_{12} Y^{r-1} u \in W_1 \). Then \( \iota_1^{-1} \) maps \( (Y, u) \) to \( ((Y_{11}, u'), (Y_{22}, u)) \). It is then easy to see that \( Y^{r} u = Y_{22} u \in W_2 \) for \( i = 0, 1, ..., r - 1 \). Therefore \( (X_{22}, u) \) is a regular semisimple element. It is not hard to see that this defines an isomorphism. Moreover it is equivariant under the action of \( U(W_1) \times U(W_2) \): on the LHS the action is the restriction of that of \( U(W) \) to the subgroup; on the RHS. Then the morphism \( \iota \) induces a morphism between the categorical quotients

\[
i_1^* : \left( \mathfrak{U}(W_1) \times W_1 \right) / U(W_1) \times \left( \mathfrak{U}(W_2) \times W_2 \right)_{rs} / / U(W_2) \to \left( \mathfrak{U}(W) \times W \right) / / U(W).
\]

Similarly, we have morphisms still denoted by \( \iota_1 \) and \( \iota_1^* \) in the general linear case \( \mathfrak{S} \). And we have an equivalent version for the restriction of the adjoint action of \( GL_{n+1,F} \) to \( GL_n,F \). We describe the analogous construction for this. Let \( x \in \mathfrak{g}_F \) be as above and denote by \( r = \text{dim} U_2 = \text{dim} V_2 \). Without loss of generality, we may assume \( U_2 = F^r \) embedded into \( F^n \) by sending \( u \) to \( (0, ..., 0, u) \). Similarly for \( V_2, U_1, V_1 \). Then we define a closed subvariety \( \Xi \) consisting of \( y = \begin{pmatrix} Y & u' \\ v' & d' \end{pmatrix} \) such that \( u', Y u', ..., Y^{n} u' \) span \( U_2 \) and \( v', v' Y, ..., v' Y^n \) span \( V_2 \). Then similarly we have an isomorphism ([44, sec. 7]):

\[
i_1 : \left( \mathfrak{g}_n \times F^{n-r} \times F_{n-r} \right) \times \left( \mathfrak{g}_r \times F^r \times F_r \right)_{rs} \times F \to \Xi
\]

such that \( \iota^{-1} \) maps \( y \) to \( ((Y_{11}, Y_{12} Y_{22}^{-1} u), (v Y_{22}^{-1} Y_{21}), (Y_{22}, u, v), d) \). And it also induces a morphism

\[
i_1^* : \left( \mathfrak{g}_n \times F^{n-r} \times F_{n-r} \right) / / GL_{n,F} \times \left( \mathfrak{g}_r \times F^r \times F_r \right)_{rs} / / GL_r \times F \to \mathfrak{g}_{n+1} / / GL_n.
\]

**Lemma 3.12.** \( \iota_1^* \) is étale for both \( \mathfrak{g}_{n+1} \) (equivalently, \( \mathfrak{S} \)) and \( \mathfrak{U}(W) \times W \).
Proof. We will the coordinates described earlier for these categorical quotients involved in the morphism \( \iota_1^\# \). By Jacobian criterion for \( \acute{\text{e}} \)taleness, it suffices to show that the Jacobian of \( \iota_1^\# \) is non-zero everywhere. Indeed we will show the Jacobian is a non-zero constant. Therefore it suffices to compute the Jacobian over the algebraic closure. In particular, it suffices to consider the equivalent question for the restriction of the adjoint representation of \( GL_{n+1,F} \) to \( GL_{n,F} \). Note that the Jacobian is a regular function on the source of the morphism \( \iota_1^\# \). It is then enough to show that it is a non-zero constant on a Zariski open subset. The categorical quotients \( gl_{n+1,F}/GL_{n,F} \) is given by

\[
\text{Spec}(F[\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \beta_2, \ldots, \beta_n]),
\]

where the regular invariant functions are defined

\[
\alpha_i = \text{tr} \wedge^i x, \beta_j = eX^j e^*, \quad x \in \mathfrak{gl}_{n+1,F},
\]

and \( e = e_n = (0, \ldots, 0, 1) \), \( e^* \) is the transpose of \( e \). Another choice of invariants is as follows. When we write

\[
x = \begin{pmatrix} X & u \\ v & d \end{pmatrix}, \quad X \in \mathfrak{gl}_{n,F}.
\]

then we may also identify the categorical quotient \( gl_{n+1,F}/GL_{n,F} \) as

\[
\text{Spec}(F[\alpha'_1, \ldots, \alpha'_n, d, \beta'_1, \ldots, \beta'_n]),
\]

where

\[
\alpha'_i = \text{tr} \wedge^i X, \beta'_j = vX^{j-1}u.
\]

In particular, the Jacobian of the isomorphism

\[
\psi = \psi_n : \text{Spec}(F[\alpha'_1, \ldots, \alpha'_n, d, \beta'_1, \ldots, \beta'_n]) \to \text{Spec}(F[\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \beta_2, \ldots, \beta_n])
\]

is a nonzero constant

\[
(3.7) \quad \frac{\partial(\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \beta_2, \ldots, \beta_n)}{\partial(\alpha'_1, \ldots, \alpha'_n, d, \beta'_1, \ldots, \beta'_n)} = \kappa_n \in F^\times.
\]

To indicate the dependence on \( n \) we may write the invariants as \( \alpha_i(n) \) etc..

We choose an auxiliary open subvariety consisting of strongly regular elements in the sense of Jacquet–Rallis [35]. More precisely they are in the \( GL_{n,F} \)-orbits to elements of the form

\[
x = x(a_1, \ldots, a_{n+1}, c_1, \ldots, c_n) := \begin{pmatrix} a_1 & 1 \\ c_1 & a_2 \\ \vdots & \vdots \\ c_{n-1} & a_n \\ c_n & a_{n+1} \end{pmatrix}, \quad a_i \in F, c_j \in F^\times.
\]
All such \( x \)'s with \( a_{n+1} = 0 \) form a locally closed subvariety denoted by \( \Theta_n \) of \( \mathfrak{gl}_{n+1,F} \). Note that \( \dim \Theta_n = 2n \). For \( d \in F \), let \( \delta_n(d) \) be the matrix with only non-zero entry \( d \) at the position \((n+1,n+1)\). For \((x,d) \in \Theta_n \times F\), taking invariants \( \alpha_i, \beta_j \) of \( x + \delta_n(d) \) yields a morphism with Zariski dense image:

\[
\xi = \xi_n : \Theta_n \times F \to \text{Spec}(F[\alpha_1, \ldots, \alpha_{n+1}, \beta_1, \beta_2, \ldots, \beta_n]).
\]

Then we claim that the Jacobian of \( \xi_n \) is given by

\[
(3.8) \quad \frac{\partial(\alpha_1, \ldots, \alpha_n, d, \beta_1, \ldots, \beta_n)}{\partial(a_1, \ldots, a_n + 1, c_1, \ldots, c_n)} = (-1)^{n(n-1)/2} \kappa_1 \kappa_2 \cdots \kappa_n c_2^2 \cdots c_n^{n-1}.
\]

We prove this by induction on \( n \). It is easy to verify this for \( n = 1 \). Now for \( n > 1 \), we may write \( \xi_n = \xi'_n \circ \psi_n \). As \( d = a_{n+1} \), we have

\[
\frac{\partial(\alpha'_1, \ldots, \alpha'_n, d, \beta'_1, \ldots, \beta'_n)}{\partial(a_1, \ldots, a_n + 1, c_1, \ldots, c_n)} = \frac{\partial(\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_n)}{\partial(a_1, \ldots, a_n, c_1, \ldots, c_n)}.
\]

Note that \( \alpha'_i(n) = \alpha_i(n - 1) \) and for \( x \in \Theta_n, \beta'_j(n)(x) = c_n \) and

\[
\beta'_j(n)(x) = c_n \beta_{j-1}(n-1)(X), \quad j \geq 2,
\]

since \((0, \ldots, 0, c_n) = c_n e_{n-1}\). Here \( X \) is the \( n \times n \) matrix by deleting the last row and the last column. This gives us

\[
\frac{\partial(\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_n)}{\partial(a_1, \ldots, a_n, c_1, \ldots, c_n)} = \frac{\partial(\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_{n-1})}{\partial(a_1, \ldots, a_n, c_1, \ldots, c_{n-1})}
\]

which is equal to

\[
(-1)^{n-1} c_n^{n-1} \frac{\partial(\alpha_1(n-1), \ldots, \alpha_n(n-1), \beta_1(n-1), \ldots, \beta_{n-1}(n-1))}{\partial(a_1, \ldots, a_n, c_1, \ldots, c_{n-1})}.
\]

By induction hypothesis, this gives the Jacobian of \( \xi'_n \):

\[
(-1)^{n(n-1)/2} \kappa_1 \kappa_2 \cdots \kappa_n c_2^2 \cdots c_n^{n-1}.
\]

Together with the Jacobian of \( \psi_n (3.7) \), we have proved (3.8).

Now let’s return to the morphism \( \iota : \Theta_{n-r} \times \Theta_r \times F \). We have an obvious isomorphism

\[
\phi : \Theta_{n-r} \times \Theta_r \times F \to \Theta_n \times F
\]

which sends the triple

\[
(x(a_1, \ldots, a_{n-r}, 0, c_1, \ldots, c_{n-r}), x(a_{n-r+1}, \ldots, a_n, 0, c_{n-r+1}, \ldots, c_n), d)
\]

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to $x(a_1, ..., a_{n+1}, c_1, ..., c'_{n-r}, c_{n-r+1}, ..., c_n)$ with

$$a_{n+1} = d, c'_{n-r} = c_{n-r} \prod_{i=1}^{r} c_{n-r+i}.$$ 

We have the product

$$\xi_{n-r} : \Theta_{n-r} \times \Theta_r \times F \to (\mathfrak{gl}_{n-r} \times F^{n-r} \times F_{n-r})//GL_{n-r} \times (\mathfrak{gl}_r \times F^r \times F_r)_{rs}//GL_r \times F.$$

Then it is easy to see that the diagram commutes:

$$\xymatrix{ \Theta_{n-r} \times \Theta_r \times F \ar[r]^\phi \ar[d]^{\xi_{n-r,r}} & \Theta_n \times F \ar[d]^{\xi_n} \\
(\mathfrak{gl}_{n-r} \times F^{n-r} \times F_{n-r})//GL_{n-r} \times (\mathfrak{gl}_r \times F^r \times F_r)_{rs}//GL_r \times F \ar[r] & \mathfrak{gl}_{n+1}//GL_n.}$$

Then the Jacobian of $\iota^\sharp_1$ restricted to the image of $\xi_{n-r,r}$ is equal to the ratio of the Jacobian of $\xi_n$ over that of $\xi_{n-r,r}$ and $\phi$. By (3.8), we obtain this ratio is a non-zero constant times

$$\frac{c_2c_3^2...c_n}{c_2c_3^2...(c_{n-r} \prod_{i=1}^{r} c_{n-r+i})^{n-r-1} \cdot c_{n-r+2}c_{n-r+3}...c_{n+1} \prod_{i=1}^{r} c_{n-r+i}} = 1.$$

This shows that the Jacobian of $\iota^\sharp_1$ is a non-zero constant on a Zariski dense subset and hence itself is a non-zero constant on the source of $\iota_1^\sharp$. This shows that $\iota_1^\sharp$ is étale. □

We now refine the morphism $\iota_1$. We return to the semisimple element $x = (X, w)$ in the unitary case. Then $X_{11}$ is semisimple in the usual sense. Therefore we may consider the Lie algebra $\mathfrak{u}(W_1)_{X_{11}}$ of the stabilizer $U(W_1)_{X_{11}}$ and an open subvariety $\mathfrak{u}(W_1)'_{X_{11}}$ of $\mathfrak{u}(W_1)_{X_{11}}$ consisting of those $Y$ such that

$$\det(ad(Y); \mathfrak{u}(W_1)/\mathfrak{u}(W_1)_{X_{11}}) \neq 0,$$

cf. [36, 14.5]. Let $\iota_2$ be the restriction of $\iota_1$ to

$$(\mathfrak{u}(W_1)'_{X_{11}} \times W_1) \times (\mathfrak{u}(W_2) \times W_2)_{rs}.$$ 

Then $\iota_2$ is $U(W_1)_{X_{11}}$-equivariant and it induces a morphism

$$\iota_2^\sharp : (\mathfrak{u}(W_1)'_{X_{11}} \times W_1)/U(W_1)_{X_{11}} \times (\mathfrak{u}(W_2) \times W_2)_{rs}://U(W_2) \to (\mathfrak{u}(W) \times W)/U(W).$$

Note that the representation of $U(W_1)_{X_{11}}$ on $\mathfrak{u}(W_1)_{X_{11}} \times W_1$ is a product of representations of the same type (including the linear case). Similar construction for $\mathfrak{gl}_{n+1}$ (equivalently, $\mathfrak{S}$).

