Spacetime is Right-handed

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Outline

1. Vectors and spinors in four dimensions
2. An alternative to the usual Wick rotation
3. Twistors
4. Unification?

Note: These slides are at
`https://www.math.columbia.edu/~woit/twistorunification/osmu.pdf`
Related announcement paper (more detailed version in progress)
`https://arxiv.org/abs/2311.00608`
Related ideas (older) on unification
`https://arxiv.org/abs/2104.05099`

Minkowski spacetime and the Lorentz group

To study Minkowski spacetime and its Lorentz group of symmetries, it's convenient to represent vectors $x = (x_0, x_1, x_2, x_3)$ as Hermitian matrices X

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x_0\mathbf{1} + x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3$$

with (indefinite) inner product given by

$$|x|^2 = -\det X = -x_0^2 + x_1^2 + x_2^2 + x_3^2$$

Elements Ω of the Lorentz group $SL(2, \mathbf{C})$ (in another definition, this is the double cover of the Lorentz group) act on Minkowski spacetime by

$$X \to \Omega X \Omega^\dagger$$

This action preserves Hermiticity of the matrix and its determinant, so preserves the inner product.

Wick rotation and Euclidean spacetime

One method to make a quantum field theory well-defined is to "Wick rotate" from x_0 to ix_0, making the inner product positive definite. Using the isomorphism of the quaternions $\mathbf{H}$ with two by two matrices

$$\mathbf{1} \leftrightarrow 1, \quad -i\sigma_1 \leftrightarrow \mathbf{i}, \quad -i\sigma_2 \leftrightarrow \mathbf{j}, \quad -i\sigma_3 \leftrightarrow \mathbf{k}$$

Wick-rotated Hermitian matrices (multiplied by $-i$) are quaternions

$$-i\begin{pmatrix} ix_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & ix_0 - x_3 \end{pmatrix} \leftrightarrow q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

These have the standard four-dimensional inner product, which is preserved by the linear transformation

$$q \to q_L x q_R^{-1}$$

where q_R, q_L are independent unit length quaternions (elements of $SU(2)$). The Wick-rotated Lorentz group is the group $Spin(4) = SU(2)_L \times SU(2)_R$ of such pairs.

Real forms of complex spacetime

Complex spacetime is $M(2, \mathbf{C})$, with inner product $(-\det)$ preserved by

$$Spin(4, \mathbf{C}) = SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$$

This is the complexification of three different "real forms":

- $M(2, \mathbf{R})$, i.e matrices of the form

  $$\begin{pmatrix} x_0 - x_3 & x_1 - x_2 \\ x_1 + x_2 & x_0 + x_3 \end{pmatrix}$$

  This has inner product $|x|^2 = -x_0^2 + x_1^2 - x_2^2 + x_3^2$ of signature $(2, 2)$, preserved by the group

  $$Spin(2, 2) = SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$$

  of independent left and right multiplication by elements of $SL(2, \mathbf{R})$
- Minkowski spacetime of Hermitian matrices. This has $(3, 1)$ signature inner product preserved by $Spin(3, 1) = SL(2, \mathbb{C})$ (see above).
- Euclidean spacetime of quaternions. This has $(4, 0)$ signature inner product preserved by $Spin(4, 0) = SU(2)_L \times SU(2)_R$ (see above).

Spinors and vectors: the conventional complex story

The conventional way to relate complex spacetime vectors and spinors is:

Complex spacetime

Complex spacetime is the tensor product two copies of $\mathbf{C}^2$

$$S_L \otimes_{\mathbf{C}} S_R$$

As a representation of $Spin(4, \mathbf{C}) = SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$, S_L here is the defining representation on the first factor, trivial on the second. S_R is trivial on the first factor, the defining representation on the second. The matrix representation as elements of $M(2, \mathbf{C})$ is the expression in coordinates of the fact that complex spacetime vectors are complex linear maps from S_R^* (dual of S_R) to S_L.

The signature $(2,2)$ real form for spinors

For real spacetimes, the way the real form works for spinors is straightforward in signature $(2,2)$. S_L and S_R are real representations on $\mathbf{R}^2$ of $Spin(2,2) = SL(2,\mathbf{R})_L \times SL(2,\mathbf{R})_R$.

Real spacetime: signature $(2,2)$

Real spacetime in this signature is the tensor product

$$S_L \otimes_{\mathbf{R}} S_R$$

where S_L and S_R are real representations on $\mathbf{R}^2$, the defining representation on one factor, trivial on the other.

