Euclidean Spinors and Twistor Unification

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Abstract

Quantum field theories are best defined in Euclidean space-time, with behavior in Minkowski space-time given by boundary values of an analytic continuation. Euclidean spinor fields are known however to have a confusing relationship to Minkowski spinor fields, due to their different behavior under space-time rotations. We argue that the necessity of picking an imaginary time direction for the analytic continuation gives a new point of view on this problem, and allows an interpretation in Minkowski space-time of one of the chiral factors of $Spin(4) = SU(2) \times SU(2)$ as an internal symmetry. The imaginary time direction spontaneously breaks this $SU(2)$, playing the role of the Higgs field.

Twistor geometry provides a compelling framework for formulating spinor fields in complexified four-dimensional space-time and implementing the above suggestion. Projective twistor space PT naturally includes an internal $SU(3)$ symmetry as well as the above $SU(2)$, and spinors on this space behave like a generation of leptons.

Since only one chirality of the Euclidean $Spin(4)$ is a space-time symmetry after analytic continuation and the Higgs field defines the imaginary time direction, the space-time geometry degrees of freedom are only a chiral $SU(2)$ connection and a spatial frame. These may allow a consistent quantization of gravity in a chiral formulation, unified in the twistor framework with the degrees of freedom of the Standard Model.

The Penrose-Ward correspondence relates gauge fields on Euclidean space-time, classically satisfying anti-self-duality equations, to holomorphic objects on projective twistor space. The above unification proposal requires implementation as a theory with gauge symmetry on PT, perhaps related to known correspondences between super Yang-Mills theories and supersymmetric holomorphic Chern-Simons theories on PT.

1 Introduction

Penrose's 1967 [28] twistor geometry provides a remarkable alternative to conventional ways of thinking about the geometry of space and time. In the usual description of space-time as a pseudo-Riemannian manifold, the spinor degree of freedom carried by all matter particles has no simple or natural explanation. Twistor geometry characterizes a point in Minkowski space-time as a complex 2-plane in $\mathbf{C}^4$, with this $\mathbf{C}^2$ providing tautologically the (Weyl) spinor degree of freedom at the point. The $\mathbf{C}^4$ is the twistor space T, and it is often convenient to work with its projective version $PT = \mathbf{C}P^3$, the space of complex lines in T. Conformal symmetry becomes very simple to understand, with conformal transformations given by linear transformations of $\mathbf{C}^4$.

A well-known issue with the twistor geometry formulation of fundamental physics is that, unlike general relativity, it is inherently parity asymmetric (Penrose refers to this as the "googly" problem, invoking a term from cricket). One aspect of this is that left and right-handed Weyl spinors have a very different nature, with spinors of only one particular handedness describing the points in space-time. While this asymmetry causes problems with describing gravity, it has often been speculated that it has something to do with the parity asymmetry of the weak interactions.

Twistor geometry most naturally describes not Minkowski space-time, but its complexification, as the Grassmanian $G_{2,4}(\mathbf{C})$ of all complex 2-planes in the twistor space T. This allows a formulation of fundamental physics in terms of holomorphic fields on $G_{2,4}(\mathbf{C})$, a rather different framework than the usual one of conventional fields on Minkowski space-time. Correlation functions can be characterized by their values on a Euclidean signature space-time, related to the Minkowski ones by analytic continuation. The Euclidean space-time correlation functions are better behaved: at non-coincident points they are legitimate functions whereas the Minkowski versions are distributions given as boundary values of holomorphic functions.

Focusing not on the usual Minkowski space-time version of twistor theory, but on its analytic continuation to Euclidean space-time, it is a remarkable fact that the specific internal symmetry groups and degrees of freedom of the Standard Model appear naturally, unified with the space-time degrees of freedom:

- Projective twistor space PT can be thought of as

  $$\mathbf{C}P^3 = \frac{SU(4)}{U(1) \times SU(3)}$$

  or as

  $$\frac{Sp(2)}{U(1) \times SU(2)}$$

  This identifies $U(1), SU(2)$ and $SU(3)$ internal symmetry groups at each point in projective twistor space.

- In Euclidean space-time quantization, the definition of the space of states requires singling out a specific direction in Euclidean space-time that will be the imaginary time direction. Lifting the choice of a tangent vector in the imaginary time direction from Euclidean space-time to PT, the internal $U(1) \times SU(2)$ acts on this degree of freedom in the same way the Standard Model electroweak symmetry acts on the Higgs field.

- The degrees of freedom of a spinor on PT transform under internal and space-time symmetries like a generation of Standard Model leptons.
- Connections for the chiral $SU(2)$ symmetry that acts on the spinor describing a point, together with frames of tangent vectors in the space directions, provide the degrees of freedom needed for a chiral version of general relativity.

