Quantum Mechanics and Representation Theory

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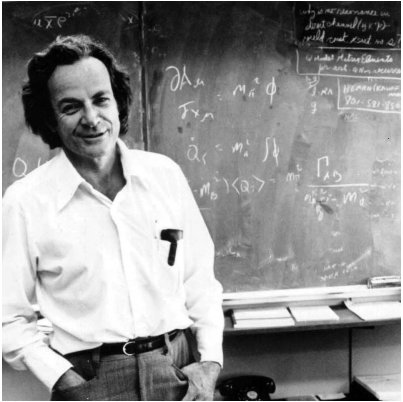
"No One Understands Quantum Mechanics"

"I think it is safe to say that no one understands quantum mechanics"
Richard Feynman
The Character of Physical Law, 1967



Understanding Quantum Mechanics

Feynman was contrasting quantum mechanics to general relativity, which could be expressed in terms of fields and differential equations. Understanding quantum mechanics requires different mathematical tools, less familiar to physicists, but widely used in mathematics.

Lie groups

Groups G that have the structure of a smooth manifold.

Unitary representations

Pairs (π, V), V a complex vector space with a Hermitian inner product, π a homomorphism

$$\pi : G \rightarrow U(V)$$

i.e. a unitary transformation $\pi(g)$ of V for each $g \in G$, with

$$\pi(g_1)\pi(g_2) = \pi(g_1 g_2)$$

Outline

Today would like to

- Explain how Lie groups and unitary representations are related to quantum mechanics, providing some sort of "understanding" of the structure of the subject.
- Advertise a book project in progress, based on a course taught last year. Hope to finish draft early next year, about 80 percent done, see my web-site at Columbia.
- Explain some unanswered questions about representation theory raised by physics.
- Quickly indicate some work in progress: possible relevance of some new ideas in representation theory ("Dirac cohomology") to physics.

What We Really Don't Understand About Quantum Mechanics

While representation theory gives insight into the basic structure of the quantum mechanics formalism, a mystery remains

The mystery of classical mechanics

We don't understand well at all how "classical" behavior emerges when one considers macroscopic quantum systems.

This is the problem of "measurement theory" or "interpretation" of quantum mechanics. Does understanding this require some addition to the fundamental formalism? **Nothing to say today about this.**

What is Quantum Mechanics?

Two Basic Axioms of Quantum Mechanics

- The states of a quantum system are given by vectors $\psi \in \mathcal{H}$ where $\mathcal{H}$ is a complex vector space with a Hermitian inner product.
- Observables correspond to Hermitian linear operators on $\mathcal{H}$

The mysterious part

An axiom of measurement theory is that an experiment will somehow put the system in a state that is an eigenvector of certain observables, the eigenvalue will be the number one measures corresponding to this observable.

Where do these axioms come from?

A unitary representation of a Lie group G gives exactly these mathematical structures:

- A complex vector space $V = \mathcal{H}$, the representation space.
- Differentiating the representation homomorphism π at the identity, one gets a "Lie algebra homomorphism"

$$\pi' : X \in \mathrm{Lie}(G) = T_e G \to \pi'(X)$$

The condition that the $\pi(g)$ be unitary implies that the $\pi'(X)$ are skew-Hermitian. Multiplying by i, the $i\pi'(X)$ are Hermitian linear operators on V.

Two-dimensional quantum systems, I

For the simplest non-trivial example of this, take a quantum system with $\mathcal{H} = \mathbf{C}^2$ (called the "qubit").
The Lie group $U(2)$ of two-by-two unitary ($U^{-1} = (\overline{U})^T$) matrices acts on $\mathcal{H}$ by the defining representation (π the identity map, π' also the identity).
Such matrices are of the form

$$U = e^{tX}$$

where X is skew-Hermitian ($\overline{X}^T = -X$).
The Lie algebra of $U(2)$ is a four-dimensional real vector space, with basis $\{i\mathbf{1}, i\sigma_1, i\sigma_2, i\sigma_3\}$, where the σ_j are Hermitian matrices, the physicist's "Pauli matrices":

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Two-dimensional quantum systems, II

States of the qubit system are vectors in $\mathcal{H} = \mathbf{C}^2$, observables are two-by-two skew-Hermitian matrices. There is a four-dimensional linear space of these.

Characteristic quantum behavior

If a state ψ is an eigenvector of one of the $i\sigma_j$, it has a well-defined value for that observable (the eigenvalue).
It can't then have a well-defined value for the other two $i\sigma_j$. These three operators are non-commuting, not simultaneously diagonalizable.