**Lemma 3.13.** The morphism $\iota_2^\sharp$ is étale for both $\mathfrak{gl}_{n+1}$ (equivalently, $\mathfrak{S}$) and $\mathfrak{u}(W)$.
Proof. We By the previous lemma, it suffices to show that the morphism

\[(\mathfrak{U}(W_1)^{r}_{X_{11}} \times W_1)//U(W_1)_{X_{11}} \to (\mathfrak{U}(W_1) \times W_1)//U(W_1)\]

is étale. It is not hard to show that the Jacobian of this morphism at the image of \((Y, u) \in \mathfrak{U}(W_1)^{r}_{X_{11}} \times W_1\) is given by, up to a sign:

\[\det(ad(Y); \mathfrak{U}(W_1)/\mathfrak{U}(W_1)_{X_{11}}).\]

This is non-zero by the definition of \(\mathfrak{U}(W_1)^{r}_{X_{11}}\).

To lighten notations, we denote

\[V = \mathfrak{U}(W) \times W, V_x = (\mathfrak{U}(W_1)^{r}_{X_{11}} \times W_1) \times (\mathfrak{U}(W_2) \times W_2),\]

\[H = U(W), H_i = U(W_i),\]

and \(H_x = U(W_1)^{r}_{X_{11}}\), which is isomorphic to the stabilizer of \(x\). Similar notations apply to \(\mathfrak{gl}_{n+1}\) (equivalently, \(\mathfrak{H}\)).

Let

\[V'_x = (\mathfrak{U}(W_1)^{r}_{X_{11}} \times W_1) \times (\mathfrak{U}(W_2) \times W_2)_{rs}.\]

Then \(x \in V'_x\). Note that \(\iota_2\) also induces a morphism

\[\iota : H \times (H_x \times H_2) \to V\]

by sending \((h, x)\) to \(h \cdot \iota_2(x)\).

Lemma 3.14. The morphism \(\iota\) is étale.

Proof. We show this in the unitary case. It suffices to show that the differential \(d\iota\) at \((1, y)\) is an isomorphism. We first assume that \(X_{11}\) is a scaler. Then \(H_x = H_1\). Suppose \(y = (Y, u) \in V_x\).

Then it is not hard to see that for \(\Delta Y = \text{diag}(\Delta Y_{11}, \Delta Y_{22}), \Delta u = (\Delta u_1, \Delta u_2),\)

\[\iota_2(Y + \Delta Y, u + \Delta u) = \iota(Y, u) + \Delta \iota_2(Y, u) + \text{higher terms},\]

\[\Delta \iota_2(Y, u) := \begin{pmatrix} \Delta Y_{11} & \phi_{\Delta u_1} \\
... & \Delta Y_{22} \end{pmatrix},\]

where the part \(...\) is determined by the hermitian condition, and \(\phi_{\Delta u_1} \in Hom(W_2, W_1)\) is the homomorphism that sends \(Y_{22}^{r-1}u_2\) to 0 for \(i = 0, 1, ..., r - 2\) and \(Y_{22}^{r-1}u_2\) to \(\Delta u_1\).

Then the differential \(d\iota\) at \((1, y)\) is given by

\[d\iota : \mathfrak{U}(W) \times_{\mathfrak{U}(W_1)^{r}_{X_{11}} \times \mathfrak{U}(W_2)} V_x \to V\]

\[\Delta X, (\Delta Y, \Delta u) \mapsto ([\Delta X, \text{diag}(Y_{11}, Y_{22})] + \Delta \iota_2(Y, u), \Delta u_2 + \Delta X \cdot u_2).\]
Here the LHS means the quotient of $(\mathcal{U}(W) \times \mathcal{V}_x)/\mathcal{U}(W_1)_{x_{11}} \times \mathcal{U}(W_2)$. By comparing the dimension, it suffices to show that $d\iota$ is injective. Suppose that $d\iota(\Delta X, (\Delta Y, \Delta u)) = 0$. By the action of $(\mathcal{U}(W) \times \mathcal{V}_x)/\mathcal{U}(W_1)_{x_{11}} \times \mathcal{U}(W_2)$, we may assume that

$$\Delta X = \begin{pmatrix} 0 & \phi \\ \vdots & 0 \end{pmatrix}, \phi \in \text{Hom}_E(W_2, W_1).$$

Then we need to show that $\phi = 0$ and $(\Delta Y, \Delta u) = 0$. From the diagonal blocks, it is easy to see that $\Delta Y = 0$. Note that now we have $\Delta X \cdot u_2 = \phi u_2 \in W_1$. Therefore $\Delta u_2 + \Delta X \cdot u_2 = 0$ implies that both $\Delta u_2 = 0$ and $\phi u_2 = 0$. Now we use the condition from the off-diagonal block to obtain

$$Y_{11} \phi - \phi Y_{22} + \phi_{\Delta u_1} = 0 \in \text{Hom}(W_2, W_1).$$

Since $(Y_{22}, u_2)$ is regular semisimple, $u_2, Y_{22}u_2, \ldots, Y_{22}^{r-1}u_2$ form a basis of $W_2$. We apply the above homomorphism to $Y_{22}^i u_2$:

$$\phi Y_{22}^{i+1} u_2 = Y_{11} \phi Y_{22}^i u_2 + \phi_{\Delta u_1} Y_{22}^i u_2.$$ 

Since $\phi u_2 = 0$, we may show that $\phi Y_{22}^{i+1} u_2 = 0$ recursively. This shows that $\phi = 0 \in \text{Hom}(W_2, W_1)$. This completes the proof when $x_{11}$ is a scalar.

Now we consider a general semisimple $X_{11}$, and $H_x = U(W_1)_{x_{11}}$. Then the assertion follows if we show that the following analogous morphism is étale

$$H_1 \times H_x (\mathcal{U}(W_1)'_{x_{11}} \times W_1) \to \mathcal{U}(W_1) \times W_1.$$ 

Similar argument as in the above works and we omit the details. \qed

**Proposition 3.15.** For both $\mathfrak{gl}_{n+1}$ (equivalently, $\mathfrak{S}$) and $\mathcal{U}(W)$, the following diagram is cartesian

$$\begin{array}{ccc}
H \times_{(H_x \times H_2)} V'_x & \longrightarrow & V \\
\downarrow & & \downarrow \pi \\
V'_x/(H_x \times H_2) & \longrightarrow & V'/H.
\end{array}$$

**Proof.** Thanks to the previous lemmas, now the proof is similar to that of [36, Lemma 14.1]. We need to show the induced morphism

$$\gamma : H \times_{(H_x \times H_2)} V'_x \to V'_x/(H_x \times H_2) \times_{V'/H} V$$

is indeed an isomorphism. It suffices to show that in an algebraic closure the induced map on the geometric points is bijective. From this we see that the question becomes equivalent for both $\mathfrak{gl}_{n+1}$ (equivalently, $\mathfrak{S}$) and $\mathcal{U}(W)$. To simplify exposition, we consider the unitary case.

Actually the bijectivity holds for any field as we now show. To show the surjectivity, we may write an element $V'_x/(H_x \times H_2) \times_{V'/H} V$ as $((Y', u'), (Y, u))$ where $(Y, u) \in V'_x$ is abused to denote its image in the quotient. Then $\iota^*_2(Y', u') = \pi(Y, u)$. This implies
that the \( u, Yu, \ldots, Y^{r-1}u \) span a hermitian subspace that is isometric to that spanned by \( u_2', Y_2' u_2, \ldots, Y_2'^{r-1}u_2' \) which is non-degenerate. Therefore by Witt’s theorem there exists an \( h \in H \) such that \( h(Y_2^i u_2') = Y^i u \) for \( 0 \leq i \leq r-1 \). We may thus assume that \( u = u'_2, Y_2 = Y_2' \). The rest follows from [36, Lem. 14.1].

To show the injectivity, without loss of generality it suffices to show that if
\[
\gamma(1, y) = \gamma(h, z),
\]
then \( h \in H_x \times H_2 \) and \( y = h z \). It suffices to show \( h \in H_x \times H_2 \) since the second assertion follows from this. Denote \( y = (Y, u) \) and \( z = (Z, w) \). Then \( h \) obviously preserves the subspace \( W_2 \) and hence \( W_1 \), too. It follows that \( h \in H_1 \times H_2 \). The rest follows from [36, Lem. 14.1]. □

We now may construct an étale Luna slice of a semisimple element \( x \). We only do so in the unitary case. Choose an étale Luna slice \( Z_2 \) of \((X_{22}, w)\) (which is \( H_2\)-regular semisimple) for the action of \( H_2 \) on \( \mathcal{U}(W_2) \times W_2 \). Then an étale Luna slice of \( x \) can be chosen to be the image of \((\mathcal{U}(W_1)_X \times W_1) \times Z_2 \) under \( \iota_2 \).

Similarly, we see that this also gives us a way to choose an analytic Luna slice once we fix a choice of analytic Luna slice for the \( H_2\)-regular semisimple element \((X_{22}, w)\).

**Smooth transfer for regular supported functions**

**Lemma 3.16.** Let \( \mathcal{V} \) be either \( \mathcal{G} \) or \( \mathcal{U}(V) \). Let \( f \in \mathcal{C}_c^\infty(\mathcal{V}(F)) \). Then the function \( \pi_{s, \omega}(f) \) is smooth on \( \mathcal{V}/H(F)_{rs} \) and compact supported on \( \mathcal{V}/H(F) \).

**Proof.** The smoothness follows from the first part of Prop. 3.11 and the fact that the stabilizer of a regular semisimple element is trivial. The support is contained in the continuous image of a compact setup, hence compact. □

**Proposition 3.17.**

1. If \( f' \in \mathcal{C}_c^\infty(\mathcal{G}_{rs}) \), then \( \pi_{s, \omega} \in \mathcal{C}_c^\infty(Q(F)_{rs}) \). And conversely given \( \phi \in \mathcal{C}_c^\infty(Q(F)_{rs}) \), there exists \( f' \in \mathcal{C}_c^\infty(\mathcal{G}_{rs}) \) such that \( \pi_{s, \omega}(f') = \phi \).

2. If \( f_w \in \mathcal{C}_c^\infty(\mathcal{U}(V)_{rs}) \), then \( \pi_s(f_w) \in \mathcal{C}_c^\infty(Q(F)_{rs}) \). And conversely given \( \phi \in \mathcal{C}_c^\infty(Q(F)_{rs}) \), there exists a tuple \( (f_w \in \mathcal{C}_c^\infty(\mathcal{U}(V)_{rs}))_{W} \) such that \( \sum_W \pi_s(f_w) = \phi \).

**Proof.** We only prove (1) and the proof of (2) is similar. By the previous lemma, it suffices to show the converse part. By the localization principle Prop. 3.8 (or rather its proof), it suffices to show that for every regular semisimple \( x \in Q(F) \), \( \phi \) is locally around \( x \) an orbital integral of a function with regular semisimple support. We now fix a regular semisimple \( x \). Note that the stabilizer of \( x \) is trivial. When choosing of the analytic slice, we may require that \( S \) is contained in the regular semisimple locus. Then the result follows from the decent of orbital integral, the second part of Prop. 3.11. □

**Theorem 3.18.** Given \( f' \in \mathcal{C}_c^\infty(\mathcal{G}_{rs}) \), there exists its smooth transfer \( (f_w) \) such that \( f_w \in \mathcal{C}_c^\infty(\mathcal{U}(V)_{rs}) \). Conversely, given a tuple \( (f_w \in \mathcal{C}_c^\infty(\mathcal{U}(V)_{rs}))_{W} \), there exists its smooth transfer \( f' \in \mathcal{C}_c^\infty(\mathcal{G}_{rs}) \).

In particular, this includes the existence of local transfer at a regular semisimple point \( z \in Q(F) \).
Reduction to local transfer around 0 of sliced representations. Fix $z \in Q(F)$. Within the fiber of $x$, there are one semisimple $H'$-orbit in $S_{n+1}$ and finitely many semisimple $H$-orbits in $\mathcal{U}(V)$. Note that there may be infinitely many non-semisimple orbits within the fiber. We have described the sliced representations of those semisimple elements and we have seen that those are still of the same type as $S$ or $U(V)$ with lower dimension and possibly extending the base field $F$ to a finite extension. So we may also speak of local transfer around zero of those sliced representations.

To compare the meaning of transfer at $z$ and zero of the sliced representations, we need to compare their transfer factors.