2 Euclidean quantum fields

> But besides this, by freeing ourselves from the limitation of the Lorentz group, which has produced all the well-known difficulties of quantum field theory, one has here a possibility — if this is indeed necessary — of producing new theories. That is, one has the possibility of constructing new theories in the Euclidean space and then translating them back into the Lorentz system to see what they imply.
>
> J. Schwinger, 1958[43]

> Should the Feynman path integral be well-defined only in Euclidean space, as axiomaticians would have it, then there seems to exist a very real problem when dealing with Weyl fields as in the theory of weak interactions or in its unification with QCD.
>
> P. Ramond, 1981[56]

> A certain sense of mystery surrounds Euclidean fermions.
>
> A. Jaffe and G. Ritter, 2008[71]

That one chirality of Euclidean space-time rotations appears after analytic continuation to Minkowski space-time as an internal symmetry is the most hard to believe aspect of the proposed framework for a unified theory outlined above. One reason for the very long time that has passed since an earlier embryonic version of this idea (see [35]) is that the author has always found this hard to believe himself. While the fact that the quantization of Euclidean spinor fields is not straightforward is well-known, Schwinger's early hope that this might have important physical significance (see above) does not appear to have attracted much attention. In this section we'll outline the basic issue with Euclidean spinor fields, and argue that common assumptions about analytic continuation of the space-time symmetry do not hold in this case. This issue becomes apparent in the simplest possible context of free field theory. There are also well-known problems when one attempts to construct a non-perturbative lattice-regularized theory of chiral spinors coupled to gauge fields.

Since Schwinger's first proposal in 1958[42], over the years it has become increasingly clear that the quantum field theories governing our best understanding of fundamental physics have a much simpler behavior if one takes time to be a complex variable, and considers the analytic continuation of the theory to imaginary values of the time parameter. In imaginary time the invariant notion of distance between different points becomes positive, path integrals often become well-defined rather than formal integrals, field operators commute, and expectation values of field operators are conventional functions rather than the boundary values of holomorphic functions found at real time.

While momentum eigenvalues can be arbitrarily positive or negative, energy eigenvalues go in one direction only, which by convention is that of positive energies. Having states supported only at non-negative energies implies (by Fourier transformation) that, as a function of complex time, states can be analytically continued in one complex half plane, not the other. A quantum theory in Euclidean space has a fundamental asymmetry in the direction of imaginary time, corresponding to the fundamental asymmetry in energy eigenvalues.

Quantum field theories can be characterized by their n-point Wightman (Minkowski space-time) or Schwinger (Euclidean space-time) functions, with the Wightman functions not actual functions, but boundary values of analytic continuations of the Schwinger functions. For free field theories these are all determined by the 2-point functions W_2 or S_2. The Wightman function W_2 is Poincaré-covariant, while the Schwinger function S_2 is Euclidean-covariant.

This simple relation between the Minkowski and Euclidean space-time free field theories masks a much more subtle relationship at the level of fields, states and group actions on these. In both cases one can construct fields and a Fock space built out of a single-particle state space carrying a representation of the space-time symmetry group. For the Minkowski theory, fields are non-commuting operators obeying an equation of motion and the single-particle state space is an irreducible unitary representation of the Poincaré group.

The Euclidean theory is quite different. Euclidean fields commute and do not obey an equation of motion. The Euclidean single-particle state space is a unitary representation of the Euclidean group, but far from irreducible. It describes not physical states, but instead all possible trajectories in the space of physical states (parametrized by imaginary time). The Euclidean state space and the Euclidean fields are not in any sense analytic continuations of the corresponding Minkowski space constructions. For a general theory encompassing the relation between the Euclidean group and Poincaré group representations, see [60].

One can recover the physical Minkowski theory from the Euclidean theory, but to do so one must break the Euclidean symmetry by choosing an imaginary time direction. In the following sections we will outline the relation between the Minkowski and Euclidean theories for the cases of the harmonic oscillator, the free scalar field theory, and the free chiral spinor field theory.