Real, complex and quaternionic representations

The Euclidean and Minkowski spacetime real forms are significantly more subtle when one looks at spinors, not just vectors, with the spinor representation not a real representation (the vectors are always a real representation). Recall the following general classification of representations of representations of a real Lie group on a complex vector space V:

- Real representations. These have an anti-linear conjugation map $\sigma : V \to V$ with $\sigma^2 = 1$ with fixed points the real subspace $V_{\mathbf{R}} \subset V$ and $V = V_{\mathbf{R}} \otimes \mathbf{C}$.
- Quaternionic representations: These have an anti-linear map σ with $\sigma^2 = -1$. This provides the **j** of an action of quaternions on V.
- Complex representations: Neither real nor quaternionic. Unlike the other two cases, in this case V and its conjugate representation are not equivalent.

The Euclidean (signature $(4,0)$) real form

For the Euclidean real form, $Spin(4,0) = SU(2)_L \times SU(2)_R$. Spinors are quaternionic, with anti-linear maps $\sigma_{L,R}$ on S_L, S_R such that $\sigma^2_{L,R} = -1$. These provide a conjugation map $\sigma = \sigma_L \otimes \sigma_R$ such that $\sigma^2 = 1$ on complex vectors $S_L \otimes S_R$. Real Euclidean space-time is the σ-invariant subspace. Choosing coordinates on spinors, real Euclidean spacetime gets identified with matrices

$$\begin{pmatrix} ix_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & ix_0 - x_3 \end{pmatrix}$$

Note that taking the Hermitian adjoint leaves the $x_0 = 0$ spatial subspace invariant but reflects ix_0 to $-ix_0$.

The Minkowski (signature $(3,1)$) real form

In the Minkowski spacetime real form, $Spin(3,1) = SL(2,\mathbf{C})$. The spinor representations are complex, with S (standard action on $\mathbf{C}^2$) and the conjugate representation $\overline{S}$ inequivalent.
To get a real representation, one has to put together S and $\overline{S}$, and define

$$\sigma : (S \oplus \overline{S}) \to (S \oplus \overline{S})$$

where σ conjugates spinor coordinates and interchanges S and $\overline{S}$, so satisfies $\sigma^2 = 1$. Then $S \oplus \overline{S}$ is the complexification of a four-dimensional real $SL(2, C)$ representation (this is the Majorana representation).
If one takes complex spacetime to be the tensor product $S \otimes_{\mathbf{C}} \overline{S}$, then the same sort of conjugation interchanging S and $\overline{S}$ applied to the tensor product provides a real structure and picks out the real Minkowski space (in coordinates, as Hermitian matrices).

The usual Wick rotation of spinors and vectors

The usual Wick rotation relation between Minkowski and Euclidean vectors and spinors proceeds by taking complexified spacetime to be

$$(S_L \otimes \mathbf{C}) \otimes (\mathbf{C} \otimes S_R)$$

where the first factor is the irreducible $SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$ representation trivial on $SL(\mathbf{C})_R$, defining representation on $SL(2, \mathbf{C})_L$, chiralities switched in the second factor.

These representations are holomorphic and one can use them to analytically continue between the various real forms. For the Minkowski real form, one needs to take

$$\Omega \in SL(2, \mathbf{C}) \rightarrow (\Omega, \overline{\Omega}) \in SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$$

i.e. the Lorentz group is a real subspace of the complexified group given by the conjugate diagonal. Then the holomorphic representation above restricts on the Lorentz group real subspace to $S \otimes \overline{S}$.

Problems with the usual Wick rotation of spinors

What has always been confusing about the standard story of Wick rotation for spinors from Minkowski to Euclidean spacetime is that the real forms, as well as the behavior under chiral transformations, are completely different. As a result, one has two problems:

- No Majorana spinors in Euclidean space, since while there is a real structure on $S \oplus \overline{S}$ in Minkowski space, instead there is a quaternionic structure on $S_L \oplus S_R$ in Minkowski space.
- No single Weyl-spinor theory in Euclidean space. In Minkowski space one can write a Lorentz invariant Lagrangian density for a single Weyl spinor field taking values in S, with the Weyl operator a vector, so in $S \otimes \overline{S}$. In Euclidean spacetime vectors are in $S_L \otimes S_R$, so one needs two kinds of chiral spinor field to get a Lorentz invariant Lagrangian density.

The usual way to handle these problems involves doubling the number of degrees of freedom in the Euclidean theory, but this makes for a confusing relationship between the two theories.

An alternative relation between Minkowski and Euclidean spacetimes

It turns out that there is an alternative way to relate vectors and spinors, which looks the same on Minkowski spacetime, very different on Euclidean spacetime. One can take the Lorentz group $SL(2,\mathbf{C})$ to be not the conjugate diagonal in the complexifed Lorentz group, but instead the chiral factor $SL(2,\mathbf{C})_R$, with complex spacetime

$$S_R \otimes \overline{S}_R$$

The $SL(2,\mathbf{C})_L$ factor in the complexified Lorentz group acts trivially: complex spacetime is now purely right-handed.
If one just looks at Minkowski spacetime, this is the same as the conventional relation between spinors and spacetime, but quite different on the complex and Euclidean spacetimes.