We may define an equivalent choice of transfer factors as follows. For $x = \begin{pmatrix} X & u \\ v & d \end{pmatrix} \in \mathfrak{gl}_{n+1,F}$ we define
\[ \nu(x) = \det(u, Xu, X^2 u, ..., X^{n-1} u). \]
Then the transfer factor can be chosen as $\eta(\nu(x)) \in \{ \pm 1 \}$.

Lemma 3.19. We may choose an $H_x$-invariant neighborhood of $x$ such that for any $y$ in this neighborhood, $\omega(y)$ is equal to a non-zero constant times $\omega(\psi(y))$.

Proof. We only treat the two basic case: (1) $r = 0$, (2) $r = n$. The general case can be reduced to those two by the same strategy as in the proof above. When $r = n$, namely $x$ is regular semisimple so that $H_x$ is trivial, the assertion follows from that there is a neighborhood of $x$ over which $\omega$ is a constant. When $r = 0$, this follows from the fact that
\[ \nu(y) = \pm \nu(\psi(y)) \det(\text{ad}(Y), \mathfrak{gl}_{n+1+Y_{11}})^{1/2} \]
where $\det(\text{ad}(Y), \mathfrak{gl}_{n+1+Y_{11}})^{1/2}$ is a square root of $\det(\text{ad}(Y), \mathfrak{gl}_{n+1+Y_{11}})$ (for example it can be given by the determinant of $\text{ad}(Y)$ on the upper triangular blocks). Since $\det(\text{ad}(X), \mathfrak{gl}_{n+1+X_{11}}) \neq 0$, we may shrink the neighborhood if necessary such that $\eta(\nu(y))$ and $\eta(\nu(\psi(y)))$ differ only by a non-zero constant.

Proposition 3.20 (Reduction to zero). Fix $z \in Q(F)$. If the local transfer around zero exists for the sliced representations, then the local transfer around $z$ exists.

Proof. By Prop. 3.11, under our choice of the étale Luna slice and hence an analytic Luna slice, the orbital integral of regular semisimple element near to a semisimple element can be written as an orbital integral of regular semisimple elements near zero of the sliced representation. By our construction above, the choice of étale Luna slices on $S_{n+1}$ and $\mathcal{U}(V)$’s can be made compatible. Moreover, the transfer factors are compatible with respect to the choice of analytic Luna slices by Lemma 3.19 above. This completes the proof.

Remark 11. We see that the reduction steps are along the same line as those in the classical endoscopic transfer by Langlands–Shelstad. The only non-trivial point is the explicit construction of the étale Luna slices which is a slightly more involved than the case of adjoint action of a reductive group on its Lie algebra.
4 Smooth transfer for Lie algebra

In this section we assume that $F$ is non-archimedean of characteristic zero, namely a finite extension of $\mathbb{Q}_p$ for some rational prime $p$.

4.1 A relative local trace formula

To simplify notations we unify the linear side and the unitary side in this subsection. Let $F$ be a field and $E$ be an étale $F$-algebra of rank two. Namely, $E$ is either a quadratic field extension or $F \times F$. Let $W$ be a free $E$-module of rank $n$ and $B : W \times W \to E$ be a non-degenerate Hermitian form. We denote by $H = H(W)$ the algebraic group $U(W)$:

$$H = U(W) = \{ h \in \text{Aut}_E(W) : (hu, hv) = (u, v) \}$$

and $\mathfrak{U}(W)$ its Lie algebra:

$$\mathfrak{U}(W) = \{ X \in \text{End}_E(W) : (Au, v) = -(u, Av) \}.$$ 

Note that we allow $E = F \times F$ in which case we have $H \simeq GL_n$.

We will use $x \to \bar{x}$ to denote the (unique) non-trivial $F$-linear automorphism of $E$. In the case $E = F \times F$, it is the permutation of the two coordinates.

We consider the representation of $H$ on an $F$-vector space:

$$\mathcal{V} = \mathfrak{U}(W) \times W.$$ 

We usually denote by $x = (X, w)$ an element of $\mathcal{V}$ for $X \in \mathfrak{U}(W), w \in W$. We define

$$\Delta(x) = \Delta(X, w) = \text{det}((X^i w, X^j w))_{i,j=0}^{n-1}.$$ 

Then $x$ is regular semisimple if and only if $\Delta(x) \neq 0$. In the case $E = F \times F$, we may equivalently consider

$$\mathfrak{gl}_n \times F_n \times F^n,$$

where $F_n$ ($F^n$, resp.) is the $n$-dimensional vector space of row (column, resp.) vectors. We consider the natural action of $GL_n$ by

$$h \cdot (X, u, v) = (h^{-1}Xh, uh^{-1}, hv).$$

Similarly we define for an element $x = (X, u, v)$

$$\Delta(x) = \Delta(X, u, v) = \text{det}(uX^{i+j}v)_{i,j=0}^{n-1}.$$ 

Then $x$ is regular semisimple if and only if $\Delta(x) \neq 0$. In either case, we denote by $D(X)$ the discriminant of $X \in \mathfrak{U}(W)$ or $\mathfrak{gl}_n$, namely

$$D(X) = \prod_{i,j} (\lambda_i - \lambda_j)^2,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the $n$ eigenvalues of $X$. 

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Upper bound of orbital integrals. We first estimate the orbital integral for a regular semisimple $x \in \mathcal{V}$:

$$O(x, f) = \int_{H} f(xh)\eta(h)dh,$$

where $\eta$ is any (unitary) character.

**Lemma 4.1.** Let $\Omega \subset M_n(F)$ be a compact open set. Let $T$ be a Cartan subgroup of $GL_n(F)$ and $\mathfrak{t} \subset M_n(F)$ be the corresponding Cartan subalgebra. Let $\varphi \in C_c^\infty(F_n \times F^n)$. Then there exists a constant $r > 0$ depending only on $n$ and a constant $C$ depending only on $n, \varphi, \Omega$ with the following property: for all regular semisimple $X \in \mathfrak{t}$, and $h \in$ such that $h^{-1}Xh \in \Omega$, we have

$$\int_T \varphi(uth, h^{-1}t^{-1}v)dt \leq C \max\{1, \log|\Delta(X, u, v)|\}^r.$$

**Proof.** If $h^{-1}Xh \in \Omega$, then for all $i, j = 0, ..., n-1$, the following vectors are in a compact set depending only on the support of $\varphi$ and $\Omega$:

$$h^{-1}t^{-1}X^iv, uX^jth.$$

Write $\delta_1 = (X^iv)_{i=0,..,n-1} \in M_n(F)$ and $\delta_2 = (uX^j)_{j=0,..,n-1} \in M_n(F)$ so that $\Delta(X, u, v) = det(\delta_1\delta_2)$. Then the condition becomes

$$h^{-1}t^{-1}\delta_1, \delta_2th.$$

are in a compact set $\Omega_1$ of $M_n(F)$ depending only on the support of $\varphi$ and $\Omega$.

We may identify $t$ with $\prod_{i=1}^n E_i$ and $T$ with $\prod_{i=1}^n E_i^\times$ where $E_i/F$ is a degree $n_i$ field extension such that $\sum n_i = n$. Let $P$ be the parabolic of $H = GL_{n-1}$ associated to the partition $\sum n_i = n$ with Levi decomposition $P = MN$ and we may assume that $T \subset M$. Then $T$ is elliptic in $M$. By Iwasawa decomposition $H = NMK$ for $K = GL_n(O_F)$, we may write $h = nmk, \delta_1 = n_1m_1k_1, \delta_2 = k_2m_2n_2$. By enlarging $\Omega_1$ if necessary, we may assume that $\Omega_1 = K\Omega_1K$. Since $h^{-1}t\delta_1 = k^{-1}m^{-1}n^{-1}tn_1m_1k_1 \in \Omega_1$, we may see that the Levi component $m^{-1}t^{-1}m_1 \in \Omega_2$ for a compact set $\Omega_2$ of $M_2(F)$. Similarly we have $m_2tm \in \Omega_2$. Let $t = (t^{(i)})_{i=1}^e$ where $t^{(i)} \in E_i^\times$ similarly for $m, m_1, m_2$. Then we have some constant $C_1, C_2 > 0$ such that for all $i = 1, 2, ..., e$

$$C_1|\det(m_1^{(i)}(m^{(i)})^{-1})| \leq |t_i| \leq C_2|\det(m_2^{(i)}m^{(i)})|^{-1}.$$

The volume of such $t_i$ is bounded above by

$$C_3 + \log \frac{C_2|\det(m_2^{(i)}m^{(i)})|^{-1}}{C_1|\det(m_1^{(i)}m^{(i)})^{-1}|} = C_4 - \log|\det(m_1^{(i)}m_2^{(i)})|$$

for constants $C_3, C_4$. In particular, the integral in the lemma is zero unless for all $i = 1, 2, ..., e$:

$$C_4 - \log|\det(m_1^{(i)}m_2^{(i)})| \geq 0$$
Under this assumption, we have a constant $C$ such that for all $i = 1, 2, ..., e$:

$$C_4 - \log|\det(m_1^{(i)} m_2^{(i)})| \leq C_5 - \sum_{i=1}^{e} \log|\det(m_1^{(i)} m_2^{(i)})|$$

which is equal to

$$C_5 - \log|\det(m_1 m_2)| = C_5 - \log|\det(\Delta(X, u, v))|.$$ 

Then it is easy to see that the integral over $T$ is either zero or bounded above by the $L^\infty$-norm of $\varphi$ times

$$(C_5 - \log|\det(\Delta(X, u, v))|)^k.$$ 

Then the lemma follows.

We say that $x = (X, w)$ is strongly regular if $x, X, w$ are all $H$-regular semisimple (i.e., $\Delta(x) \neq 0, (w, w) \neq 0$ and $D(X) \neq 0$).

**Proposition 4.2.** There exists a constant $C$ depending on $f$ and integer $r > 0$ such that for all $x \in V$ strongly regular:

$$|O(x, f)| \leq C\max\{1, |\log|\Delta(x)||^r\}\max\{1, |D(X)|^{-1/2}\}.$$ 

**Proof.** We only give the proof for the linear case. The unitary case is similar (perhaps easier). We choose a (finite) set of representatives of Cartan subalgebra $t$. We may assume that $X \in t$ and $f = \phi \otimes \varphi$ where $\phi \in C^\infty_c(M_n(F))$ and $\varphi$ as in the previous lemma. Then we have Now we have

$$|O(x, f)| \leq C_6\max\{1, |\log|\Delta(x)||\}\int_{H/T} |\phi|(ht_0h^{-1})dh.$$ 

By the bound of Harish-Chandra on the usual orbital integral, the above integral is bounded by a constant times $\max\{1, |D(X)|^{-1/2}\}$. Since there are only finitely many $t_i$, we may choose a uniform constant $C$ to complete the proof.

**Local integrability** We want to show the orbital integral is a locally integrable function on $V$. The following result is probably well-known. But we could not find a reference so we decide to include a proof here.

**Lemma 4.3** (Igusa integral). Let $P(x) \in F[x_1, ..., x_m]$ be a polynomial. Then there exists $\epsilon > 0$ such that

$$\int_{\Omega_F} |P(x)|^{-\epsilon}dx < \infty.$$ 

If $P$ is homogeneous, then there exists $\epsilon > 0$ such that the function $|P(x)|^{-\epsilon}$ is locally integrable everywhere.
Proof. The first assertion implies the second one. If $P$ is homogenous of degree $k$, assuming the first assertion we want to show that $|P|^{-\epsilon}$ is locally integrable around any $x_0 \in F^m$. In deed, we may assume that $x_0 \in \mathbb{Z}^{-n}O^m$ for some $n > 0$. By homogeneity, we have

$$\int_{\mathbb{Z}^{-n}O^m} |P(x)|^{-\epsilon} dx = |\mathbb{Z}|^{-mn+kn} \int_{O^m} |P(x)|^{-\epsilon} dx < \infty.$$  

This shows the local integrality around $x_0$.