2.1 The harmonic oscillator

The two-point Schwinger function for the one-dimensional quantum harmonic oscillator of frequency ω ($\omega > 0$) is

$$S_2(\tau) = \frac{1}{(2\pi)(2\omega)} e^{-\omega|\tau|}$$

with Fourier transform

$$\widetilde{S_2}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\tau} S_2(\tau) d\tau = \frac{1}{(2\pi)^{3/2}} \frac{1}{s^2 + \omega^2}$$

In the complex $z = t + i\tau$ plane, S_2 can be analytically continued to the upper half plane as

$$\frac{1}{(2\pi)(2\omega)} e^{i\omega z}$$

and to the lower half plane as

$$\frac{1}{(2\pi)(2\omega)} e^{-i\omega z}$$

The Wightman functions are the analytic continuations to the t (real z) axis, so come in two varieties:

$$W_2^-(t) = \lim_{\epsilon \to 0^+} \frac{1}{(2\pi)(2\omega)} e^{i\omega(t+i\epsilon)}$$

and

$$W_2^+(t) = \lim_{\epsilon \to 0^+} \frac{1}{(2\pi)(2\omega)} e^{-i\omega(t-i\epsilon)}$$

The conventional interpretation of $W_2^{\pm}$ is not as functions, but as distributions, given as the boundary values of holomorphic functions. Alternatively (see appendix A), one can interpret $W_2^{\pm}(t)$ as the lower and upper half-plane holomorphic functions defining a hyperfunction. Like distributions, hyperfunctions can be thought of a elements of a dual space to a space of well-behaved test functions, in this case a space of real analytic functions. The Fourier transform of W_2 is then the hyperfunction

$$\widetilde{W_2}(E) = \frac{i}{(2\pi)^{3/2}} \frac{1}{\omega^2 - E^2}$$

which is a sum of terms $\widetilde{W}_2^{\pm}(E)$ supported at $\omega > 0$ and $-\omega < 0$. Note that the convention that e^{-iEt} has positive energy means that Fourier transforms of positive energy functions are holomorphic for $\tau < 0$.

The physical state space of the harmonic oscillator is determined by the single-particle state space $\mathcal{H}_1 = \mathbf{C}$. $\mathcal{H}_1$ is the state space for a single quantum, it can be thought of as the space of positive energy solutions to the equation of motion

$$(\frac{d^2}{dt^2} + \omega^2)\phi = 0 \tag{2.1}$$

$\mathcal{H}_1$ can also be constructed using W_2^+, by defining

$$\begin{aligned}(f, g) =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(t_2)} W_2^+(t_2 - t_1) g(t_1) dt_1 dt_2 \\ =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(t_2)} \frac{e^{-i\omega(t_2 - t_1)}}{(2\pi)(2\omega)} g(t_1) dt_1 dt_2 \\ =& \frac{1}{2\omega} \overline{\widetilde{f}(\omega)} \widetilde{g}(\omega)\end{aligned}$$

for f, g functions in $\mathcal{S}(\mathbf{R})$ and taking the space of equivalence classes

$$\mathcal{H}_1 = [f] \in \{f \in \mathcal{S}(\mathbf{R})\}/\{(f, f) = 0\}$$

One can identify such equivalence classes as

$$[f] = \frac{1}{\sqrt{2\omega}} \widetilde{f}(\omega)$$

$\mathcal{H}_1$ is $\mathbf{C}$ with standard Hermitian inner product

$$\langle [f], [g] \rangle = \frac{1}{2\omega} \overline{\widetilde{f}(\omega)} \widetilde{g}(\omega)$$

Note that it doesn't matter whether one takes real or complex valued functions f, in either case one gets the same quotient complex vector space $\mathcal{H}_1$.

Given $\mathcal{H}_1$ and the inner product $\langle \cdot, \cdot \rangle$, the full state space $\mathcal{H}$ is an inner product space given by the Fock space construction, with

$$\mathcal{H} = S^*(\mathcal{H}_1) = \bigoplus_{k=0}^{\infty} S^k(\mathcal{H}_1)$$

In this case the symmetrized tensor product $S^k(\mathcal{H}_1)$ of k copies of $\mathcal{H}_1 = \mathbf{C}$ is just again $\mathbf{C}$, the states with k-quanta. A creation operator $a^\dagger(f)$ (for f real) acts by symmetrized tensor product with $[f]$ and $a(f)$ is the adjoint operator. One can define an operator

$$\widehat{\phi}(f) = a(f) + a^\dagger(f)$$

and then

$$\langle 0|\widehat{\phi}(f)\widehat{\phi}(g)|0\rangle = \langle [f], [g]\rangle$$

$\widehat{\phi}(t)$ should be interpreted as an operator-valued distribution, writing

$$\widehat{\phi}(f) = \int_{-\infty}^{\infty} \widehat{\phi}(t) f(t) dt$$

$\widehat{\phi}$ satisfies the equation of motion 2.1.