Relating Quantum Mechanics and Representations

Basic Principle

Quantum mechanical systems carry unitary representations π of various Lie groups G on their state spaces $\mathcal{H}$. The corresponding Lie algebra representations π' give the operators for observables of the system.

Significance for physicists

Identifying observables of one's quantum system as coming from a unitary representation of a Lie group allows one to use representation theory to say many non-trivial things about the quantum system.

Significance for mathematicians

Whenever physicists have a physical system with a Lie group G acting on its description, the state space $\mathcal{H}$ should provide a unitary representation of G. This is a fertile source of interesting unitary representations of Lie groups.

Example: translations in space, $G = \mathbf{R}^3$

Physics takes place in a space $\mathbf{R}^3$. One can consider the Lie group $G = \mathbf{R}^3$ (the "space translation group").
The quantum state space $\mathcal{H}$ will provide a unitary representation of this group. The Lie algebra representation operators are called the "momentum operators"

$$P_j, \quad j = 1, 2, 3$$

These commute, so can be simultaneously diagonalized, in a basis of $\mathcal{H}$ of states called the "momentum basis."
Basis elements have well-defined values for the components p_j of the momentum vector (the eigenvalues of the operators P_j).
By Heisenberg uncertainty, these states are not "localized", carry no information about position.

Wavefunctions

For a single quantum particle moving in $\mathbf{R}^3$, one can take the state space to be

$$\mathcal{H} = L^2(\mathbf{R}^3)$$

The unitary representation of $G = \mathbf{R}^3$ is given by taking the action on functions on $\mathbf{R}^3$ induced from the action on $\mathbf{R}^3$ by translations.
If $\psi(\mathbf{x}) \in \mathcal{H}$, translation by $\mathbf{a}$ gives

$$\pi(\mathbf{a})\psi(\mathbf{x}) = \psi(\mathbf{x} - \mathbf{a})$$

Taking the derivative π' of this to get the Lie algebra representation, one gets the differentiation operators $\frac{\partial}{\partial x_j}$. The physicist's momentum operators have a factor of i, are taken to be

$$P_j = -i\frac{\partial}{\partial x_j}$$

Example: Quanta, $G = U(1)$

Many physical systems have a group $G = U(1)$ (circle group of phase rotations) acting on the system. Any unitary representation of $U(1)$ is a direct sum of one-dimensional representations π_n on $\mathbf{C}$ where

$$\pi_n(e^{i\theta}) = e^{in\theta}$$

for $n \in \mathbf{Z}$.

The Lie algebra representation is given by an anti-Hermitian operator with eigenvalues in, $n \in \mathbf{Z}$. The physicist's Hermitian observable operator is called N, and has eigenvalues n.

Where "quantum" comes from

In a very real sense, this is the origin of the name "quantum": many systems have a $U(1)$ group acting, so states are characterized by an integer, which counts "quanta". Sometimes this has an interpretation as "charge".

Example: Rotations, $G = SO(3)$ or $G = SU(2)$

The group $G = SO(3)$ acts on $\mathbf{R}^3$ by rotations about the origin, and it has a double cover $G = SU(2)$.

Unitary representations of $SU(2)$ break up into direct sums of irreducible components π_n on $\mathbf{C}^{n+1}$, where $n = 0, 1, 2, \ldots$ (for n even these are also representations of $SO(3)$). Physicists call these the "spin $\frac{n}{2}$" representations.

Recall that a basis of the Lie algebra of $SU(2)$ is given by $\{i\sigma_1, i\sigma_2, i\sigma_3\}$. The corresponding observables are the operators

$$J_j = -i\pi'_n(i\sigma_j)$$

These are called the "angular momentum operators" for spin $\frac{n}{2}$. They do not commute and cannot be simultaneously diagonalized.

Example: time translations, $G = \mathbf{R}$

Time evolution is given by a unitary representation of the group $G = \mathbf{R}$ of translations in time. The corresponding Lie algebra operator is

$$-iH$$

where H is a Hermitian operator called the "Hamiltonian" operator. An eigenstate of H with eigenvalue E is said to have energy E.
The fact that $-iH$ is the operator one gets by differentiating time translation means one has

The Schrödinger equation

The fundamental dynamical equation of the theory is determined by the Hamiltonian operator, it is

$$\frac{\partial}{\partial t}\psi = -iH\psi$$

on states $\psi \in \mathcal{H}$. This is called the Schrödinger equation.