A right-handed Euclidean spacetime

Since the defining and conjugate representations are isomorphic as representations of $SU(2)_R \subset SL(2,\mathbf{C})_R$ the complexified Euclidean spacetime is the $SU(2)_R$ representation

$$S_R \otimes \overline{S}_R = S_R \otimes S_R = \mathbf{C} \oplus V$$

This is a reducible representation, with a trivial one-dimensional representation and a complex three dimensional representation V transforming in the vector representation of $SU(2)_R$. This is a real representation, complexification of a four real dimensional representation with one direction (imaginary time) invariant under $SU(2)_R$, the other three giving purely spatial vectors.

In this new alternative Wick rotation, Euclidean spacetime has a different geometry than expected, with a distinguished direction and only the chiral right-handed half of the usual $Spin(4)$ symmetry acting non-trivially.

Euclidean quantum field theory

That the Wick rotated spacetime in this alternative has a distinguished direction fits in well with the very different structure of Euclidean quantum field theory. To relate a Euclidean quantum field theory to a physical Minkowski spacetime QFT, one must choose a distinguished imaginary time direction. Then the construction of the physical state space is done (Osterwalder-Schrader) by using reflection in the imaginary time direction. As we saw earlier, in the explicit representation of Euclidean spacetime as two-by-two complex matrices, the Minkowski conjugation (Hermitian conjugation of matrices) is exactly reflection in imaginary time.

In a Minkowski spacetime formulation of quantum field theory the definition of the state space does not need a choice of time direction and is Lorentz ($SL(2, \mathbf{C})$) invariant. In Euclidean quantum field theory, the definition of the state space is only invariant under purely spatial rotations. In the usual geometry of Euclidean spacetime spatial rotations are the diagonal subgroup of $SU(2)_L \times SU(2)_R$. In the alternative geometry considered here, spatial rotations are identified with $SU(2)_R$.

Twistor theory

Twistor geometry is a different way of thinking about the geometry of space-time, first proposed in 1967 by Penrose. It naturally provides a joint complexification of Minkowski and Euclidean space-time and a way to look at analytic continuation between them.

$4d$ conformal symmetry is most easily understood using twistors, especially if one works with the conformal compactification of space-time (S^4 instead of $\mathbb{R}^4$ in the Euclidean case).

Most discussions for physicists focus on the Minkowski version, we're interested in the Euclidean version and how it is related to the Minkowski version.

Twistor geometry

In twistor theory one takes as fundamental twistor space $T = \mathbb{C}^4$ (or its projective version $PT = \mathbb{C}P^3$, the complex lines in T).

Compactified complex spacetime is the Grassmannian $Gr_{2,4}(\mathbf{C})$ of possible $\mathbf{C}^2 \subset \mathbf{C}^4$.

Right-handed spinors are tautological: the spinor space S_R at a point is the spacetime point (warning, changing by a dual from spinor case).

Left-handed spinors S_L at the point S_R are of a different nature: the quotient T/S_R.

Linear maps from S_R to S_L at the point S_R are the fiber of the holomorphic tangent bundle of $Gr_{2,4}(\mathbf{C})$.

The complex conformal group is $SL(4, \mathbf{C}) = Spin(6, \mathbf{C})$, acting linearly on T.

Twistor geometry: real forms

Just as in spinor case there are three real forms:

- Real: conformal group $SL(4, \mathbf{R}) = Spin(3,3)$ acting on $Gr_{2,4}(\mathbf{R})$
- Quaternionic: conformal group $SL(2, \mathbf{H}) = Spin(5,1)$ acting on $\mathbf{HP}^1 = S^4$, which is compactified Euclidean spacetime.
- Complex: conformal group $SU(2,2) = Spin(4,2)$ acting on compactified Minkowski spacetime.

Euclidean real form

In the Euclidean case, twistor space is a quaternionic representation of the conformal group. $T = \mathbb{C}^4 = \mathbb{H}^2$ and $S^4 = \mathbb{H}P^1$, quaternionic projective space. The conformal group $Spin(5,1) = SL(2,\mathbb{H})$ acts transitively on PT and S^4 through its linear action on $\mathbb{H}^2$.

