Wick Rotating Spinors and Twistors

Peter Woit

Columbia University
Mathematics Department

Centre de Physique Théorique, Aix-Marseille
April 3, 2026

Outline

1. Wick rotation
2. Spacetime and its complexification
3. Spinors
4. Twistors
5. Wick rotation using twistors

Note: These slides are at
https://www.math.columbia.edu/~woit/twistorunification/marseille.pdf

By "Wick rotation", one usually means taking

$$t \to -it$$

with the following motivation:

- Perturbatively, Feynman diagram calculations no longer have integrands with poles on the real axis, removing inherent ambiguities in such calculations.
- Nonperturbatively, path integrals become much better defined. In certain cases, the imaginary time path integral becomes a partition function, which can be studied using the methods of statistical mechanics.

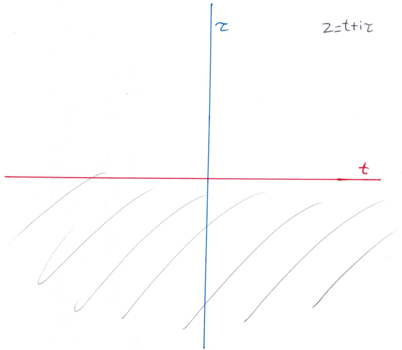
Positivity of the energy and holomorphicity in complex time

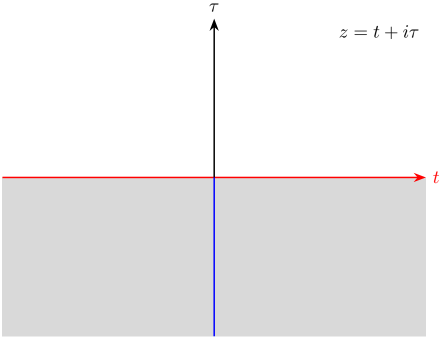
Positivity of the energy implies

- If a function $f(t)$ has positive energy ($\widetilde{f}(E) = 0$ for $E < 0$), then one can try to extend it to complex time $z = t + i\tau$ by inverse Fourier transform
$$f(z) = \int_0^\infty e^{-izE}\widetilde{f}(E)dE$$
Since
$$e^{-izE} = e^{-itE}e^{\tau E}$$
the integrand is well-behaved for $\tau < 0$ (Paley-Wiener). As a result the integral gives a function holomorphic in z in the lower half plane.
- The operator $U(z) = e^{-izH}$ is holomorphic in z in the lower-half z plane.

Wick rotation: time dependence in physics should be thought of as holomorphic dependence on complex time in the lower half plane.

Lower half plane





Euclidean Quantum Field Theory

Julian Schwinger (1957): it is the imaginary time theory that is well-defined. In such a theory the Minkowski spacetime metric becomes the Euclidean metric

$$-dt^2+dx^2+dy^2+dz^2 \to -d(it)^2+dx^2+dy^2+dz^2 = dt^2+dx^2+dy^2+dz^2$$

The imaginary time theory is called a "Euclidean quantum field theory" and is determined by its imaginary time operator vacuum expectation operators ("Schwinger functions").

Minkowski spacetime and matrices

To study spacetime vectors four dimensions it is very convenient to write them as matrices, for instance representing $x = (x_0, x_1, x_2, x_3)$ by Hermitian matrices

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x_0 \mathbf{1} + x_1 \boldsymbol{\sigma}_1 + x_2 \boldsymbol{\sigma}_2 + x_3 \boldsymbol{\sigma}_3$$

The Minkowski inner product is

$$|x|^2 = -\det X = -x_0^2 + x_1^2 + x_2^2 + x_3^2$$

Elements Ω of the Lorentz group $SL(2, \mathbf{C})$ act on Minkowski spacetime by

$$X \to \Omega X \Omega^\dagger$$

This action preserves Hermiticity of the matrix and its determinant, so preserves the inner product.

Complex spacetime and complex spacetime rotations

Complexifying by allowing x_0, x_1, x_2, x_3 to be complex numbers, one gets complex spacetime $\mathbf{C}^4$, identified with the space of all complex two by two matrices. Now, given any pair Ω_L, Ω_R of elements of $SL(2, \mathbf{C})$, the transformation

$$X \to \Omega_L X \Omega_R^{-1}$$

preserves the determinant and thus the inner product (extended linearly to complex coordinates). One has

$$Spin(4, \mathbf{C}) = SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$$

The spin group is the double cover of the rotation group. Four dimensions is very special: it is the only dimension in which the complex spin group is not simple, but decomposes into two commuting pieces.