One can use the Schwinger function S_2 to set up a Euclidean (imaginary time τ) Fock space, taking $\mathcal{E}_1$ to be the space of real-valued functions in $\mathcal{S}(\mathbf{R})$ with inner product

$$\begin{aligned}(f, g)_{\mathcal{E}_1} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau_2) S_2(\tau_2 - \tau_1) g(\tau_1) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau_2) \frac{e^{-\omega|\tau_2-\tau_1|}}{(2\pi)(2\omega)} g(\tau_1) d\tau_1 d\tau_2\end{aligned}$$

The Fock space will be

$$\mathcal{E} = S^*(\mathcal{E}_1 \otimes \mathbf{C})$$

based on the complexification of $\mathcal{E}_1$, with operators $a_E^\dagger(f), a_E(f), \widehat{\phi}_E(f), \widehat{\phi}_E(\tau)$ defined for $f \in \mathcal{E}_1$. Expectation values of products of fields $\widehat{\phi}_E(f)$ for such real-valued f can be given a probabilistic interpretation (see for instance [16]).

Note that the imaginary time state space and operators are of a quite different nature than those for real time. The operators $\widehat{\phi}_E(\tau)$ do not satisfy an equation of motion, and commute for all τ. They describe not the annihilation and creation of a single quantum, but an arbitrary path in imaginary time of a configuration-space observable. The state space is much larger than the real-time state space, with $\mathcal{E}_1$ infinite dimensional as opposed to $\mathcal{H}_1 = \mathbf{C}$.

One way to reconstruct the physical real-time theory from the Euclidean theory is to consider the fixed τ subspace of $\mathcal{E}_1 \otimes \mathbf{C}$ of complex functions localized at τ_0. Here $f(\tau) = a\delta(\tau - \tau_0)$ for $a \in \mathbf{C}$ and one defines a Hermitian inner product on $\mathcal{E}_1 \otimes \mathbf{C}$ by

$$(f, g)_{\mathcal{E}_1 \otimes \mathbf{C}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(\tau_2)} S_2(\tau_2 - \tau_1) g(\tau_1) d\tau_1 d\tau_2$$

While elements of $\mathcal{E}_1$ satisfy no differential equation and have no dynamics, one does have an action of time translations on $\mathcal{E}_1$, with translation by τ_0 taking $a\delta(\tau)$ to $a\delta(\tau - \tau_0)$. Since the inner product satisfies

$$(a\delta(\tau), b\delta(\tau - \tau_0)) = abe^{-\omega|\tau_0|}$$

one sees that one can define a Hamiltonian operator generating imaginary time translations on these states by taking H to be multiplication by ω. The imaginary time translation operator $e^{-\tau_0\omega}$ can be analytically continued from $\tau_0 > 0$ to real time t as

$$U(t) = e^{-it\omega}$$

Another way to reconstruct the real-time theory is the Osterwalder-Schrader method, which begins by picking out the subspace $\mathcal{E}_1^+ \subset \mathcal{E}_1 \otimes \mathbf{C}$ of functions supported on $\tau < 0$. Defining a time reflection operator on $\mathcal{E}_1$ by

$$\Theta f(\tau) = f(-\tau)$$

one can define

$$(f, g)_{OS} = (\Theta f, g)_{\mathcal{E}_1 \otimes \mathbf{C}}$$

The physical $\mathcal{H}_1$ can then be recovered as

$$\mathcal{H}_1 = \frac{\{f \in \mathcal{E}_1^+\}}{\{(f, f)_{OS} = 0\}}$$

Note that for $f, g \in \mathcal{E}_1^+$ one has (since f, g are supported for $\tau < 0$)

$$\begin{aligned}
(f, g)_{OS} =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(-\tau_2)} \frac{e^{-\omega|\tau_2-\tau_1|}}{(2\pi)(2\omega)} g(\tau_1) d\tau_1 d\tau_2 \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(\tau_2)} \frac{e^{\omega(\tau_2+\tau_1)}}{(2\pi)(2\omega)} g(\tau_1) d\tau_1 d\tau_2 \\
=& \frac{1}{(2\pi)(2\omega)} \int_{-\infty}^{\infty} \overline{f(\tau_2)} e^{\omega\tau_2} d\tau_2 \int_{-\infty}^{\infty} g(\tau_1) e^{\omega\tau_1} d\tau_1 \\
=& \frac{1}{2\omega} \overline{\widetilde{f}(-i\omega)} \widetilde{g}(-i\omega)
\end{aligned}$$

This gives a map

$$f \in \mathcal{E}_1^+ \to [f] \in \mathcal{H}_1$$

similar to that of the real-time case

$$f \to [f] = \frac{1}{\sqrt{2\omega}} \widetilde{f}(-i\omega) = \frac{1}{\sqrt{2\omega}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\omega\tau} f(\tau) d\tau$$