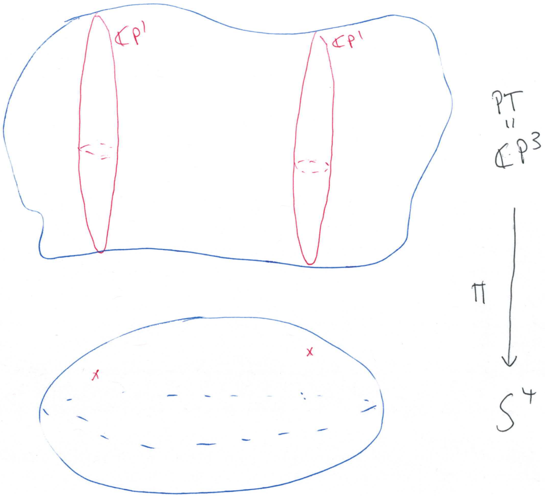
One has a fibration with fibers $\mathbb{C}P^1$

$$\begin{array}{ccc} \mathbb{C}P^1 & \longrightarrow & PT = \mathbb{C}P^3 \\ & & \Big\downarrow \pi \\ & & S^4 \end{array}$$

where the map π takes a complex line in $\mathbb{C}^4$ to the quaternionic line it generates.

This deserves a picture:

Euclidean twistor fibration: a picture



Two interpretations of PT

PT is the projective spin bundle $P(S_R)$

The fiber at a point is the $\mathbb{C}P^1$ of projective S_R space.

PT is the bundle of complex structures on S^4

The $\mathbb{C}P^1 = S^2$ fiber above a point on S^4 can be identified with the possible choices of complex structure on the tangent space at the point.

These definitions generalize PT to give a twistor space for any Riemannian manifold in $d = 4$. If the metric is ASD, this twistor space is a complex manifold and allows study of the Riemannian geometry using holomorphic methods.

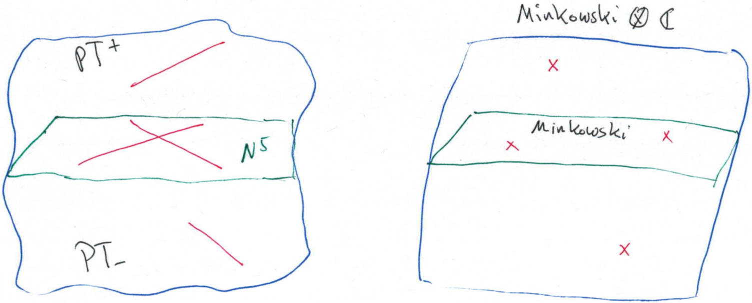
Minkowski real form

Just as in the spinor case, the Minkowski real form of twistor theory is neither real nor quaternionic. To define a conjugation σ, one can't just use T but has to introduce a conjugate T. This conjugation gives a $(2,2)$ signature Hermitian form Φ on $\mathbf{C}^4$. Compactified Minkowski space-time is the subspace of $\mathbb{C}^2 \subset \mathbb{C}^4$ on which $\Phi = 0$. This is acted on by the Minkowski conformal group $SU(2,2)$.

The $\mathbb{C}P^1 = S^2$ in PT corresponding to a point in Minkowski space can be identified with the "celestial sphere" of light rays through that point. When two points are light-like separated, the corresponding $\mathbb{C}P^1$'s intersect.

Another picture ($N^5 \subset PT$ is the subset on which $\Phi = 0$):

Minkowski space-time twistors: a picture



Alternative Wick rotation in twistor space

From the twistor space point of view, instead of the usual holomorphic tangent bundle of $Gr_{2,4}(\mathbf{C})$, one can take the tangent space to be the non-holomorphic bundle with fiber $S_R \otimes \overline{S}_R$. Restricted to $\Phi = 0$, this is the usual tangent bundle of Minkowski spacetime. Restricted to $\mathbf{H}P^1 \subset Gr_{2,4}(\mathbf{C})$ it is the tangent bundle of S^4, but now with a distinguished direction.

Unification: spacetime physics

Remarkably, the spacetime physics of the Standard Model and gravity can be written down just using the right-handed spinor geometry:

- Matter fields are built out of chiral Weyl spinors.
- Gauge field dynamics can be derived from a Lagrangian just using the self-dual part of the field-strength two-form. This is the part that just transforms under $SL(2, C)_R$.
- One can formulate general relativity in a purely chiral way.

Unification: internal symmetries

The internal symmetries of the Standard Model appear naturally in the Euclidean spacetime twistor geometry:

- The electroweak $SU(2)$ is the $SU(2)_L$ of the conventional Euclidean rotation group, but in the alternative Wick rotation acting trivially on spacetime, so an internal symmetry.
-
$$PT = \frac{SU(4)}{U(1) \times SU(3)}$$
so, there is a $U(1) \times SU(3)$ acting as an internal symmetry on PT.

Work in Progress

Now starting to

- Write a longer version of the short paper "Spacetime is right-handed", giving details of the material discussed here.
- Return to earlier work on "Euclidean twistor unification", with new perspective.