Lie Group Representations and Symmetries

When the action of a Lie group G on a quantum system commutes with the Hamiltonian operator, G is said to be a "symmetry group" of the system, acting as "symmetries" of the quantum system. Then one has

Conservation Laws

Since the observable operators $\mathcal{O}$ corresponding to Lie algebra elements of G commute with H, which gives infinitesimal time translations, if a state is an eigenstate of $\mathcal{O}$ with a given eigenvalue at a given time, it will have the same property at all times. The eigenvalue will be "conserved."

Degeneracy of Energy Eigenstates

Eigenspaces of H will break up into irreducible representations of G. One will see multiple states with the same energy eigenvalue, with dimension given by the dimension of an irreducible representation of G.

Lie Group Representations Are Not Always Symmetries

The state space of a quantum system will be a unitary representation of Lie groups G, even when this action of G on the state space is not a symmetry, i.e. doe not commute with the Hamiltonian.

The basic structure of quantum mechanics involves a unitary group representation in a much more fundamental way than the special case where there are symmetries. This has to do with a group that already is visible in classical mechanics. This group does not commute with any non-trivial Hamiltonian, but it plays a fundamental role in the theory.

Classical (Hamiltonian) Mechanics

The theory of classical mechanical systems, in the Hamiltonian form, is based on the following structures

- An even dimensional vector space $\mathbf{R}^{2n}$, called the "phase space" M, with coordinate functions that break up into position coordinates $q_1, \cdots, q_n$ and momentum coordinates $p_1, \cdots, p_n$.
- The "Poisson bracket", which takes as arguments two functions f, g on M and gives a third such function

$$\{f, g\} = \sum_{j=1}^{n} (\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j})$$

A state is a point in M, observables are functions on M. There is a distinguished function, h, the Hamiltonian, and observables evolve in time by

$$\frac{df}{dt} = \{f, h\}$$

The Group of Canonical Transformations

Sophus Lie first discovered Lie groups in the context of Hamiltonian mechanics. It turns out that there is an infinite-dimensional group acting on a phase space M, known to physicists as the group of "canonical transformations".

For mathematicians, this is the group $G = Symp(M)$ of symplectomorphisms, the group of diffeomorphisms of M preserving the symplectic form, a two-form ω on M given by

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j$$

It turns out that the Lie algebra of this group is given by the functions on M, with the Poisson bracket giving the Lie bracket, the structure which reflects the infinitesimal group law near the identity of the group. The Poisson bracket has the right properties to be the Lie bracket of a Lie algebra: it is anti-symmetric, and satisfies the Jacobi identity.

First-year Physics Example

A classical particle of mass m moving in a potential $V(q_1, q_2, q_3)$ in $\mathbf{R}^3$ is described by the Hamilton function h on phase space $\mathbf{R}^6$

$$h = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(q_1, q_2, q_3)$$

where the first term is the kinetic energy, the second the potential energy. Calculating Poisson brackets, one finds

$$\frac{dq_j}{dt} = \{q_j, h\} = \frac{p_j}{m} \implies p_j = m\frac{dq_j}{dt}$$

and

$$\frac{dp_j}{dt} = \{p_j, h\} = -\frac{\partial V}{\partial q_j}$$

where the first equation says momentum is mass times velocity, and the next is Newton's second law ($F = -\nabla V = m\mathbf{a}$).

Heisenberg Commutation Relations

The quantum theory of a single particle includes not just momentum operators P, but also position operators Q, satisfying the Heisenberg commutator relation

$$[Q, P] = i\mathbf{1}$$

Soon after the discovery (1925) by physicists of this relation, Hermann Weyl realized that it is the Lie bracket relation satisfied by a certain nilpotent Lie algebra, now called the "Heisenberg Lie algebra" (it's isomorphic to the Lie algebra of strictly upper-triangular 3 by 3 matrices). There's a corresponding group, the Heisenberg group (sometimes called the "Weyl group" by physicists).

Dirac and Quantization

Dirac noticed the similarity of the Poisson bracket relation $\{q,p\}=1$ and the Heisenberg operator relation $[Q,P]=i$ and proposed the following method for "quantizing" any classical mechanical system

Dirac Quantization

To functions f on phase space, quantization takes

$$f \to \mathcal{O}_f$$

where $\mathcal{O}_f$ are operators satisfying the relations

$$\mathcal{O}_{\{f,g\}} = -i[\mathcal{O}_f, \mathcal{O}_g]$$

Quantization and Symplectomorphisms

Dirac's proposal can be stated very simply in terms of representation theory, it just says

Dirac quantization

A quantum system is a unitary representation of the group of symplectomorphisms of phase space.