2.2 Relativistic scalar fields

The theory of a mass m free real scalar field in $3+1$ dimensions can be treated as a straightforward generalization of the above discussion of the harmonic oscillator, treating time in the same way, spatial dimensions with the usual Fourier transform. Defining

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$$

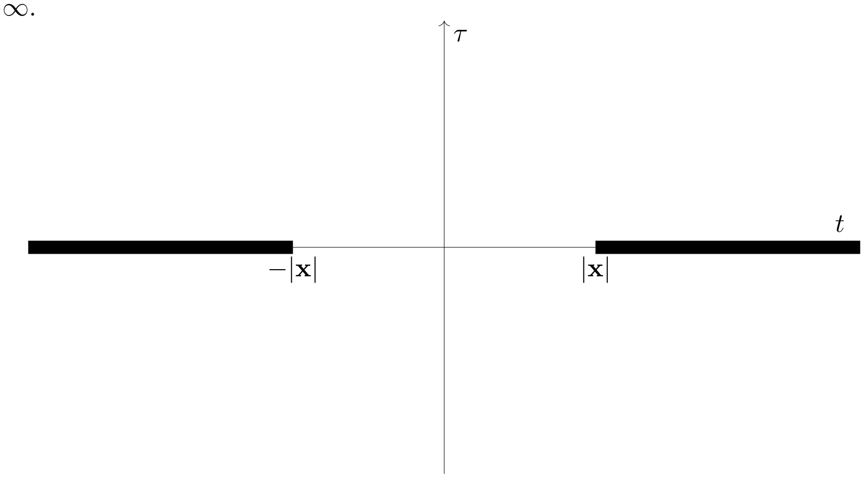
the Fourier transform of the Schwinger function is

$$\widetilde{S_2}(s, \mathbf{p}) = \frac{1}{(2\pi)^3} \frac{1}{s^2 + \omega_{\mathbf{p}}^2}$$

and the Schwinger function itself is

$$
\begin{aligned}
S_2(\tau, \mathbf{x}) =& \frac{1}{(2\pi)^2} \int_{\mathbf{R}^4} e^{i(\tau s + \mathbf{x}\cdot\mathbf{p})} \frac{1}{(2\pi)^3} \frac{1}{s^2 + \omega_{\mathbf{p}}^2} ds d^3\mathbf{p} \\
=& \frac{m}{(2\pi)^3 \sqrt{\tau^2 + |\mathbf{x}|^2}} K_1(m\sqrt{\tau^2 + |\mathbf{x}|^2})
\end{aligned}
$$

where K_1 is a modified Bessel function. This has an analytic continuation to the $z = t + i\tau$ plane, with branch cuts on the t axis from $|\mathbf{x}|$ to ∞ and $-|\mathbf{x}|$ to $-\infty$.



The Wightman function $W_2^+(t, \mathbf{x})$ will be defined as the limit of the analytic continuation of S_2 as one approaches the t-axis from negative values of τ. This will be analytic for spacelike $t < |\mathbf{x}|$, but will approach a branch cut for timelike $t > |\mathbf{x}|$. The Fourier transform of $W_2^\pm$ will be, as a hyperfunction (in the time-energy coordinate)

$$
\widetilde{W}_2(p) = \frac{1}{(2\pi)^3} \frac{i}{\omega_{\mathbf{p}}^2 - E^2}
$$

or, as a distribution, the delta-function distribution

$$
\widetilde{W}_2^+(p) = \frac{1}{(2\pi)^2} \theta(E) \delta(E^2 - \omega_{\mathbf{p}}^2)
$$

on the positive energy mass shell $E = +\omega_{\mathbf{p}}$. Here $W_2^+(x)$ is

$$
W_2^+(t, \mathbf{x}) = \frac{1}{(2\pi)^4} \int_{\mathbf{R}^3} \frac{1}{2\omega_{\mathbf{p}}} e^{-i\omega_{\mathbf{p}} t} e^{i\mathbf{p}\cdot\mathbf{x}} d^3\mathbf{p}
$$

As in the harmonic oscillator case, one can use it to reconstruct the single particle state space $\mathcal{H}_1$, defining

$$
(f, g) = \int_{\mathbf{R}^4} \int_{\mathbf{R}^4} f(x) W_2^+(x - y) g(y) d^4x d^4y
$$

for $f, g \in \mathcal{S}(\mathbf{R}^4)$ ($\mathbf{R}^4$ is Minkowski space), and equivalence classes

$$\mathcal{H}_1 = [f] \in \{f \in \mathcal{S}(\mathbf{R}^4)\}/\{(f,f)=0\}$$

The inner product on $\mathcal{H}_1$ is given by

$$\langle [f],[g]\rangle = \int_{\mathbf{R}^4} \theta(E)\delta(E^2-\omega_{\mathbf{p}}^2)\overline{\widetilde{f}(p)}\widetilde{g}(p)d^4p$$