To show the first assertion, we may assume that $P \in O_F[x]$ and $F = \mathbb{Q}_p$. Now following Igusa ([7]) we may define

$$\tilde{N}_n := \sharp \{ x \in (\mathbb{Z}_p/p^n)^m \mid f(x) \equiv 0 \ mod p^n \}.$$  

Let $w_n = \tilde{N}_n/p^{nm}$. Then $w_{n+1} < w_n$ and we want to prove that there exists $\epsilon > 0$ such that

$$\sum_n p^{nm}(w_n - w_{n+1}) < \infty.$$  

And define the Poincare series

$$\tilde{P}(T) = \sum_{n=0}^{\infty} \tilde{N}_n T^n.$$  

By the rationality of $\tilde{P}(T)$ proved firstly by Igusa ([7]), we may write

$$\tilde{P}(T) = Q(T) \prod_{i,j} (1 - \alpha_{i,j} B^\beta_i T)^{-k_{i,j}}$$  

where $\beta_i \in \mathbb{R}, k_i \in \mathbb{N}_{>0}, \beta_i$ are distinct. $\alpha_{i,j}$ are roots of unit, and $Q(T)$ is a polynomial ([7, Remark 3.3]). We must have all $\beta_i \leq m$ since $\tilde{N}_n \leq |(\mathbb{Z}_p/p^n)^m| = q^{nm}$ for all $n$. If all $\beta_i < m$, then we certainly can choose $\epsilon > 0$ such that all $\beta_i < m - \epsilon$ and hence

$$\tilde{N}_n = O(p^{nm(\epsilon - \epsilon)}).$$  

Assume now that $\beta_0 = m$ and all other $\beta_i < m$. Since $|\tilde{N}_n| \leq p^{nm}$ we must have all $k_{0,j} = 1$ (i.e., no multiplicity). Then for all $\epsilon < m - \max_{i \neq 0} \beta_i$ and $a_j \in \mathbb{C}$ such that

$$|\tilde{N}_n - p^{nm} \sum_j a_j \alpha_{0,j}^n| = O(p^{nm(\epsilon - \epsilon)})$$  

when $n$ large. Let $w'_n = \sum_j a_j \alpha_{0,j}^n$. Since $\alpha_{i,j}$ are roots of unit, $w'_n$ is periodic, say with period $N$. Then by

$$|w_n - w'_n| = O(p^{-\epsilon})$$  

we have

$$w_n - w_{n+1} \leq w_n - w_{n+N} \leq |w'_n - w'_{n+N}| + O(p^{-\epsilon}) = O(p^{-\epsilon}).$$  

And this finishes the proof.
Remark 12. The same holds for archimedean local fields and they are called local zeta function.

Corollary 4.4. There exists \( \epsilon > 0 \) such that the function

\[ m_\epsilon : x \mapsto |D(X)|^{-1/2-\epsilon} \log |\Delta(x)| \]

is locally integrable on \( \mathcal{V} \).

Proof. Let \( \Omega \) be a compact subset of \( \mathcal{V} \). In Young’s inequality

\[ ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b, p, q > 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \]

we let \( p = 1 + \epsilon_1 \) to obtain

\[ m_\epsilon(x) \leq \frac{|D(X)|^{-(1/2+\epsilon)(1+\epsilon_1)}}{1 + \epsilon_1} + \frac{|\log |\Delta(x)||^q}{q}. \]

We now need to use the Lie algebraic version of [25, Theorem 15], namely: there exists \( \epsilon_2 > 0 \) such that the function \( X \mapsto |\Delta(X)|^{-1/2-\epsilon_2} \) is locally integrable on \( \mathfrak{U}(W) \). This implies that for an appropriate choice of \( \epsilon, \epsilon_1 \), the first term above is locally integrable on \( \mathfrak{U}(W) \times W \). The Lemma above implies the second term is also locally integrable. This completes the proof.

In summary we have showed that

Corollary 4.5. For any \( f \in C_c^\infty(\mathcal{V}) \), we have

- The absolute value of the orbital integral \( x \mapsto |O(x, f)| \) is locally integrable on \( \mathcal{V} \).
- If \( X \in \mathfrak{gl}_n(F) \) (\( \mathfrak{U}(W) \) in the unitary case) is regular semisimple, \( w \mapsto |O((X, w), f)| \) is locally integrable on \( W \).
- If \( w \in W \) is regular semisimple, \( X \mapsto |O((X, w), f)| \) is locally integrable on \( \mathfrak{gl}_n(F) \) (\( \mathfrak{U}(W) \) in the unitary case).

A relative local trace formula We now show a local trace formula for the representation of \( H \) on \( \mathcal{V} \). We would like to consider partial Fourier transform with respect to an invariant subspace \( \mathcal{V}_0 \) of \( \mathcal{V} \). We fix an \( H \)-invariant bilinear form on \( \mathcal{V} \) such that its restriction to any invariant subspace is non-degenerate (obviously such form exists). We define a partial Fourier transform:

\[ \mathcal{F}_{\mathcal{V}_0} f(x) = \int_{\mathcal{V}_0} f(y) \psi(<x, y>) dy, \quad f \in C_c^\infty(\mathcal{V}). \]
Choose the self-dual measure on $V_0$. Then the fact that Fourier transform is an isometry on $L^2$ space can be written as:

$$\int_{V_0} f_1(x)\overline{f_2(x)}dx = \int_{V_0} F_{V_0}f_1(x)\overline{F_{V_0}f_2(x)}dx.$$  

And it is clear for two orthogonal subspace $V_0, V_1$:

$$F_{V_0+V_1} = F_{V_0} \circ F_{V_1} = F_{V_1} \circ F_{V_0}.$$  

Returning to our case, $V$ is either $\mathfrak{g}l_n \times F_n \times F_n$ or $\mathfrak{U}(W) \times W$ for a hermitian space $W$. In each case we have an abelian 2-group of Fourier transforms generated by the two partial transforms $F_{\mathfrak{g}l_n}, F_{F_n \times F_n}$ ($F_{\mathfrak{U}(W)}, F_W$, resp.). We let $\hat{f}$ denote the Fourier transform with respect to $\psi$ and $\check{f}$ the Fourier transform with respect to $\psi^{-1}$.

**Theorem 4.6.** Let $V$ be either $\mathfrak{g}l_n \times F_n \times F_n$ or $\mathfrak{U}(W) \times W$ and $V_0$ an invariant-subspace with an invariant complementary space $V_0^\perp$. We write $x = (y, z)$ according to the decomposition $V = V_0 \oplus V_0^\perp$. Fix a regular semisimple $z \in V_0^\perp$. For $f_1, f_2 \in C^\infty_c(V)$, define

$$T(f_1, f_2) = \int_{V_0} \left( \int_{H(F)} f_1((y, z)^h)\eta(h)dh \right) f_2(y, z)dy,$$

where $\eta$ is trivial in the unitary case. Then we have

$$T(f_1, f_2) = T(\hat{f}_1, \check{f}_2).$$

**Proof.** We consider the unitary case. The linear case is similar. The idea is the same as in Harish-Chandra’s work on the representability of the character of a supercuspidal representation. Take a sequence of increasing compact subsets $\Omega_i$ of $H$ such that $H = \cup \Omega_i$. Define

$$O_i(x, f) = \int_{\Omega_i} f(x^h)dh.$$  

Then it is clear for a regular semisimple $X$:

$$O(x, f) = \lim_{i \to \infty} O_i(x, f).$$  

Note that we have for any $f_1, f_2 \in C^\infty_c(V)$:

$$\int_{V_0} f_1((y, z)^h)f_2(y, z)dy = \int_{V_0} \hat{f}_1((y, z)^h)\check{f}_2(y, z)dy.$$  

noting that the Fourier transform commutes with the $H$-translation. Therefore we have

$$\lim_{i \to \infty} \int_{\Omega_i} \left( \int_{V_0} f_1((y, z)^h)f_2(y, z)dy \right) dh = \lim_{i \to \infty} \int_{\Omega_i} \left( \int_{V_0} \hat{f}_1((y, z)^h)\check{f}_2(y, z)dy \right) dh.$$  

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As $\Omega_i$ is compact, we may interchange the order of integration.

Obviously we have $|O_i((y, z), f_1)| \leq |O_i((y, z), |f_1|)|$. When $z$ is regular semisimple, by Corollary 4.5 the function $y \mapsto |O((y, z), |f_1|)$ is locally integrable. By Lebesgue dominated convergence we obtain

$$\lim_{i \to \infty} \int_{\Omega_i} \left( \int_{V_0} f_1((y, z)^h) f_2(y, z) dy \right) dh = \int_{V_0} O((y, z), f_1) f_2(y, z) dy = T(f_1, f_2).$$

Similarly we get $T(\widehat{f}_1, \widehat{f}_2)$ from the RHS. This completes the proof.

\[ \square \]

**A consequence.** Now we consider the case $n = 1$. We deduce the representability of the Fourier transform of an orbital integral on $W = M \times M^*$ where $M$ is a one dimensional $\mathbb{F}$-vector space. And let $H = GL(M) \simeq GL_1, \mathbb{F}$ act on $W$.

**Corollary 4.7.** For any quadratic character $\eta$ (possibly trivial), the Fourier transform of the orbital integral

$$\widehat{O}_w(f) = \omega(w) \int_H \widehat{f}(w^h) \eta(h) dh$$

(here $\omega(w)$ is a transfer factor) is represented by a locally constant $H$-invariant function on the regular semisimple locus $W_{rs}$ denoted by $\kappa^n(w, \cdot)$; namely for any $f \in C_c^\infty(W)$ we have

$$\widehat{O}_w(f) = \int_W f(w') \omega(w') \kappa^n(w, w') dw'.$$

Similar result holds for the unitary case. Moreover, we may let $F$ be a product of fields and we have similar results.

**Proof.** The proof is along the same line as the proof of Harish-Chandra of the representability of Fourier transform of orbital integrals ([26, Lemma 1.19, pp.12]) noting that the Howe’s finiteness conjecture holds for the $H$-action on $W$ ([43, Theorem 6.1]). In the next two subsections we will prove a more explicit result when $\eta$ is nontrivial quadratic.

\[ \square \]

Obviously the kernel function $\kappa^n(w, w')$ can be viewed as a locally constant function on $W_{rs} \times W_{rs}$ invariant under $H \times H$.

**Remark 13.** Unfortunately, the Howe’s finiteness conjecture (cf. [43, Theorem 6.1]) fails for the $H$-action on $\mathfrak{gl}_n \times F_n \times F^n$ when $n \geq 2$. Therefore the proof of Harish-Chandra ([26]) does not say anything on the representability of the Fourier transform of $H$-orbital integral.
4.2 A Davenport–Hasse type relation

We show a Davenport–Hasse type relation between two “Kloosterman sums”. It will be used to show that the Fourier transforms preserve transfer when $n = 1$.

We first recall the definition and some basic properties of the Langlands’ constant. For a fixed non-trivial character $\psi$ of $F$, and a field extension $K/F$ of local fields, we define a character of $K$ by $\psi_K = \psi \circ Tr_{K/F}$. Let $1_K$ be the trivial representation of the Weil group $W_K$. Then the Langlands constant is defined to be

$$\lambda_{K/F}(\psi) := \epsilon(\text{Ind}_{K/F}^K 1, s, \psi)$$

which is a constant independent of $s$. In particular, it is given by

$$\lambda_{K/F}(\psi) := \epsilon(\text{Ind}_{K/F}^K 1, 1/2, \psi).$$

The character of $W_{F}^\times \simeq F^\times$:

$$\eta_{K/F}(\psi) := \det(\text{Ind}_{K/F}^K 1)$$

is of order two. For $a \in F^\times$ we denote by $\psi_a$ the character of $F$ defined by $\psi_a(x) = \psi(ax)$. We then have

$$\lambda_{K/F}(\psi_a) = \eta_{K/F}(a) \lambda_{K/F}(\psi).$$

Moreover we have

$$\lambda_{K/F}(\psi)^2 = \eta_{K/F}(-1).$$

In particular, $\lambda_{K/F}(\psi) \in \mu_4$ is a fourth root of unity.

We first show that the epsilon factor is in fact a sort of Gauss sum. Let $\psi$ be a nontrivial additive character and $dx$ be the self-dual measure on $F$. We choose a Haar measure on $F^\times$ to be $d^\times x$. For a quasi-character $\chi$ of $F^\times$, we may define its real exponent $\text{Re}(\chi)$ to be the unique real number $r$ such that $|\chi(x)| = |x|^r$ for all $x \in F^\times$. We denote $\hat{\chi} = \chi^{-1} \cdot |·|$.