Wick rotation and complex spacetime: the standard story

The standard story of Wick rotation starts from a Lorentz-invariant Minkowski spacetime starting point, analytically continuing through complex spacetime to get to a Euclidean spacetime. This proceeds by

- Use the Fourier transform formula and the condition of support inside the positive energy-momentum lightcone to analytically continue to a "tube" in complex spacetime.
- Use the action of $Spin(4, \mathbf{C})$ and locality to analytically continue to the Euclidean region of complex spacetime.

While the Minkowski spacetime theory is defined in a Lorentz-invariant way, if you want to connect the Euclidean spacetime theory to physics, you need to break $Spin(4)$ invariance, picking the direction of imaginary time in which one will Wick rotate back. Equivalently, to define the physical state space, one needs to specify the "Osterwalder-Schrader" reflection operation in imaginary time.

Spinors in complex spacetime

In complex spacetime, there are two kinds of spinors:

- S_R, right-handed spinors, a copy of $\mathbf{C}^2$ acted on by $SL(2,\mathbf{C})_R$
- S_L, left-handed spinors, a copy of $\mathbf{C}^2$ acted on by $SL(2,\mathbf{C})_L$

The identification of complex spacetime with two by two matrices is an identification with the tensor product $S_L \otimes S_R$, or more concretely these matrices are in $Hom(S_R^*, S_L)$: linear maps from the dual of S_R to S_L.

The usual 4-component spinor formalism puts these together as

$$\begin{pmatrix} S_L \\ S_R^* \end{pmatrix}$$

with a vector v acting by the matrix

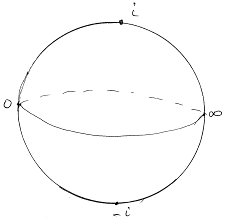
$$\not{v} = v^\mu \gamma_\mu = \begin{pmatrix} 0 & v_0 + \mathbf{v}\cdot\boldsymbol{\sigma} \\ -v_0 + \mathbf{v}\cdot\boldsymbol{\sigma} & 0 \end{pmatrix}$$

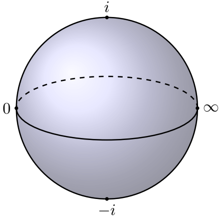
Upper right corner is v as a matrix taking S_R^* to S_L.

Visualizing spinors

To visualize a spinor $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, think about the Riemann sphere with coordinate $z = z_1/z_2$. This is the space $\mathbf{C}P^1$ of complex lines in spinor space. $SL(2,\mathbf{C})$, $SU(2)$ and $SL(2,\mathbf{R})$ act by conformal transformations:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot z = \frac{\alpha z + \beta}{\gamma z + \delta}$$





$SL(2,\mathbf{C})$ and $SU(2)$ act transitively, $SL(2,\mathbf{R})$ acts with three orbits (upper/lower hemispheres. equator).

Spinors in real spacetimes

Our spacetime is not complex, but real: Minkowski spacetime with metric of signature $(3,1)$, Before turning to spinors in this case, it's worth thinking about two other "real forms", things which after complexification become the complex spacetime story:

- Take the matrices defining spacetime vectors to be real matrices. The determinant will give a metric of signature $(2,2)$. These will be preserved by the subgroup $Spin(2,2) = SL(2,\mathbf{R})_L \times SL(2,\mathbf{R})_R$ of $Spin(4,\mathbf{C})$. Spinors will be the real 2d reps. of the $SL(2,\mathbf{R})$s. This seems to have nothing to do with physics.
- Take the matrices defining spacetime vectors to be quaternions $\mathbf{H}$. The determinant will have signature $(4,0)$. These will be preserved by the subgroup $Sp(1)_L \times Sp(1)_R$ of left and right multiplication by unit quaternions. Spinors will be quaternions. This is supposed to be related to the physical Minkowski case by Wick rotation.