where $p=(E,\mathbf{p})$ and θ is the Heaviside step function. Elements $[f]$ of $\mathcal{H}_1$ can be represented by functions $\widetilde{f}$ on $\mathbf{R}^3$ of the form

$$\widetilde{f}(\mathbf{p}) = \widetilde{f}(\omega_{\mathbf{p}},\mathbf{p})$$

In this representation, $\mathcal{H}_1$ has the Lorentz-invariant Hermitian inner product

$$\langle [f],[g]\rangle = \int_{\mathbf{R}^3} \overline{\widetilde{f}(\mathbf{p})}\widetilde{g}(\mathbf{p})\frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}}$$

Using the Fock space construction (as in the harmonic oscillator case, where $\mathcal{H}_1 = \mathbf{C}$), the full physical state space is

$$\mathcal{H} = S^*(\mathcal{H}_1) = \bigoplus_{k=0}^{\infty} S^k(\mathcal{H}_1)$$

with creation operators $a^\dagger(f)$ acting by symmetrized tensor product with $[f]$. $a(f)$ is the adjoint operator and one can define field operators by

$$\widehat{\phi}(f) = a(f) + a^\dagger(f)$$

Writing these distributions as $\widehat{\phi}(t,\mathbf{x})$, one recovers the usual description of Wightman functions as

$$W_2^+(x-y) = \langle 0|\widehat{\phi}(x)\widehat{\phi}(y)|0\rangle$$

The operators $\widehat{\phi}(x)$ satisfy the equation of motion

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\widehat{\phi} = 0$$

and $\widehat{\phi}(x), \widehat{\phi}(y)$ commute for x and y space-like separated, but not for time-like separations (due to the branch cuts described above).

The Euclidean (imaginary time) theory has the Fock space

$$\mathcal{E} = S^*(\mathcal{E}_1 \otimes \mathbf{C})$$

where $\mathcal{E}_1$ is the space of real-valued functions in $\mathcal{S}(\mathbf{R}^4)$ (now $\mathbf{R}^4$ is Euclidean space) with inner product

$$(f,g)_{\mathcal{E}_1} = \int_{\mathbf{R}^4}\int_{\mathbf{R}^4} f(x)S_2(x-y)g(y)d^4x d^4y$$

This Fock space comes with operators $a_E^\dagger(f), a_E(f), \widehat{\phi}_E(f), \widehat{\phi}_E(x)$ defined for $f \in \mathcal{E}_1$. Expectation values of products of fields $\widehat{\phi}_(f)$ for such real-valued f can be given a probabilistic interpretation in terms of a Gaussian measure on the distribution space $\mathcal{S}'(\mathbf{R}^4)$ (for details, see [16]).

As in the harmonic oscillator case, there are two ways to recover the real time theory from the Euclidean theory. In the first, one takes $\mathcal{H}_1 \subset \mathcal{E}_1$ to be the functions on Euclidean space-time localized at a specific value of τ, say $\tau = 0$, of the form

$$f(\tau, \mathbf{x}) = \frac{1}{2\pi}\delta(\tau)F(\mathbf{x})$$

Evaluating the inner product for these, one finds

$$(f,g)_{\mathcal{E}_1} = \int_{\mathbf{R}^3} \overline{\widetilde{F}(\mathbf{p})}\widetilde{G}(\mathbf{p})\frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}}$$

which is the usual Lorentz-invariant inner product. The rotation group $SO(3)$ of spatial rotations acts on this $\tau = 0$ subspace of $\mathcal{E}_1$ and this action passes to an action on the physical $\mathcal{H}_1$. Time translations act on $\mathcal{E}_1$ and one can use the infinitesimal action of such translations to define the Hamiltonian operator on $\mathcal{H}_1$.

To recover the physical state space from the Euclidean theory by the Osterwalder-Schrader method, one has to start by picking an imaginary time direction in the Euclidean space $\mathbf{R}^4$, with coordinate τ. One can then restrict to the subspace $\mathcal{E}_1^+ \subset \mathcal{E}_1$ of functions supported on $\tau < 0$. Defining a time reflection operator on $\mathcal{E}_1$ by

$$\Theta f(\tau, \mathbf{x}) = f(-\tau, \mathbf{x})$$

one can define

$$(f,g)_{OS} = (\Theta f, g)_{\mathcal{E}_1}$$

The physical $\mathcal{H}_1$ can be recovered as

$$\mathcal{H}_1 = \frac{\{f \in \mathcal{E}_1^+\}}{\{(f,f)_{OS} = 0\}}$$

In both the Euclidean and Minkowski space-time formalisms one has a unitary representation of the space-time symmetry groups (the Euclidean group $E(4)$ and the Poincaré group P respectively) on the spaces $\mathcal{E}_1, \mathcal{H}_1$ and the corresponding Fock spaces. In the Minkowski space-time case this is an irreducible representation, while in the Euclidean case it is far from irreducible, and the representations in the two cases are not in any sense analytic continuations of each other.

