Twistor Unification

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Outline

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1. States and Imaginary Time Quantum Theory
2. Spinors in Minkowski and Euclidean space
3. Twistor Geometry

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Note: These slides and paper with details (soon on arXiv) at
`https://www.math.columbia.edu/~woit/twistors-oist.pdf`
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If $\widehat{f}(E)$ is supported on $E > 0$, the Fourier transform

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(E) e^{iEt} dE$$

gives a well-defined holomorphic function on the upper complex $t + i\tau$ plane (more specifically, see Paley-Wiener theorems).

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- In real time you get functions like e^{iEt} and have to worry about what happens at $\pm\infty$. In one sign of imaginary time, $e^{-E\tau}$ well-behaved.
- Real time Wightman functions are not functions but distributions, best thought of as boundary values of holomorphic functions. Real-time quantum fields only commute for space-like separations. In imaginary time, quantum fields always commute. They can be simultaneously diagonalized at all points, and often treated by the methods of classical statistical mechanics.

States in real time

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For instance, for a relativistic scalar field, $\mathcal{H}_1$ is the space of functions on the positive mass shell, square integrable with respect to the Lorentz-invariant inner product.

Note that there is no need to specify a particular time-like direction as the time direction, or to specify a $t = 0$ hypersurface.

States in imaginary time

Quantization in imaginary time works differently. You can't just take square integrable solutions of the analytically continued wave equation as $\mathcal{H}_1$ (there aren't any) and use Fock space methods. You must choose a time direction and then define states asymmetrically, using e.g. $\tau > 0$.

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(Euclidean) canonical formalism: you must work not with momentum space functions on a mass-shell but with functions of imaginary time for $\tau > 0$. The inner product used start with a Euclidean invariant one, but then adds a time-reflection operation in the definition of the inner product on physical states. This must satisfy Osterwalder-Schrader positivity to get unitarity.

Spinors in Minkowski space

In Minkowski space the Lorentz group is $SL(2,\mathbb{C})$, and Weyl spinors transform as either the standard representation on $\mathbb{C}^2$ or as the complex conjugate representation.

Spinors in Euclidean space

In Euclidean space, the space-time rotation group is $Spin(4) = SU(2) \times SU(2)$. There are now two completely different kinds of chiral spinors, not related by conjugation.

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One motivation for twistors: a possibly consistent framework for better understanding the analytic continuation problem for spinors.

Twistors: some history and background

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Suggested reference: *Twistor Geometry and Field Theory* by Ward and Wells.

Spacetime and the Grassmannian $G_{2,4}(\mathbb{C})$

Twistors and space-time

Conformally compactified, complexified Minkowski space is the Grassmanian $G_{2,4}(\mathbb{C})$ of complex 2-planes in $\mathbb{C}^4$. Or, equivalently, the space of complex projective lines ($\mathbb{C}P^1$s) in complex projective 3-space.

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- Compactified Minkowski space and compactified Euclidean space (S^4) are two real slices of the same complex manifold, with the spinor bundle a holomorphic vector bundle. This provides the context needed for better understanding analytic continuation of spinor fields.

Relating (projective) twistor space and space-time

$$P(S) = \{\mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^4\}$$

$$PT = \{\mathbb{C} \subset \mathbb{C}^4\} \xleftarrow{\ \mu\ } P(S) \xrightarrow{\ \nu\ } M = \{\mathbb{C}^2 \subset \mathbb{C}^4\}$$

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$$pt. \in PT \xrightarrow{\nu \circ \mu^{-1}} \mathbb{C}P^2 \subset M \quad (\text{"}\alpha - plane\text{"})$$

$$\mathbb{C}P^1 \subset PT \xleftarrow{\mu \circ \nu^{-1}} pt. \in M$$

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Holomorphic objects on PT

The sheaf cohomology group

$$H^1(\widehat{U}, \mathcal{O}(-k-2))$$

where $\widehat{U} = \mu \circ \nu^{-1}(U)$.

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The Ward correspondence identifies holomorphic anti-self-dual $GL(n, \mathbb{C})$ connections on $U \subset M$ and holomorphic rank n vector bundles on $\widehat{U} \subset PT$.

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- Conformal compactification of Euclidean space (S^4)

Minkowski space twistors

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One way to define the Minkowski real form: choose a nondegenerate signature $(2, 2)$ Hermitian form Φ on C^4. This picks out (compactified) Minkowski space as the subspace of $\mathbb{C}^2 \subset \mathbb{C}^4$ on which $\Phi = 0$. On PT, it picks out a subspace $N \subset PT$ of $\mathbb{C} \subset \mathbb{C}^4$ on which $\Phi = 0$.

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$\Phi = 0$ determines a real form $SU(2,2) = Spin(4,2)$ of $SL(4,\mathbb{C})$ that acts transitively on (compactified) Minkowski space. This is the conformal group, it also acts on solutions to massless wave equations.

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Just as in the conventional story, the Euclidean signature quantization requires picking a time direction, equivalently a $t = 0$ subspace (equator of S^4) that will be shared with Minkowski space. The asymmetry in imaginary time now corresponds to the fact that Euclidean picture states will be defined in terms of spinor fields on only one hemisphere of the S^4.

Euclidean space twistors

Euclidean signature twistor geometry is much simpler than the Minkowski space version. Restricted to S^4, the map μ is an isomorphism, so there is a single fibration

$$\begin{array}{ccc} \mathbb{C}P^1 & \longrightarrow & P(S) = PT = \mathbb{C}P^3 \\ & & \downarrow \pi \\ & & S^4 \end{array}$$

Should draw a picture here!