Dirac quantization is just the infinitesimal or Lie algebra aspect of this. The Lie algebra representation is a homomorphism taking functions on phase space to operators, with the Poisson bracket going to the commutator. Unfortunately it turns out this doesn't work....

Bad News and Good News

It turns out that if one tries to follow Dirac's suggestion one finds

Bad News, Groenewold-van Hove

No-go theorem: there is a representation that quantizes polynomial functions on phase space of degree up to two, but this can't be done consistently for higher degrees.

but also

Good News, Stone-von Neumann

The quantization of quadratic polynomials is unique up to unitary equivalence. For a phase space M of dimension $2n$, one gets a unitary representation not of the infinite-dimensional symplectomorphism group, but of the group whose Lie algebra is the quadratic polynomials: a semi-direct product of a Heisenberg group and a symplectic group (actually, of a double-cover, called the metaplectic group)

The Friedrichs-Segal-Berezin-Shale-Weil representation

This unique representation of the metaplectic group is perhaps the central object in quantum theory, has been studied by a long list of mathematicians and mathematical physicists.

It even appears in various places in number theory (as the Weil representation), where one application is in the theory of theta functions.

Over p-adic fields, it is used to construct supercuspidal representations of p-adic groups.

Quantum Field Theory

Modern fundamental quantum theories are "quantum field theories", with phase spaces that are infinite-dimensional function spaces.

The Stone-von Neumann theorem is not true in infinite dimensions: one gets inequivalent representations, depending on how one constructs the representation from the phase space.

This is one of the main sources of difficulty in quantum field theory, related to the problem of "renormalization" (if one approximates the phase space by something finite-dimensional, and tries to take the limit, the limit is singular).

Infinite-dimensional symmetry groups in physics

The best fundamental physics theories (the Standard Model of particle physics, and General Relativity) involve infinite dimensional groups

- The diffeomorphism group $Diff(M)$, where M is one's space (e.g.$\mathbf{R}^3$)
- So-called gauge groups, which are locally groups of maps from M to compact Lie groups like $U(1)$ or $SU(2)$.

Open Question

What can one say about representations of these groups?

Much is known for M one-dimensional (representation theory of affine Lie algebras, the Virasoro algebra), where similar methods to ones that work in finite-dimensions are available (analogs of Borel or parabolic subalgebras). These are a major part of the subject of "Conformal Field Theory", which are quantum field theories in this dimension. Little is known in higher dimension.

BRST and Lie algebra cohomology

Physicists use homological methods for dealing with the quantum theory of gauge groups, under the name "BRST Cohomology". This is a well-known technique in mathematics: replace modules for an algebra by complexes of modules, functors on modules by "derived functors".
For representations of Lie algebras, one gets Lie algebra cohomology: for a Lie algebra $\mathfrak{g}$ and a representation V there are cohomology groups

$$H^*(\mathfrak{g}, V)$$

These are derived functors of the functor of taking the $\mathfrak{g}$-invariant piece of a representation, so in degree 0 one has

$$H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$$

For semi-simple Lie algbras, this functor is exact, so higher cohomology not so interesting. For solvable Lie algebras (including the Borel subalgebra of a semi-simple Lie algebra), one gets interesting invariants of representations and one can develop the representation theory of semi-simple Lie algebras this way.

Dirac cohomology

A recent development in representation theory is the replacement of Lie algebra cohomology methods by something called "Dirac cohomology". This replacement involves

$$\text{Koszul resolution using } \Lambda^*(\mathfrak{g}) \rightarrow \text{use of spinors}$$

$$d \rightarrow \text{Dirac Operator } D$$

$$\mathbf{Z}-\text{graded } H^*(\mathfrak{g}, V) \rightarrow \mathbf{Z}_2 - \text{graded } H^*_D(\mathfrak{g}, V)$$

So far this has just been used in mathematics, with mathematicians borrowing the physicist's Dirac operator. Can these techniques be applied back in physics (replacing the BRST technique)?

Summary

Representation theory is a subject that brings together many different areas of mathematics, providing an often surprising "unification" of mathematics.

It is remarkable that exactly this sort of mathematics underlies and gives us some understanding of our most fundamental physical theory, quantum mechanics.

There likely is much more to be learned about the relation of fundamental physics and mathematics, with representation theory like to play a major role.

Thanks for your attention!