The spatial Euclidean group $E(3)$ is in both $E(4)$ and P, and the two methods for passing from the Euclidean to Minkowski space theory preserve this group action. For translations in the remaining direction, one can fairly readily define the Hamiltonian operator using the semi-group of positive imaginary time translations in Euclidean space, then multiply by i and show that this generates real time translations in Minkowski space-time.

More delicate is the question of what happens for group transformations in other directions in $SO(3,1)$ (the boosts) and $SO(4)$. In the Minkowski theory, boosts act on $\mathcal{H}_1$, preserving the inner product, so one has a unitary action of the Poincaré group on $\mathcal{H}_1$ and from this on the full state space (the Fock space). But while elements of $SO(4)$ not in the spatial $SO(3)$ act on $\mathcal{E}_1$ preserving $(\cdot,\cdot)_{\mathcal{E}_1}$, they do not preserve the positive time subspace $\mathcal{E}_1^+$ and do not commute with the time reflection operator Θ. One can construct operators on $\mathcal{E}_1^+$ giving infinitesimal generators corresponding to directions in the Lie algebra complementary to the Lie algebra of $SO(3)$, and then show that these can be analytically continued and exponentiated to give the action of boosts on $\mathcal{H}_1$. That this can be done was first shown by Klein and Landau in 1982 (by a not completely straight-forward argument, see [21]).

2.3 Spinor fields

While scalar field theories and pure gauge theories have well-understood and straightforward formulations in Euclidean space-time, the question of how to define spinor quantum field theories in Euclidean space-time has always been (see the quote from Jaffe and Ritter above) much more problematic. At the end of this paper one can find a fairly complete bibliography of attempts to address this question over the years, none of which provide a fully satisfactory answer. Schwinger's earliest work argued that in Euclidean space a doubling of the spinor degrees of freedom was necessary, and a version of Euclidean spinor fields due to Osterwalder and Schrader [47] that includes such a doubling has been the conventionally accepted best solution to the definitional problem.

We'll consider the theory of a chiral (Weyl) spinor field in Minkowski space, and then see what problems arise when one tries to find a corresponding Euclidean field theory. It is well-known (see the quote at the beginning of this section from [56]) that a problem arises immediately if one tries to write down a Euclidean path integral for such a theory: there is no way to write an $SO(4)$ invariant Lagrangian just using one chirality.

The equation of motion for a right-handed Weyl spinor is

$$\left(\frac{\partial}{\partial t} + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\psi(t,\mathbf{x}) = 0$$

or, in energy-momentum space

$$(E - \boldsymbol{\sigma}\cdot\mathbf{p})\widetilde{\psi}(E,\mathbf{p}) = 0 \tag{2.2}$$

Since one has

$$(E + \boldsymbol{\sigma}\cdot\mathbf{p})(E - \boldsymbol{\sigma}\cdot\mathbf{p}) = E^2 - |\mathbf{p}|^2$$

solutions in energy-momentum space will also satisfy

$$(E^2 - |\mathbf{p}|^2)\widetilde{\psi}(E,\mathbf{p}) = 0$$

and be supported on the light-cone $E = \pm|\mathbf{p}|$.

The momentum space Wightman function for the Weyl spinor theory will be the hyperfunction

$$\widetilde{W_2}(E,\mathbf{p}) = \frac{-i}{(2\pi)^3}\frac{1}{E-\boldsymbol{\sigma}\cdot\mathbf{p}} = \frac{-i}{(2\pi)^3}\frac{E+\boldsymbol{\sigma}\cdot\mathbf{p}}{E^2-|\mathbf{p}|^2}$$

or equivalently the distribution

$$\widetilde{W_2^+}(E,\mathbf{p}) = \frac{1}{(2\pi)^2}\theta(E)(E+\boldsymbol{\sigma}\cdot\mathbf{p})\delta(E^2-|\mathbf{p}|^2)$$