Lemma 4.8. The gamma factor as a meromorphic function of $\chi$ (namely, its value at $\chi| · |^s$ is meromorphic for $s \in \mathbb{C}$) is given by

$$\gamma(\chi, \psi) = \int_{F^\times} \psi(x) \hat{\chi}(x) \frac{dx}{|x|} = \int_F \psi(x) \chi^{-1}(x) dx.$$ 

Here the RHS is interpreted as

$$\int_{|x| < C} \psi(x) \hat{\chi}(x) |x|^s \frac{dx}{|x|} \big|_{s=0}$$

for any $C$ large enough. In particular, it is holomorphic when $\text{Re}(\chi) < 1$ and given by an absolute convergent integral

$$\gamma(\chi, \psi) = \int_{|x| < C} \psi(x) \chi^{-1}(x) dx$$

when $C$ large enough.
Proof. By definition
\[ \gamma(\chi, \psi) = \frac{\zeta(\hat{f}, \hat{\chi})}{\zeta(f, \chi)}. \]
Note that it does not depend on the choice of measure on \( F^\times \). We choose \( \frac{dx}{|x|} \). We choose \( f_n = 1_{1+\varpi^n \mathcal{O}_F} \). Then we have
\[ \hat{f}_n(x) = \psi(x)1_{\varpi^n \mathcal{O}_F}(x) = \psi(x)|\varpi|^n1_{\mathcal{O}_F}(x\varpi^n) \]
and if \( n \) is larger than the conductor of \( \chi \):
\[ \zeta(f_n, \chi) = |\varpi|^n vol(\mathcal{O}_F). \]
There is an integer \( m \) be such that \( 1_{\varpi^m \mathcal{O}_F} = vol(\mathcal{O}_F) \cdot 1_{\varpi^{-m} \mathcal{O}_F} \). Then we have
\[
\zeta(\hat{f}_n, \hat{\chi}) = \int_{F^\times} \psi(x)|\varpi|^n1_{\mathcal{O}_F}(x\varpi^n)\hat{\chi}(x) \frac{dx}{|x|} = vol(\mathcal{O}_F)|\varpi|^n \int_{\varpi^{-m} \mathcal{O}_F} \psi(x)\hat{\chi}(x) \frac{dx}{|x|}.
\]
The RHS is interpreted in the sense of analytic continuation. Therefore we have for \( n \) large enough:
\[ \gamma(\chi, \psi) = \int_{\varpi^{-m} \mathcal{O}_F} \psi(x)\hat{\chi}(x) \frac{dx}{|x|}. \]
\[ \square \]

We have some sort of Kloosterman sum. Let \( E \) be a quadratic extension and let \( E^1 \) be the kernel of the norm map. Consider an analogue of Kloosterman sum: for \( a = A\bar{A} \in \text{Norm}(E^\times) \):
\[ \Phi(a) = \Phi_{E/F}(a) := \int_{E^1} \psi_E(Ax) d^\times x. \]
The measure on \( E^1 \) is chosen such that for all \( \phi \in \mathcal{C}_c^\infty(E) \):
\[ \int_{E^\times} \phi(X) \frac{dX}{|X|_E} = \int_{F^\times} \int_{E^1} \phi(At) dtd^\times a \]
where \( dX \) is the self-dual measure on \( E \) with respect to the character \( \psi_E \) and \( NA = a \).

Let \( \eta \) be the quadratic character of \( F^\times \) associated to \( E/F \) by local class field theory. And define
\[ \Psi(a) = \Psi_{E/F}(a) := \int_{F^\times} \psi(x + \frac{a}{x}) \eta(x) d^\times x. \]
The RHS is similarly interpreted as
\[ \int_{1/C < |x| < C} \psi(x + \frac{a}{x}) \eta(x) d^\times x \]
for large enough $C$ (depending on $a$). Note that this integral becomes a constant when $C$ is large enough.

We also define some auxiliary functions:

$$
\Psi_C(a) := \int_{|a|/C < |y| < C} \psi(a/y + y)\eta(y) d^x y.
$$

And we formally set $\Psi_\infty = \Psi$. Obviously $\Psi_C$ converges to $\Psi_\infty$ pointwisely as $C \to \infty$.

**Lemma 4.9.** The function $\Psi_C$ (possibly $C = \infty$) is a locally constant function on $F^\times$. And there are constant $\beta_1, \beta_2$ independent of $C$ such that

$$
|\Psi_C(a)| \leq \beta_1 |\log |a|| + \beta_2
$$

for all $a \in F^\times$.

**Proof.** Let $B$ be such that $\psi(x) = 1$ when $|x| \leq B$. If $|a| < B^2$, either $|x| < B$ or $|a/x| < B$. So we may bound the integral by

$$
|\int_{|x| < B, |a|/C < |x| < C} \psi(a/x)\eta(x) d^x x| + |\int_{|x| \geq B, |a|/C < |x| < C} \psi(x)\eta(x) d^x x|.
$$

It is easy to see that the second term is bounded above by a constant. The first term is equal to

$$
|\int_{|x| > a/B, |a|/C < |x| < C} \psi(x)\eta(x) d^x x|
$$

which is bounded by

$$
|\int_{a/B < |x| < B, |a|/C < |x| < C} \psi(x)\eta(x) d^x x| + |\int_{|x| \geq B, |a|/C < |x| < C} \psi(x)\eta(x) d^x x|.
$$

The first term is at most

$$
\int_{a/B < |x| < B} \frac{1}{d^x x}
$$

which is of the form $\beta_1 |\log |a|| + \beta_2$ for some constants $\beta_1, \beta_2$. Now up to adding a constant to $\beta_2$, we complete the proof.

□

A key technical lemma is to estimate the asymptotic behavior of $\Psi_C(a)$ and $\Phi(a)$ as $|a| \to \infty$.

**Lemma 4.10.** There is a constant $A$ independent of $C$ such that when $|a| > A$, there is a constant $\alpha$ independent of $C$ such that

$$
|\Psi_C(a)| < \alpha |a|^{-1/4}
$$

and

$$
\Phi(a) < \alpha |a|^{-1/4}.
$$
Proof. The proof follows the strategy of that of [50, Prop. VIII.1]. We only prove the case for $\Psi_C$ since the same proof with simple modification also applies to $\Phi$.

Denote by $v$ the valuation on $F$. We may choose a constant $c$ such that whenever $m \geq c$, the exponential map

$$\varpi^m \mathcal{O}_F \to F^\times$$

$$t \mapsto e^t = \sum_{i \geq 0} \frac{t^i}{i!}$$

converges and we have for $t \in \varpi^m \mathcal{O}_F$:

\[
\begin{align*}
\left(\ast\right) & \quad \begin{cases} 
  v\left(\frac{e^t + e^{-t}}{2} - 1\right) = v\left(t^2 / 2\right). \\
  v\left(\frac{e^t - e^{-t}}{2}\right) = v\left(t\right).
\end{cases}
\end{align*}
\]

Let $K_m$ be the image of $\varpi^m \mathcal{O}_F$. It is easy to see that $K_m = 1 + \varpi^m \mathcal{O}_F$.

We also choose an integer (the conductor of $\psi$) $d$ such that $\psi(a) = 1$ if $v(a) \leq d$ and $\psi(a) \neq 1$ for some $x$ with $v(x) = d - 1$.

Now we choose $\ell \in \mathbb{Z}$ such that

\[
\left(\ast\ast\right) \quad \ell > 4c - 2d + 10.
\]

Now assume that $v(a) < -\ell$. To explain the idea and to warm up, we first show that when $v(a) < -\ell$, $\Psi_C(a) = 0$ unless $a \in F^\times$. Suppose that $a$ is not a square, then $|a/x \pm x| = \max\{|x|, |a/x|\} \geq |a|^{1/2}$.

$$\Psi_C(a) = \sum_{v(a/C) < i < v(C)} \int_{v(x) = i} \psi(ax^{-1} + x) \eta(x) d^x x$$

For a fixed $i$, let $n > c$ be such that

$$n + \min\{i, v(a) - i\} < d, 2n + \min\{i, v(a) - i\} > d.$$

Such $n$ obvious exists due to (\ast\ast). Then we have a nontrivial character of $\varpi^n \mathcal{O}_F$:

$$t \mapsto \psi(ax^{-1} e^{-t} + xe^t) \eta(x) = \psi((ax^{-1} - x)t + (ax^{-1} + x)t^2 / 2 + \ldots) = \psi((ax^{-1} - x)t) \eta(x).$$

Hence the integral over $v(x) = i$ can be break into a sum of integrals over $xK_n$ where $x$ runs over $\varpi^n \mathcal{O}_F/K_n$. But obviously each term is of the form

$$|x|^{-1} \int_{K_n} \psi(ax^{-1}k^{-1} + xk) \eta(x) d^x k = |x|^{-1} \int_{\varpi^n \mathcal{O}_F} \psi(ax^{-1} e^{-t} + xe^t) \eta(x) = |x|^{-1} \eta(x) \psi((ax^{-1} - x)t) dt = 0.$$
Now we may assume that \(a = b^2\) is a square. A change of variable yields:

\[
\Psi_C(a) = \eta(b) \int_{|b/C| < x < |C/b|} \psi(b(x^{-1} + x))\eta(x)d^\infty x.
\]

For a fixed \(x\), we look for an integer \(n\) such that

\[
\begin{cases}
n + v(b) + v(x - x^{-1}) < d, \\
2n + v(b) > d.
\end{cases}
\]

For example we may take

\[
n = 1 + \left[\frac{(d - v(b))/2}{2}\right] \quad \text{(due to (**))}.
\]

Then the second condition becomes:

\[
(***) \quad v(x - x^{-1}) < d - 1 - v(b) - [(d - v(b))/2] = [(d - 1 - v(b))/2].
\]

If this last inequality holds, the \(v(\chi k - x^{-1}k^{-1}) = v(x - x^{-1})\) if \(k \in K_n\): this is obvious if \(v(x) \neq 0\); if \(v(x) = 0\), then

\[
(x - x^{-1}) - (\chi k - x^{-1}k^{-1}) = x(1 - k)(1 + x^{-2}k^{-1})
\]

has valuation at least \(v(1 - k) \geq n > v(x - x^{-1})\). In particular, for those \(x\) satisfying (***)

we may write the integral as a disjoint union of the form

\[
\sum_x x^{-1} \eta(x)x^{-1} \int_{x^n \mathcal{O}_F} \psi(b(x^{-1} - x)t)dt = 0.
\]

Therefore, we have proved that the only possible non-zero contribution comes from those \(x\) violating (***)

In particular, \(|\Psi_C(a)|\) is bounded by the volume of \(x\) such that \(v(x - x^{-1}) \geq [(d - 1 - v(b))/2]\). It is easy to see that the volume is bounded by \(\alpha|b|^{-1/2} = \alpha|a|^{-1/4}\) for some constant \(\alpha\) independent of \(C\).

\[\square\]

Remark 14. Just like [50, Prop. VIII.1], the same argument in the proof above actually yields a formula of \(\Psi(a)\) and \(\Phi(a)\) for large \(a\).

Lemma 4.11. When \(\frac{3}{4} < \text{Re}(\chi) < 1\), we have

\[
\gamma(\chi, \psi)\gamma(\chi \eta, \psi) = \int_F \Psi(a)\chi^{-1}(a)da
\]

where the integral converges absolutely. Similarly, \(\frac{3}{4} < \text{Re}(\chi) < 1\) we have

\[
\gamma(\chi E, \psi_E) = \int_F \Phi(a)\chi^{-1}(a)da.
\]

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Proof. We have for $C$ large enough
\[
\gamma(\chi, \psi) \gamma(\chi \eta, \psi) = \int_{|x| < C} \psi(x) \hat{\chi}(x) d^x x \int_{|x| < C} \psi(y) \hat{\eta}(y) d^y y.
\]
It is equal to
\[
\int_{|x|, |y| < C} \psi(x + y) \hat{\chi}(xy) \eta(y) d^x x d^y y
= \int_{F^\times} \left( \int_{|a|/C < |y| < C} \psi(a/y + y) \eta(y) d^y y \right) \hat{\chi}(a) d^a a
= \int_{F^\times} \Psi_C(a) d^a a.
\]
Note that $|\Phi_C(a)|$ is bounded by $\beta_1 \log |a| + \beta_2$ when $a$ is small and bounded by $\alpha |a|^{1/4}$ when $a$ is large. The result now follows from Lebesgue’s dominance convergence theorem. Similarly the second result follows noting that $|\Phi(a)|$ is bounded by a constant and when $a$ is small and bounded by $\alpha |a|^{1/4}$ when $a$ is large. \hfill \qed

**Theorem 4.12.** We have
\[
\Psi(a) = \Phi(a) \lambda_{E/F}(\psi).
\]

**Proof.** Note that $L(\text{Ind}_{K/F} 1_{K'}, s) = L(1_{K'}, s)$. Thus we have an equivalent form of Langlands constant:
\[
\lambda_{K/F}(\psi) := \frac{\gamma(\text{Ind}_{K/F} 1_{K}, s, \psi)}{\gamma(1_{K}, s, \psi_K)}.
\]
Then for all characters $\chi$ of $F^\times$ such that $\frac{3}{4} < Re(\chi) < 1$, we have
\[
\int_{F} \Psi(a) \chi^{-1}(a) da = \lambda_{E/F}(\psi) \int_{F} \Phi(a) \chi^{-1}(a) da.
\]
And both integrals converges absolutely. Now the theorem follows easily. \hfill \qed

### 4.3 A property under base change

For later use we need a property of the Langlands constant of a quadratic extension under base change.