Spinors in Minkowski spacetime

Earlier we saw that Minkowski spacetime can be identified with Hermitian matrices X, with $\Omega \in SL(2, \mathbf{C})$ (the Lorentz group) acting by $X \to \Omega X \Omega^\dagger$. There is only one $SL(2, \mathbf{C})$, and one spinor representation, which we'll call S. The conjugate representation $\overline{S}$ is (unlike for $SU(2)$) inequivalent to S. Spacetime vectors (Hermitian matrices) are elements of $S \otimes \overline{S}$ (equivalently, maps from $\overline{S}^*$ to S), invariant under the conjugation map that interchanges S and $\overline{S}$.

- Four-spinor formalism: use four-component spinors $\begin{pmatrix} S \\ \overline{S}^* \end{pmatrix}$
- Two-spinor formalism: use two kinds of two-component spinors, z^A are components of S, $z^{\dot{A}}$ are components of $\overline{S}$ ($A = 1, 2$).

Relating the two, in components a four-spinor is $\begin{pmatrix} z^A \\ w_{\dot{A}} \end{pmatrix}$

Wick-rotating spinors, the usual story

The usual way to relate spinors in Minkowski and Euclidean spacetimes is to embed the Lorentz $SL(2, \mathbf{C})$ in $Spin(4, \mathbf{C}) = SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$ by

$$\Omega \to (\Omega, \overline{\Omega})$$

then analytically continue in $Spin(4, \mathbf{C})$ between this $SL(2, \mathbf{C})$ (thought of as a real group) and $SU(2)_L \times SU(2)_R$. One identifies $S, \overline{S}$ as the restriction to this $SL(2, \mathbf{C})$ of the holomorphic representations S_L and S_R. Problems with this:

- 4d geometry breaks up nicely into two sectors (L, R, or self-dual and anti-self-dual) in Euclidean spacetime, but not in Minkowski spacetime (no SD or ASD Yang-Mills or gravity solutions).
- Wick rotation needs to relate spinor representations with very different properties. Wick rotating a Weyl spinor field becomes impossible.

Wick rotating Weyl spinor fields

The equation of motion for a right-handed Weyl spinor field (in energy-momentum space) is

$$(E - \boldsymbol{\sigma} \cdot \boldsymbol{p})\widehat{\widetilde{\psi}}(E, \mathbf{p}) = 0$$

Such a theory describes the fundamental matter fields of the Standard model: relativistic massless particles with

- Positive energy $E = |\mathbf{p}|$, particles with helicity (eigenvalue of $\frac{\sigma \cdot \boldsymbol{p}}{2|\mathbf{p}|}$) $+\frac{1}{2}$.
- Negative energy $E = -|\mathbf{p}|$, anti-particles with helicity $-\frac{1}{2}$.

The two by two matrix $(E - \boldsymbol{\sigma} \cdot \boldsymbol{p})$ transforms like a vector. In the usual Wick rotation this is $S_L \otimes S_R$. To get an invariant action you need to introduce S_L valued fields, and there is no way to Wick rotate a single Weyl spinor field.

Right-handed spacetime?

One way to solve this Wick rotation problem is to just insist on only using one kind of spinor, which motivates the following.

A slogan

"Spacetime is right-handed", meaning that spacetime geometry will be based on spinors, but only one kind, spinors of $SL(2, \mathbf{C})_R$.

If you just think about QFT in Minkowski spacetime, this proposal to use only one $SL(2, \mathbf{C})$ to describe spacetime geometry is not distinguishable from the usual story about Minkowski spacetime, which only involves one $SL(2, \mathbf{C})$. The proposal only becomes nontrivial if you think about Wick rotation.

Wick-rotated right-handed spacetime

In complex space-time, vectors are now in the representation $S_R \otimes \overline{S}_R$ os $Spin(4, \mathbf{C})$ instead of $S_L \otimes S_R$. If you restrict to Euclidean spacetime, where $Spin(4) = SU(2)_L \times SU(2)_R$ acts on vectors, they are not the usual 4-dimensional vectors. Since $\overline{S}_R$ is equivalent to S_R as a representation of $SU(2)_R$, vectors are in the representation

$$S_R \otimes S_R = \mathbf{1} \oplus V_3$$

$SU(2)_L$ acts trivially and the four dimensions break up into one on which $SU(2)_R$ also acts trivially (this is the imaginary time direction) and three on which $SU(2)_R$ acts as the usual three-dimensional rotations. This opens up a very interesting possibility: in the Wick-rotated theory, the symmetry group $SU(2)_L$ has become an internal symmetry.
The problem is that one has given up something fundamental to our usual understanding of Wick rotation and of QFT: the analytic continuation through the usual complex spacetime. A possible framework replacing the usual one is twistor theory, our next topic.