Euclidean space twistors, quaternions and complex structures

Euclidean space twistors are best understood using quaternions

Identifying $\mathbb{C}^4 = \mathbb{H}^2$ and noting that $S^4 = \mathbb{H}P^1$, the projection

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is just the map taking a complex line to the quaternionic line it generates.

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PT is the bundle of complex structures on S^4

The $\mathbb{C}P^1 = S^2$ fiber above a point on S^4 can be identified with the possible choices of complex structure on the tangent space at the point. For a general Riemannian manifold in $d = 4$, if the metric is ASD, the bundle of complex structures is a complex manifold and allows study of the geometry using holomorphic methods.

Twistor unification: internal symmetries

Twistor unification proposal

At short distances, unification via a conformally invariant theory of space-time and chiral spinors, based on twistor geometry. The fundamental structure is the projective twistor space PT, with points in space-time given by complex projective lines in PT, Euclidean and Minkowski signature real slices related by analytic continutation as described above.

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The $U(1) \times SU(2)$ is the internal electroweak symmetry.
- Using $T = \mathbb{C}^4$
$$PT = \frac{SU(4)}{U(1) \times SU(3)}$$
the $SU(3)$ is the color symmetry.

Electroweak symmetry breaking

In a Euclidean signature theory, the definition of the state space breaks the full Euclidean space-time symmetry ($SO(4)$), by a choice of time direction. In the twistor picture, this choice of time direction corresponds to the choice of a signature $(2,2)$ Hermitian form Φ, with PT now broken up into pieces given by the sign of Φ

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which are the inverse images of Euclidean time on S^4 positive, zero, or negative.

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Up on PT, the tangent vector pointing in the unit time direction lies in a tangent space that is identified with $\mathbb{C}^2$ (by the complex structure given by the point in the fiber). The internal electroweak symmetry group $U(2)$ acts on this by the defining representation. This is exactly the transformation property of the usual Higgs field that conventionally breaks the electroweak symmetry.

Spinors on $PT = CP^3$

Taking PT as fundamental, instead of the Dirac operator acting on spinor fields on Euclidean space-time S^4, one should look at fields on PT. One can analyze the structure of spinors on PT and find that they are in some sense products of

- Pull-backs to PT of the spinors on S^4
- Spinors on the fiber CP^1

The second of these gives the Weyl-spinor degree of freedom that one also sees as a chiral spinor in Minkowski space and is invariant under the internal $U(2)$. The first however transforms non-trivially under $U(2)$, with the right degrees of freedom to give a generation of leptons (including a neutral right-handed neutrino).

What's missing for a full unified theory?

The geometrical framework described so far provides an elegant unification of precisely the known internal and space-time symmetries of the Standard Model. There is still a lot more needed to have a full theory, in particular:

Matter degrees of freedom on PT

While a spinor on PT gives a single generation of leptons, one also want quarks, and three generations. Quarks are rather naturally introduced, since a point in PT corresponds to a decomposition

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with the internal $SU(3)$ acting trivially on the first factor, as the fundamental representation on the second factor. More seriously, it is very unclear where multiple generations might come from. Using quaternions and complex numbers, one has not fully exploited all the possible structures on the real eight-dimensional space T. In terms of unit vectors, S^7 carries several different kinds of geometry

$$S^7 = Spin(8)/Spin(7) = Spin(7)/G_2 = Spin(6)/SU(3) = Spin(5)/Sp(1)$$

In particular, we have used complex ($Spin(6) = SU4)$) and quaternionic ($Spin(5) = Sp(2)$) aspects of the geometry, but not the octonionic aspects that appear in $S^7 = Spin(7)/G_2$.

Matter field dynamics on PT

In the usual twistor formalism, instead of space-time equations of motion, one has purely holomorphic characterizations of the fields on PT, eg. as H^1 sheaf cohomology groups. One likely needs something different than just sheaf cohomology to replace equations of motion on PT. Since one wants fields that take values in spinors on PT, presumably what is needed is some version of the Dirac equation on PT.

Gauge field dynamics

The Penrose-Ward correspondence relates holomorphic structures on PT to ASD connections on space-time. One needs some version of this that reflects the full Yang-Mills equations.

Quantum gravity

One can naturally consider the problem of incorporating gravity into this framework from the point of view that takes the spin connection and the vierbeins as fundamental variables. The new factors here are that, formulated in Euclidean space, only half of the usual spin connection is present (the other is the internal $SU(2)$), so one seems to have a "self-dual" sort of gravity theory. In addition, one of the vierbeins is distinguished, playing the role of the Higgs field. One would like to find a dynamics for these degrees of freedom that is well-defined at short distances, gives the usual Einstein-Hilbert effective action at large distances, while giving an appropriate dynamics to the Higgs field.

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- The Euclidean space quantization is more interesting and more non-trivial than usual thought, not just a matter of adding factors of $\sqrt{-1}$ in the right places.
- Twistor geometry is a compelling way to thing about $4d$ space-time geometry, especially the geometry of spinors.

Conclusions

I hope that I've at least convinced you of three things:

- The Euclidean space quantization is more interesting and more non-trivial than usual thought, not just a matter of adding factors of $\sqrt{-1}$ in the right places.
- Twistor geometry is a compelling way to thing about $4d$ space-time geometry, especially the geometry of spinors.
- There are intriguing prospects here for unifying space-time and internal symmetries in an unexpected manner, well worth further investigation.