If $K = K_1 \times K_2 \times ... \times K_m$, we define $\lambda_{K/F}(\psi)$ as the product $\prod_{i=1}^{m} \lambda_{K_i/F}(\psi)$. Similarly, $\epsilon(\text{Ind}_{K/F} 1_{K}, 1/2, \psi)$ is by definition $\prod_{i=1}^{m} \epsilon(\text{Ind}_{K_i/F} 1_{K_i}, 1/2, \psi)$.

Recall that for an arbitrary field extension $K/F$ of degree $d$, we may define a discriminant $\delta_{K/F} \in F^\times/(F^\times)^2$ as follows. Choose a $F$-basis $a_1, ..., a_d$ of $K$ and let $\sigma_1, ..., \sigma_d$ be all $F$-embeddings of $K$ into an algebraic closure of $F$. Then it is easy to see that
\[
det(\sigma_j(a_i))_{1 \leq i, j \leq d} \in F^\times.
\]
And if we change the $F$-basis, this only changes by a square in $F^\times$. So we define $\delta_{K/F}$ to be the class in $F^\times/(F^\times)^2$ of $\det(\alpha_i^{d_j})^2$. If $K = F(\alpha)$ and $\alpha$ has minimal polynomial $f$, then $\delta_{K/F}$ is the class of the discriminant of the polynomial $f$. In particular, in this case we can choose a representative of $\delta_{K/F}$ such that it lies in $(-1)^{d(d-1)/2}N_{K/F}K^\times$. Finally, for a quadratic character $\eta$ of $F^\times$, it makes sense to evaluate $\eta(\delta_{K/F})$. Finally it is evident how to extend the definition to a product of fields $K = K_1 \times K_2 \times ... \times K_m$.

The major property we need is the following.

**Theorem 4.13.** Let $E$ be a quadratic extension and let $F'/F$ be a field extension of degree $d$. Let $E' = E \otimes_F F'$. Then we have

$$\lambda_{E'/F'}(\psi_{F'}) = \lambda_{E/F}(\psi_F)\eta_{E/F}(\delta_{F'/F}).$$

First of all we have the following simple observation. If we replace $\psi$ by $\psi_a$, $a \in F^\times$, the RHS changes by a factor $\eta_{E/F}(a)^d$ and the LHS changes by a factor $\eta_{E'/F'}(a) = \eta_{E/F}(N_{F'/F}a) = \eta_{E/F}(a^d)$. Therefore it suffices to prove the identity for any one choice of $\psi$.

Note that the theorem is equivalent to

$$\epsilon(\eta_{E'/F'}, 1/2, \psi_{F'}) = \epsilon(\eta_{E/F}, 1/2, \psi_F)^d \cdot \eta_{E/F}(\delta_{F'/F}).$$

**Lemma 4.14.** If $E/F$ is unramified, then the theorem holds.

**Proof.** For a $p$-adic local field $F$ we let $\varpi_F$ denote a fixed uniformizer. The level of a character $\psi$ is by definition the maximal integer $k$ such that $\psi$ is trivial on $\varpi_F^{-k}O_F$. We choose $\psi$ to have level zero. Since $\eta_{E/F}$ is unramified, we have

$$\epsilon(\eta_{E/F}, 1/2, \psi_F) = 1.$$

Now $\psi_{F'}$ has level denoted by $k$ which is equal to the valuation of the different $D_{F'/F}$, namely $D_{F'/F} = (\varpi_{F'})^k$. Let $e$ be the ramification index and $f = d/e$.

**Case I:** $f$ is even. Then $E \subset F'$ is a subfield. In particular, $N_{F'/F}F'^\times \subset N_{E/F}E^\times$. In this case, LHS is equal to $1$ as $\eta_{E'/F'}$ is trivial. As we may choose a representative of $\delta_{F'/F}$ in $(-1)^{d(d-1)/2}N_{K/F}K^\times \subset N_{E/F}E^\times$, we see that $\eta_{E/F}(\delta_{F'/F}) = 1$.

**Case II:** $f$ is odd. Then $\epsilon(\eta_{E/F}, 1/2, \psi_F) = (-1)^k$. Note that $N_{F'/F}(\varpi_{F'}) = (\varpi_F)^f$. So we have $N_{F'/F}D_{F'/F} = (\varpi_F)^{k'f}$. Since $f$ is even, $kf$ and $k$ have the same parity. As the valuation of $N_{F'/F}D_{F'/F}$ has the same parity as the valuation of $\delta_{F'/F}$, we see that $\eta_{E/F}(\delta_{F'/F}) = (-1)^{k'f} = (-1)^k$ as desired.

**Lemma 4.15.** If $E/F$ is archimedean, then the theorem holds.

**Proof.** The case $F = \mathbb{C}$ is obvious. Now we assume that $F = \mathbb{R}$ and $E = \mathbb{C}$. Then the only non-trivial case we need to consider is when $F' = \mathbb{C}$. Then the LHS is equal to one. For the RHS, we have

$$\lambda_{E/F}(\psi)^2 = \eta_{E/F}(-1) = -1.$$

And

$$\eta_{E/F}(D_{F'/F}) = -1.$$

This completes the proof.
We now treat the general case for $E/F$. We use a global argument. To do so we want to
globalize the quadratic extension $E/F$. We use the following lemma from [12].

**Lemma 4.16.** Let $E/F$ be a quadratic extension of non-archimedean local fields. Then there
exists a totally real number field $F$ with $F_{v_0}$ as its completion at a place $v_0$ of $F$, and a quadratic
totally imaginary extension $E$ of $F$ with corresponding completion $E$ such that $E$ is unramified
over $F$ at all finite places different from $v_0$.

Resulting from the fact that the global epsilon factor satisfies

$$
\epsilon(Ind_E/F 1_E, s, \psi) = \epsilon(1_E, s, \psi),
$$

we have a product formula

$$
\prod_v \lambda_{E/F}(\psi_v) = 1,
$$

where $E_v$ is a product of field extensions of $F_v$ and $v$ runs over all places of $F$. Also choose
an extension $F'$ of $F$ such that $v_0$ is inert and $F_{v_0}' \simeq F'$. Such choice obviously exists. Then
it is clear that the theorem holds for all $E_v/F_v$ at those $v \neq v_0$. By the global identity and
$\prod_v \eta_{E/F}(\delta_{F'/F}) = 1$, we immediate get the desired identity at the place $v_0$. This completes the
proof of Theorem 4.13.

### 4.4 All Fourier transforms preserve transfer

Now we need to consider simultaneously the linear and the unitary case. Let $E/F$ be a fixed
quadratic field extension. We set up some notations. We will use $\mathcal{V}'$ to denote $gl(n, F) \times F^n \times
F_n$. There are two isomorphism classes of hermitian spaces which we will denote by $W_1, W_2$
respectively. Then we let $\mathcal{V}_i$ denote $\mathfrak{U}(W_i) \times W_i$ for $i = 1, 2$.

Note that we have an obvious way to match the partial Fourier transforms on $\mathcal{V}'$ and $\mathcal{V}_i$.
Recall that all Haar measures to define Fourier transform are chosen to be self-dual.

**Theorem 4.17.** For any a fixed Fourier transforms $\mathcal{F}$, there exists a constant $\nu \in \mu_4$ depending only on $n, \psi, E/F$ and the Fourier transform $\mathcal{F}$ with the following property: if $f \in \mathcal{C}_c^\infty(\mathcal{V})$
and the pair $(f_i \in \mathcal{C}_c^\infty(\mathcal{V}_i))_{i=1,2}$ are transfer of each other, so are $\nu \mathcal{F}(f)$ and $(\mathcal{F}(f_i))_{i=1,2}$.

We now let $\mathcal{F}_a$ ($\mathcal{F}_b$, $\mathcal{F}_c$, resp.) be the Fourier transform with respect to the total space
(the subspace $\mathfrak{U}(W)$, $W$, resp.). Consider the following three statements:

- **A**: For all $E/F$, $\psi$, and $W_i$ of dimension $n$, there is a constant $\nu \in \mu_4$ with the property:
if $f \in \mathcal{C}_c^\infty(\mathcal{V}')$ and $f_i \in \mathcal{C}_c^\infty(\mathcal{V}_i)$ ($i = 1, 2$) match, then so do $\nu \mathcal{F}_a(f)$ and $\mathcal{F}_a(f_i)$.

- **B**: For all $E/F$, $\psi$, and $W_i$ of dimension $n$, there is a constant $\nu \in \mu_4$ with the property:
if $f \in \mathcal{C}_c^\infty(\mathcal{V}')$ and $f_i \in \mathcal{C}_c^\infty(\mathcal{V}_i)$ ($i = 1, 2$) match, then so do $\nu \mathcal{F}_b(f)$ and $\mathcal{F}_b(f_i)$.

- **C**: For all $E/F$, $\psi$, and $W_i$ of dimension $n$, if $f \in \mathcal{C}_c^\infty(\mathcal{V}')$ and $f_i \in \mathcal{C}_c^\infty(\mathcal{V}_i)$ ($i = 1, 2$) match, then so do $\lambda_{E/F}(\psi)^{-n} \mathcal{F}_c(f)$ and $\mathcal{F}_c(f_i)$.
\[ A_{n-1} \Rightarrow B_{n}. \] In this part, we will use \( \hat{f} \) to denote the Fourier transforms with respect to \( \mathfrak{gl}_n \) and \( u(W) \). We first consider the linear side. We let \( W = F_n \times F^n \) (we use \( W \) as it is the counterpart in the general linear case of the Hermitian space \( W \)) and consider it as an \( F \times F \)-module of rank \( n \) with the hermitian form \( (w, w) = wv \) if \( w = (u, v) \in F_n \times F^n \).

By the local trace formula, for \( w \in W \) with \( (w, w) \neq 0 \), we have for any \( f \in \mathcal{C}_c^{\infty} (\mathcal{V}') \) and \( g \in \mathcal{C}_c^{\infty} (\mathfrak{gl}_n (F)) \):

\[
(4.3) \quad \int_{\mathfrak{gl}_n (F)} O_{X,w}^0 (f) \omega(X, w) g(X) dX = \int_{\mathfrak{gl}_n (F)} O_{X,w}^0 (\hat{f}) \omega(X, w) \hat{g}(X) dX.
\]

Let \( (w) \perp \) be the orthogonal complement of \( (F \times F)w \) in \( W \). Up to the \( H = GL_{n,F} \)-action, we may choose \( w \) of the form \((e, dw')\) where \( e = (0, \ldots, 0, 1) \) and \( d \in F^\times \) is the hermitian norm \((w, w)\). Then the stabilizer of \( w \) in \( H \) can be identified with \( GL_{n-1,F} \) (with the embedding into \( GL_{n,F} \) as before). If \( h \in GL_{n-1,F} \), we have \( O_{X,w}^0 (f) = O_{X,wh}^0 (f) = O_{X,h,w}^0 (f) \). Then we may rewrite the integral in LHS of the local trace formula (Theorem 4.6) as

\[
\int_{\mathfrak{gl}_n (F)} O_{X,w}^0 (f) \omega(X, w) g(X) dX = \int_{Q(F)} O_{X,w}^0 (f) \left( \int_{GL_{n-1,F}} \omega(X^h, w) g(X^h) dh \right) dq(X),
\]

where \( Q_{n-1} = \mathfrak{gl}_n / GL_{n-1} \), \( q \) is the quotient morphism and the measure is a suitable one on \( Q(F) \) such that for all \( g \in \mathcal{C}_c^{\infty} (\mathfrak{gl}_n (F)) \):

\[
\int_{\mathfrak{gl}_n (F)} g(X) dX = \int_{Q(F)} \left( \int_{GL_{n-1,F}} g(X^h) dh \right) dq(X).
\]

Note that \( \omega(X^h, w) = \eta(h) \omega(X, w) \) for \( h \in GL_{n-1,F} \). Now it is easy to see that when we restrict the transfer factor \( \omega(X, w) \) to \( X \in \mathfrak{gl}_n \), it is a constant multiple of the transfer factor we have used to define the \( GL_{n-1,F} \)-orbital integral. Moreover, this constant depends only on \( w \) and is denoted by \( c_w \). So we have

\[
\int_{GL_{n-1,F}} \omega(X^h, w) g(X^h) dh = \omega(X, w) \int_{GL_{n-1,F}} g(X^h) \eta(h) dh = c_w O_{X,w}^0 (g).
\]

Then this depends only on the \( GL_{n-1,F} \)-orbit of \( X \).

Similar result holds for the RHS. The constant \( c_w \) is then canceled. Replacing \( g \) by \( \hat{g} \), we may rewrite the local trace formula as

\[
(4.4) \quad \int_{Q(F)} O_{X,w}^0 (f) O_X^0 (\hat{g}) dq(X) = \int_{Q(F)} O_{X,w}^0 (\hat{f}) O_X^0 (g) dq(X).
\]

Note that the Fourier transform here is \( \mathcal{F}_b \) for \( GL_n \)-action on \( \mathcal{V}' \) but it is the \( \mathcal{F}_a \) for the \( GL_{n-1} \)-action on \( \mathfrak{gl}_n \).