Twistor theory

Twistor geometry is a different way of thinking about the geometry of spacetime, first proposed in 1967 by Penrose. I'll give an explanation of it that emphasizes the following:

- It generalizes the notion of a spinor, in a way that also includes spacetime itself, with conformal symmetry built in at a fundamental level.
- It embodies the principle of "spacetime is right-handed", since it inherently chiral: a point in spacetime is tautologically the space of right-handed spinors at the point.

Twistor geometry of complex spacetime

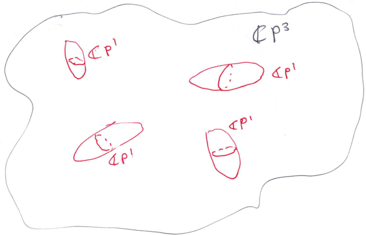
One can think of twistors as a generalization of spinors: instead of just a spinor space $S = \mathbf{C}^2$ with $SL(2, \mathbf{C})$ acting, one has a twistor space $T = \mathbf{C}^4$, with $SL(4, \mathbf{C})$ acting.
The possible $S = \mathbf{C}^2 \subset T = \mathbf{C}^4$ through the origin provide the points of complex spacetime:

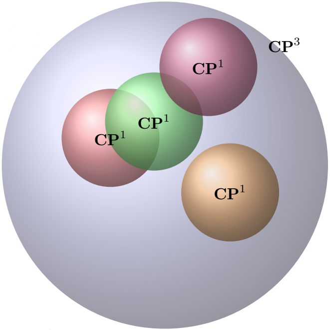
a point in spacetime IS the space S of (right-handed) spinors at the point.

The usual name for this space is $Gr(2, 4, \mathbf{C})$, the Grassmannian of complex two planes in $\mathbf{C}^4$. The version of spacetime one gets this way is conformally compactified complex spacetime, with the group $SL(4, \mathbf{C})$ acting on it by complex conformal transformations. This is also the group $Spin(6, \mathbf{C})$: twistors are spinors for six dimensional geometry.

Visualizing spacetime points in twistor space

Just as it is a good idea to visualize spinors by projectivizing and thinking about the Riemann sphere $PS = \mathbf{C}P^1$, it is a good idea to visualize twistor space as $PT = \mathbf{C}P^3$, with spacetime points embedded $\mathbf{C}P^1$s.





Twistor geometry: real forms

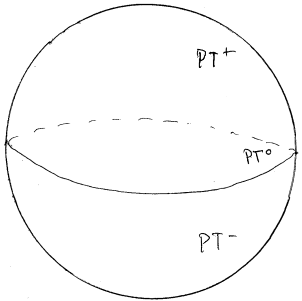
In the spinor case there are two real forms of $SL(2, C) = Spin(3, \mathbf{C})$ acting on $PS = \mathbf{C}P^1$: $SU(2) = Spin(3)$ which acts transitively, and $SL(2, \mathbf{R}) = SU(1, 1) = Spin(2, 1)$ which acts with three orbits. In the twistor case, there are four real forms of $SL(4, \mathbf{C}) = Spin(6, \mathbf{C})$ acting on $\mathbf{C}P^3$:

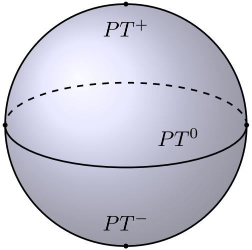
- $SU(4) = Spin(6)$
- $SL(4, \mathbf{R}) = Spin(3, 3)$
- $SL(2, \mathbf{H}) = Spin(5, 1)$
- $SU(2, 2) = Spin(4, 2)$

The first two don't seem to be relevant to physics, the third will describe Euclidean signature geometry, the last Minkowski signature geometry. Wick rotation is about the relation of the last two.

The Minkowski real form

If one picks a non-degenerate Hermitian form Φ of signature $(2, 2)$ on T, it will be preserved by the subgroup $SU(2, 2)$ of $SL(4, \mathbf{C})$. This will act on PT with three orbits PT^+, PT^-, PT^0 corresponding to lines in T on which Φ is positive, negative or zero. This is closely analogous to the action of $SU(1, 1)$ on PS, where the three orbits are the upper/lower hemispheres and the equator.





Minkowski spacetime in twistor theory

In twistor theory, compactified Minkowski spacetime is identified with those $\mathbf{C}P^1 s$ inside PT that lie in the five-dimensional subspace PT^0.

The celestial sphere

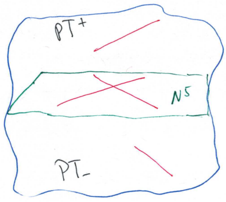
The $\mathbf{C}P^1$ in PT^0 corresponding to a point in Minkowski space can be identified with the "celestial sphere" of light rays through that point.

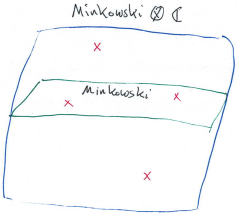
PT^0 is the space of all light rays in compactified Minkowski spacetime. It is of the form $PT^0 = S^3 \times S^2$, although to write it explicitly this way, you need to choose a spacelike S^3. Then light rays are parametrized by points in S^3 and their celestial spheres.

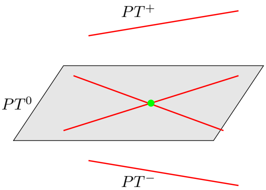
When two points are light-like separated, the corresponding $\mathbb{C}P^1$'s intersect.

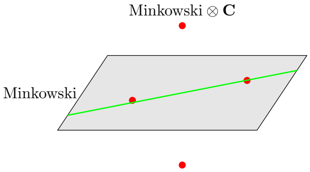
Minkowski space-time twistors: a picture

Another picture ($N^5 = PT^0 \subset PT$ is the subset on which $\Phi = 0$):









The Euclidean real form

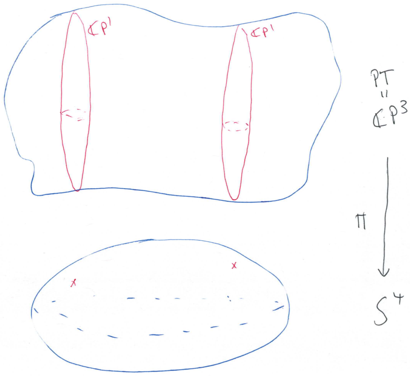
If one chooses an identification $T = \mathbf{C}^4 = \mathbf{H}^2$, then for each point on PT there is a distinguished $\mathbf{C}P^1$ going through it, giving a fibering

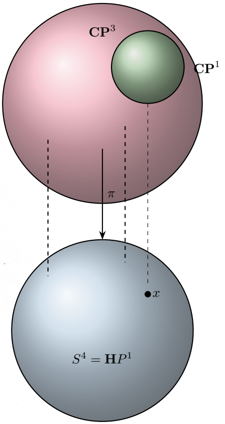
$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & PT = \mathbf{C}P^3 \\ & & \downarrow \pi \\ & & S^4 = \mathbf{H}P^1 \end{array}$$

where the map π takes a complex line in $\mathbb{C}^4$ to the quaternionic line it generates.
The conformal group $Spin(5,1) = SL(2,\mathbb{H})$ acts transitively on PT and S^4 through its linear action on $\mathbb{H}^2$.

Euclidean twistor fibration: a picture

Some pictures of the Euclidean twistor fibration:





Two interpretations of PT

PT is the projective spin bundle PS_R

The fiber at a point is the $\mathbb{C}P^1$ of projective S_R space.

PT is the bundle of complex structures on S^4

The $\mathbb{C}P^1 = S^2$ fiber above a point on S^4 can be identified with the possible choices of complex structure on the tangent space at the point.

These definitions generalize PT to give a twistor space for any Riemannian manifold in $d = 4$. If the metric is ASD, this twistor space is a complex manifold and allows study of the Riemannian geometry using holomorphic methods.

In twistor theory in Minkowski spacetime, positive energy physics is reformulated in terms of boundary values on PT^0 of holomorphic quantities on PT^+. This is already like the more abstract version of Wick rotation (e^{-izH} in lower half plan).

If you pick an imaginary time direction, you get a Euclidean spacetime twistor theory that encapsulates the same content as above.

Work in progress on this theory, its $SU(2)_L$ symmetry seems to corresponds to something that appears in Minkowski spacetime as an internal symmetry, spontaneously broken.