Now for the unitary case we also have a similar equality for \( i = 1, 2 \) (without the issue of transfer factors)

\[
\int_{Q(F)} O_{X,w} (f_i) O_X (\hat{g}_i) dq(X) = \int_{Q(F)} O_{X,w} (\hat{f}_i) O_X (g_i) dq(X).
\]
Here the stabilizer of \( w \) in \( \Omega(W) \) replaces \( GL_{n-1,F} \). Note that we have identified \( Q \) with \( Q_i \) as before.

Now suppose that \( f \leftrightarrow (f_i) \). We want to show that for some constant \( \nu \):

\[
\nu O_{X_i, w_i}^n(\hat{f}_i) = O_{X, w}^n(\hat{f})
\]

for any strongly regular semisimple \((X^0, w^0) \leftrightarrow (X_i^0, w_i^0)\). This would imply the equality for all matching regular semisimple elements by the local constancy of orbital integrals.

We may choose \( g \leftrightarrow (g_i) \) such that

- Both are supported in the regular semisimple locus.
- There exists a small (open and compact) neighborhood \( \omega \) of \( q(X^0) \in Q(F) \) with the following property: (1) the functions on \( \omega \) given by \( q(X) \mapsto O_{X, w}^n(\hat{f}) \) and \( q(X_i) \mapsto O_{X_i, w_i}^n(\hat{f}_i) \) are constant; (2) the functions on \( Q(F) \) given by \( q(X) \mapsto O_X^n(g) \) and \( q(X_i) \mapsto O_{X_i}(g_i) \) are the characteristic function \( 1_\omega \).

This is clearly possible by Lemma 3.16.

For such choice we have

\[
\int_Q O_{X, w}^n(f) O_X^n(\hat{g}) dq(X) = O_X^n(\hat{g})(\int_\omega dq(X)), \quad i = 1, 2.
\]

Now by \( A_{n-1} \), we have for some constant \( \nu \)

\[
\nu O_X^n(\hat{g}) = O_{X_i}(\hat{g}_i)
\]

whenever \( X \leftrightarrow X_i \). Now the desired equality (4.5) follows immediately for the same constant \( \nu \) as in \( A_{n-1} \). This shows that \( A_{n-1} \Rightarrow B_n \).

\( C_1 \leftrightarrow C_n \) It suffices to show \( C_1 \Rightarrow C_n \). Now we will use \( \hat{f} \) to denote the Fourier transform with respect to \( F_n \times F^n \) and \( W \) respectively. We want to show that if \( f \) and \( f_i \) match, then for strongly regular \((X, w) \leftrightarrow (X_i, w_i)\), we have

\[
\lambda_{E/F}(\psi)^n O_{X, w}^n(\hat{f}) = O_{X_i, w_i}(\hat{f}_i).
\]

For \( X \in \mathfrak{gl}_n(F) \) regular semisimple, let \( T \) be the centralizer of \( X \) in \( H \) and \( t \) its Lie algebra. Then \( T \) is isomorphic to \( \prod_j F_j^{r_j} \) for some field extensions \( F_j/F \) of degree \( n_j \) with \( \sum_{j=1}^r b_j = n \).

For \( X_i \in \mathfrak{u}(W_i) \) regular semisimple, let \( T_i \) be the centralizer of \( X_i \) in \( H_i \) and \( t_i \) its Lie algebra. Let \( E_j := E \otimes_F F_j \). As \( X, X_i \) have the same characteristic polynomial, we know that \( T_i \) is isomorphic to \( \prod_j \text{Res}_{F_j/F} E_j^1 \) for the same tuple of field extensions \( F_j/F \). Here \( E^1 \) is the kernel (as an algebraic group) of norm homomorphism \( N_{E/F} : E^\times \to F^\times \). Let \( F' := \prod_i F_j, E' := F' \otimes_F E \). By [3, Sec.5,p.18], \( W \) is a rank one Hermitian space over \( E' \) with unitary group \( U(W, E'/F') \cong T \). We may identify \( F' \) with the sub algebra \( F[X] \subset \mathfrak{gl}_n \).

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For a more intrinsic exposition, we let $M \times M^*$ denote $F_n \times F^n$ and $\mathfrak{gl}_n = End(M)$. We may describe the transfer factor as follows:

$$\omega(X, u, v) = \eta \left( \frac{u \wedge Xu \wedge X^2u \ldots \wedge X^{n-1}u}{\omega_0} \right), \quad X \in End(M), \; u, v \in M^*,$$

where $\omega_0$ a fixed generator of the line $\wedge^n_F M$. Obviously if we change the generator $\omega_0$, the transfer factor only changes by a constant in $\{\pm 1\}$.

Then under the action of $F' = F[X]$, $M$ is a free $F'$-module of rank one. In this way, $M^* = Hom_F(M, F)$ is canonically isomorphic to $Hom_{F'}(M, F')$. Indeed, we may define an $F'$-linear pairing $(\cdot, \cdot)_{F'} : M \times M' \to F'$ such that for all $\lambda \in F', x \in M, y \in M'$ we have

$$tr_{F'/F}(\lambda x, y)_{F'} = (\lambda x, y)_F.$$

Fixing a generator of $F' = F[X]$, $M \simeq M'$ we define a transfer factor $\omega(w) \in \{\pm 1\}$. We also have a compatibility $\eta_{E/F}(N_{F'/F}x) = \eta_{E'/F'}(x)$ and $N_{F'/F}x = det(x)$ when $x \in F' = F[X]$.

We then have an inversion formula as follows.

**Lemma 4.18.** For a regular semisimple $X \in \mathfrak{gl}_n(F)$ with centralizer $T \simeq \prod_j \text{Res}_{F_j/F} GL_1$ as above, let $\hat{\kappa}^n(w, w')$ be the locally constant $T \times T$-invariant function on $W_{rs} \times W_{rs} \to \mathbb{C}$ given by Corollary 4.7. Then we have

$$\hat{O}_{X,w}^n(f) = \eta_{E/F}(\delta_{F'/F}) \int_{Q(F')} O_{X,w'}^n(f) \kappa^n(w, w') dq(w'),$$

where $Q = (M \times M^*)/\text{Res}_{F'/F} GL_1$.

**Proof.** Without loss of generality, we may assume that $f = \phi \otimes \varphi, \phi \in C_c^\infty(\mathfrak{gl}_n(F)), \varphi \in C_c^\infty(M \times M^*)$. We now see that the orbital integral can be rewritten as

$$O_{X,w}^n(\phi \otimes \bar{\varphi}) = \omega(X, w) / \omega(w) \int_{T \backslash H} \phi(X^h) \eta(h) O_w^0(h \bar{\varphi}) dh$$

$$= \omega(X, w) / \omega(w) \int_{T \backslash H} \phi(X^h) \eta(h) O_w^n(h \bar{\varphi}) dh,$$

where we write $h \varphi(w) = \varphi(w^{h^{-1}})$.

By Corollary 4.7, we have

$$O_w^n(\bar{\varphi}) = \int_{Q(F')} O_{w'}^n(\varphi) \kappa^n(w, w') dq(w').$$

(4.7)

Reversing the argument we obtain

$$O_{X,w}^n(\phi \otimes \bar{\varphi}) = \omega(X, w) \omega(w) \int_{Q(F')} \omega(X, w') \omega(w') O_{X,w'}^n(\phi \otimes \varphi) \kappa^n(w, w') dq(w').$$
We leave the reader to verify that
\[\omega(X, w)\omega(X, w') = \eta(\delta_X)\omega(w)\omega(w')\]
where \(\delta(X)\) is the discriminant of the characteristic polynomial of \(X\). In particular, \(\eta(\delta_X) = \eta(\delta_{F'/F})\). Note that the product \(\omega(X, w)\omega(X, w') (\omega(w)\omega(w'), \text{resp.})\) does not even depend on the choice of the generator of \(\Lambda^*_F M (\Lambda^*_F M = M, \text{resp.})\).

Similarly we also have an inversion formula for the unitary case with a different kernel function \(\kappa_i(w_i, w'_i)\). Finally we note that the kernel functions are given by the Kloosterman sums relative to \(E'/F'\). By Theorem 4.12, we have
\[\kappa^i(w, w') = \lambda_{E'/F'}(\psi_{F'})\kappa_i(w_i, w'_i)\]
whenever \(w, w'\) match \(w_i, w'_i, i = 1, 2\).

Now the proof of \(C_1 \Rightarrow C_n\) follows from the inversion formulae and the base change property (Theorem 4.13):
\[\lambda_{E'/F'}(\psi_{F'}) = \lambda_E^F(\psi_F)\eta(\delta_{F'/F}).\]

Remark 15. It is easy to see that for fixed \(X, X_i\), the statement \(C_1\) implies that up to a constant multiple the partial Fourier transform have matching orbital integrals for those elements with first components \(X, X_i\). The lengthy computation of the Davenport–Hasse relations is to show that this constant, a priori depending on \(X, X_i\), is indeed independent of the choice of \(X, X_i\).

\(B_n + C_n \Rightarrow A_n\) This is obvious since \(\mathcal{F}_a = \mathcal{F}_b\mathcal{F}_c\).

### 4.5 Completion of the proof of Theorem 2.6

The following homogeneity result enables us to deduce the existence of transfer from the compatibility with Fourier transform. In §3.1, we have reduced the existence of transfer on groups to the Lie algebra version, namely it suffices to show Conjecture \(\mathcal{C}_{n+1}\). Obviously, the statement is equivalent to the corresponding assertion on the following subspaces:

\[\mathfrak{sl}_n \times F_n \times F^n\]
and for Hermitian \(W\)
\[\mathfrak{su}(W) \times W.\]

The reason is that the group \(GL_n (U(W), \text{resp.)}\) acts trivially on the orthogonal complement.

We let \(\mathcal{V}\) be either \(\mathfrak{sl}_n(F) \times F^n \times F_n\) or \(\mathfrak{su}(W) \times W\) and let \(\eta\) be the quadratic character associated to \(E/F\) in the former case and the trivial character in the latter.

**Theorem 4.19** (Aizenbud-Gourevitch). *There is no distribution \(T\) on \(\mathcal{V}\) with the following property*
(1) $T$ is $(H, \eta)$-invariant (hence so is $\hat{T}$).

(2) $T$, $\hat{T}^V$, $\hat{T}^W$, $\hat{T}^u(W)$ are all supported in the nilpotent cone $N$.

More precisely, it is proved in [1, Theorem 6.2.1] for the case $\eta = 1$. But the same proof goes through for the nontrivial quadratic $\eta$.

**Corollary 4.20.** Let

$$C_0 = \cap_T \text{Ker}(T)$$

where $T$ runs over all $(H, \eta)$-invariant distributions. Then we have $C_c^\infty(V)$ is the sum of $C_0$ and the image of all Fourier transforms of $C_c^\infty(V - N)$. Equivalently, any $f \in C_c^\infty(V)$ can be written as

$$f = f_0 + f_1 + \hat{f}_2^V + \hat{f}_3^W + \hat{f}_4^u(W)$$

where $f_0 \in C_0$, $f_i \in C_c^\infty(V - N)$, $i = 1, 2, 3$.

**Proof.** Let $\mathcal{C}$ be the subspace spanned by $C_0$ and the image of all Fourier transforms of $C_c^\infty(V - N)$. If the quotient $L = C_c^\infty(V)/\mathcal{C}$ is not trivial, then there must exist a nontrivial linear functional on $L$. This induces a distribution $T$ on $V$. As $T$ is zero on $C_0$, $T$ is $(H, \eta)$-invariant. As $T$ is zero on $C_c^\infty(V - N)$, it is supported on $N$. Similarly, all Fourier transforms $\hat{T}^V$ etc. are all $(H, \eta)$-invariant and supported on $N$. This contradicts Theorem 4.19 above.

Finally we may prove Theorem 2.6:

**Proof.** To abuse notations, we still use $Q = A^{2n}$ to denote the categorical quotient of $V$ by $H$. By the localization principle Prop. 3.8, it is enough to prove the existence of local transfers at all $z \in Q(F)$. We will prove this by induction on $n$. If $z \in Q(F)$ is non-zero, the stabilizer of $z$ is strictly smaller than $H$. By Prop. 3.20, the local transfer around $z$ is implied by the local transfer around 0 for the sliced representations. This representations are (possibly a product) of the same type with smaller dimension (with possibly a base change of the base field $F$). Then by induction hypothesis, we may assume the existence of local transfers at all non-zero semisimple $z \in Q(F)$. Therefore, by induction hypothesis, we have the existence of smooth transfer for functions supported away from the nilpotent cone. Now by Corollary 4.20, it suffices to show the existence of smooth transfer for $\hat{f}$ for every one of the three Fourier transforms and $f$ supported away from the nilpotent cone.. But this follows from Theorem 4.17. We thus complete the proof.

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A Appendix: Spherical characters for a strongly tempered pair
By Atsushi Ichino, Wei Zhang

Let $F$ be a non-archimedean local field. We consider a pair $(G, H)$ where $G$ is a reductive group and $H$ is a subgroup. We will also denote by $G, H$ the sets of $F$-points. We will assume that $H$ a spherical subgroup in the sense that $X := G/H$ with the $G$-action from left is a spherical variety. Following [47, sec. 6], we say that the pair $(G, H)$ is strongly tempered if for any tempered unitary representation $\pi$ of $G$, and any $u, v \in \pi$, the associated matrix coefficient $\phi_{u,v}(g) := \langle \pi(g)u, v \rangle$ (where $\langle \cdot, \cdot \rangle$ is the hermitian $G$-invariant inner product) satisfies

$$\phi_{u,v}|_H \in L^1(H).$$

To check whether a pair $(G, H)$ is strongly tempered, one uses the Harish-Chandra function $\Xi$. Let $\pi_0$ be the normalized induction of the trivial representation of a Borel $B$ of $G$ and let $v_0$ be the unique spherical vector such that $\langle v_0, v_0 \rangle = 1$. Then $\Xi$ is the matrix coefficient:

$$\Xi(g) = \langle gv_0, v_0 \rangle.$$  

We have $\Xi(g) \geq 0$ for any $g \in G$. Then $(G, H)$ is strongly tempered if $\Xi|_H \in L^1(H)$.

The major examples are those appeared in the Gan–Gross–Prasad conjecture:

- Let $V$ be an orthogonal space of dimension $n + 1$ and $W$ a codimension one subspace (both non-degenerate). Let $SO(V)$ and $SO(W)$ the corresponding special orthogonal groups. Let $G = SO(W) \times SO(V)$ and $H \subset G$ is the graph of the embedding $SO(W) \hookrightarrow SO(V)$ induced by $W \hookrightarrow V$.

- $G = GL_n \times GL_{n+1}$, $H$ is the graph of the embedding of $GL_n \hookrightarrow GL_{n+1}$ given by $g \mapsto \text{diag}(g, 1)$.

- Let $E/F$ be a quadratic extension of fields. Let $V$ be a hermitian space of dimension $n + 1$ and $W$ a codimension one subspace (both non-degenerate). Let $U(V)$ and $U(W)$ the corresponding unitary groups. Let $G = U(W) \times U(V)$ and $H \subset G$ is the graph of the embedding $U(W) \hookrightarrow U(V)$ induced by $W \hookrightarrow V$.

A proof of the fact that these pair $(G, H)$ are strongly tempered can be found in [28] for the orthogonal case and [23] for the linear and unitary cases.

Assume that $(G, H)$ are strongly tempered. Then for any tempered representation $\pi$ of $G$, following [28] we may define a matrix coefficient integration

$$\nu(u, v) := \int_H \langle \pi(h)u, v \rangle dh.$$  

Obviously $\nu \in \text{Hom}_{H \times H}(\pi \otimes \tilde{\pi}, \mathbb{C})$. The integral is absolutely convergent by the strong temperedness.

The following is a conjecture of Ichino–Ikeda in their refinement of the Gan–Gross–Prasad conjecture.
**Theorem A.1** (Sakellaridis–Venkatesh,[47]). Assume that $(G,H)$ is strongly tempered and $X = G/H$ is wavefront. Let $\pi$ be an irreducible representation. Assume that $\pi$ is a discrete series representation or $\pi = \text{Ind}_P^G(\sigma)$ for a discrete series representation $\sigma$ of the Levi of a parabolic subgroup $P$. Then $\text{Hom}_H(\pi, \mathbb{C}) \neq 0$ if and only if $\nu$ does not vanish identically.

The result is also proved in the orthogonal case by Waldspurger.

Let $\pi$ be a representation as in the Theorem above. We further assume that

$$\dim \text{Hom}_H(\pi, \mathbb{C}) \leq 1.$$ 

All the examples we list earlier satisfies this condition.

Let $\ell \in \text{Hom}_H(\pi, \mathbb{C})$. Then we define a spherical character associated to $\ell$ to be a distribution on $G$ such that

$$\theta_{\pi,\ell}(f) := \sum_{v \in \mathcal{B}(\pi)} \ell(\pi(f)v)\overline{\ell(v)}, \quad f \in C_c^\infty(G),$$

where $\mathcal{B}(\pi)$ is an orthonormal basis of $\pi$. The distribution is bi-$H$-invariant for the left and right translation by $H$. Obviously the distribution $\theta_{\pi,\ell}$ is non-zero if and only if the linear functional $\ell$ is non-zero.

This notes is to prove the following:

**Theorem A.2.** Assume that $\ell \neq 0$. Fix any open dense subset $G_r$ of $G$. Then the restriction of the distribution $\theta_{\pi,\ell}$ to $G_r$ is non-zero. Equivalently, there exists $f \in C_c^\infty(G_r)$ such that $\theta_{\pi,\ell}(f) \neq 0$.

By Theorem of Sakellaridis–Venkatesh above, there exist $v_0 \in \pi$ such that

$$\ell(v) = \int_H \langle \pi(h)v, v_0 \rangle dh, \quad v \in \pi.$$

**Lemma A.3.** For all $f \in C_c^\infty(G)$, we have

$$\theta_{\pi,\ell}(f) = \int_H \int_H \left( \int_G f(g)\Phi(h_2gh_1)dg \right) dh_1dh_2,$$

where

$$\Phi(g) = \langle \pi(h)v_0, v_0 \rangle.$$

**Proof.** The proof is analogous to that of [40, Thm. 6.1]. We may rewrite $\theta$ as

$$\theta_{\pi,\ell}(f) = \sum_{v \in \mathcal{B}(\pi)} \ell(\pi(f)v)\overline{\ell(v)}$$

$$= \sum_{v \in \mathcal{B}(\pi)} \int_H \langle v, \pi(f^*)\pi(h^{-1})v_0 \rangle dh \int_H \langle \pi(h^{-1})v_0, v \rangle dh.$$
By \( \varphi = \sum_{v \in B(\pi)} \langle \varphi, v \rangle v \) for any \( \varphi \in \pi \), we have
\[
\theta_{\pi, \ell}(f) = \sum_{v \in B(\pi)} \int_H \int_H \langle \pi(h_1^{-1})v_0, \pi(f^*)\pi(h_2^{-1})v_0 \rangle dh_1 dh_2
= \int_H \int_H \left( \int_G f(g) \langle \pi(h_2gh_1)v_0, v_0 \rangle dg \right) dh_1 dh_2.
\]

We consider the orbital integral of the matrix coefficient \( \Phi \) as in Lemma A.3:
\[
O(g, \Phi) = \int_H \int_H \Phi(h_1gh_2) dh_1 dh_2,
\]
as well as the orbital integral \( O(g, \Xi) \) of the Harish-Chandra function \( \Xi \).

**Lemma A.4.** The function \( g \mapsto O(g, \Xi) \) on \( G \) is locally \( L^1 \). Equivalently, for any \( f \in C_c^\infty(G) \),
the following integral is absolutely convergent
\[
\int_G |f(g)O(g, \Xi)| dg < \infty.
\]

**Proof.** Consider a special maximal compact open subgroup \( K \) of \( G \) such that we have the following relation for a suitable measure on \( K \):
\[
\int_K \Xi(gkg') dk = \Xi(g)\Xi(g').
\] (A.1)

Such \( K \) exists ([46]). Without loss of generality, we may assume that \( f \) is the characteristic function of \( KgK \) for some \( g \in G \). We then have
\[
\int_G |f(g)O(g, \Xi)| dg = \int_K \int_K \int_H \int_H \Xi(h_1k_1gh_2h_2) dh_1 dh_2 dk_1 dk_2.
\]

By (A.1) this is equal to
\[
\left( \int_K \int_H \Xi(h_1k_1g) dh_1 dk_1 \right) \left( \int_H \Xi(h_2) dh_2 \right).
\]

By (A.1) again, we obtain
\[
\Xi(g) \left( \int_H \Xi(h_1) dh_1 \right) \left( \int_H \Xi(h_2) dh_2 \right) < \infty.
\]

By Fubini theorem, this shows
\[
\int_G |f(g)O(g, \Xi)| dg < \infty.
\]
\[\square\]
Lemma A.5. Let $\Phi$ be the matrix coefficient as in Lemma A.3. The function $g \mapsto O(g, \Phi)$ on $G$ is locally $L^1$ and for any $f \in C_c^\infty(G)$, we have

$$\theta_{\pi,\ell}(f) = \int_G f(g)O(g, \Phi) dg.$$ 

Proof. For any tempered representation $\pi$ of $G$ and a matrix coefficient $\phi_{u,v}$ associated to $u, v \in \pi$, we have

$$|\phi_{u,v}(g)| \leq c\Xi(g), \quad g \in G$$

for some constant $c > 0$. This shows that $O(\cdot, \Phi) \in L^1(G)$.

Choose an increasing sequence of open compact subsets of $H$:

$$\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_n \subset \ldots \subset H, \quad \cup_n \Omega_n = H.$$ 

Now we define a partial orbital integral

$$O_n(g, \Phi) := \int_{\Omega_n} \int_{\Omega_n} \Phi(h_1gh_2) dh_1 dh_2.$$ 

It is clearly that $O_n(\cdot, \Phi)$ converges to $O(\cdot, \Phi)$ point-wisely. They are bounded point-wisely

$$|O_n(g, \Phi)| \leq O(g, \Xi), \quad g \in G.$$ 

Fix $f \in C_c^\infty(G)$. By Lemma A.4, the function $g \mapsto f(g)O(g, \Xi)$ is in $L^1(G)$. By Lebesgue’s dominated convergence theorem,

$$\lim_{n \to \infty} \int_G f(g)O_n(g, \Phi) dg = \int_G f(g)O(g, \Phi) dg.$$ 

On the other hand

$$\int_G f(g)O_n(g, \Phi) dg = \int_G f(g) \left( \int_{\Omega_n} \int_{\Omega_n} \Phi(h_1gh_2) dh_1 dh_2 \right) dg$$

$$= \int_{\Omega_n} \int_{\Omega_n} \int_G f(g) \Phi(h_1gh_2) dg dh_1 dh_2.$$ 

as $\Omega_n$ is compact as well as the support of $f$. This shows that

$$\lim_{n \to \infty} \int_G f(g)O_n(g, \Phi) dg = \int_H \int_H \left( \int_G f(g) \Phi(h_2gh_1) dg \right) dh_1 dh_2.$$ 

This is equal to $\theta_{\pi,\ell}(f)$ by Lemma A.3. ∎
Now we prove Theorem A.2. Since $\theta_{\pi,\ell}$ is non-zero, there exists $f \in C_c^\infty(G)$ such that

$$\theta_{\pi,\ell}(f) \neq 0.$$ 

Equivalently, by Lemma A.5,

$$\int_G f(g)O(g, \Phi)dg \neq 0.$$

As $G_r$ is open and dense, we may choose $f_n \in C_c^\infty(G_r)$, such that point-wisely on $G_r$ we have

$$\lim_{n \to \infty} f_n = f.$$

Without loss of generality, we may assume that $f \geq 0$ point-wisely and $f - f_n \geq 0$ point-wisely. Then we have

$$|(f(g) - f_n(g))O(g, \Phi)| \leq 2f(g)|O(g, \Phi)|$$

which is integrable on $G$. By Lebesgue’s dominated convergence theorem, we have

$$\lim_{n \to \infty} \int_G (f(g) - f_n(g))O(g, \Phi)dg = 0.$$

Since $\int_G f(g)O(g, \Phi)dg \neq 0$, we have for $n$ large enough,

$$\int_G f_n(g)O(g, \Phi)dg \neq 0,$$

or equivalently,

$$\theta_{\pi,\ell}(f_n) \neq 0.$$

As $f_n \in C_c^\infty(G_r)$, this completes the proof.

**Remark 16.** When comparing with the property of the usual character as established by Harish-Chandra, some questions remain for the spherical characters in this notes. For example, is the function $g \mapsto O(g, \Phi)$ continuous or even locally constant on an open dense subset? If $\pi$ is super-cuspidal, one may prove that the function $g \mapsto O(g, \Phi)$ is locally constant on some open dense subset in the unitary case listed in the beginning.

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