This is matrix-valued, and on solutions to 2.2 gives the inner product

$$\begin{aligned}\langle\widetilde{\psi}_1,\widetilde{\psi}_2\rangle &= \int_{\mathbf{R}^4}\widetilde{\psi}_1^\dagger(E,\mathbf{p})(E+\boldsymbol{\sigma}\cdot\mathbf{p})\widetilde{\psi}_2(E,\mathbf{p})\theta(E)\delta(E^2-|\mathbf{p}|^2)dEd^3\mathbf{p}\\ &= \int_{\mathbf{R}^3}\widetilde{\psi}_1^\dagger(\mathbf{p})(|\mathbf{p}|+\boldsymbol{\sigma}\cdot\mathbf{p})\widetilde{\psi}_2(\mathbf{p})\frac{d^3\mathbf{p}}{2|\mathbf{p}|}\\ &= \int_{\mathbf{R}^3}\widetilde{\psi}_1^\dagger(\mathbf{p})\widetilde{\psi}_2(\mathbf{p})d^3\mathbf{p}\end{aligned}$$

Here $\widetilde{\psi}(\mathbf{p}) = \widetilde{\psi}(|\mathbf{p}|,\mathbf{p})$.

The last expression is manifestly invariant under spatial ($Spin(3)$) rotations, but not Lorentz ($Spin(3,1) = SL(2,\mathbf{C})$) transformations. One can see Lorentz invariance using the first expression, since for $\Omega \in SL(2,\mathbf{C})$ one has

$$(\Omega^\dagger)^{-1}(E+\boldsymbol{\sigma}\cdot\mathbf{p})\Omega^{-1} = E' + \boldsymbol{\sigma}\cdot\mathbf{p}'$$

where $E', \mathbf{p}'$ are the Lorentz-transformed energy-momenta

$$(E',\mathbf{p}') = \Lambda^{-1}\cdot(E,\mathbf{p})$$

($\Lambda \in SO(3,1)$ corresponds to $\Omega \in Spin(3,1)$ in the spin double cover).

Note that the operator $E+\boldsymbol{\sigma}\cdot\mathbf{p}$ is just the momentum space identification of Minkowski space-time $\mathbf{R}^{3,1}$ with 2 by 2 hermitian matrices:

$$x = (t,x_1,x_2,x_3) \leftrightarrow M = \begin{pmatrix} t+x_3 & x_1-ix_2\\ x_1+ix_2 & t-x_3\end{pmatrix}$$

with the Minkowski norm given by $-\det M$. One can identify complexified Minkowski space-time $\mathbf{R}^{3,1}\otimes\mathbf{C} = \mathbf{C}^4$ with all 2 by 2 complex matrices by:

$$(t+i\tau,z_1,z_2,z_3) \leftrightarrow M = \begin{pmatrix} t+i\tau+z_3 & z_1-iz_2\\ z_1+iz_2 & t+i\tau-z_3\end{pmatrix}$$

Euclidean space-time $\mathbf{R}^4$ will get identified with complex matrices of the form

$$(\tau,x_1,x_2,x_3) \leftrightarrow M = \begin{pmatrix} i\tau+x_3 & x_1-ix_2\\ x_1+ix_2 & i\tau-x_3\end{pmatrix}$$

and analytic continuation between Euclidean and Minkowski space takes place on functions of such matrices.

The group $Spin(4, \mathbf{C}) = SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$ acts on complex matrices by

$$M \to g_L M g_R^{-1}$$

preserving the determinant (here $g_L, g_R \in SL(2, \mathbf{C})$). The subgroup $SL(2, \mathbf{C})$ such that $g_R = (g_L^\dagger)^{-1}$ is the Lorentz group $Spin(3, 1)$ that preserves Minkowski space-time, the subspace of hermitian matrices. The subgroup

$$SU(2)_L \times SU(2)_R = Spin(4)$$

such that $g_L \in SU(2)_L$ and $g_R \in SU(2)_R$ preserves the Euclidean space-time.

If one tries to find a Schwinger function S_2 related by analytic continuation to W_2 for the Weyl spinor theory, the factor $E + \boldsymbol{\sigma} \cdot \mathbf{p}$ in the expression for W_2 causes two sorts of problems:

- After analytic continuation to Euclidean space-time it takes spinors transforming under $SU(2)_R$ to spinors transforming under a different group, $SU(2)_L$. If the only fields in the theory are right-handed Weyl spinor fields, the Schwinger function cannot give an invariant inner product.
- After analytic continuation the self-adjoint factor $E + \boldsymbol{\sigma} \cdot \mathbf{p}$ is neither self-adjoint nor skew-adjoint. This makes it difficult to give S_2 an interpretation as inner product for a Euclidean field theory.

The first problem can be addressed by introducing fields of both chiralities, giving up on having a theory of only one chirality of Weyl spinors. The adjointness problem however still remains. Schwinger and later authors have dealt with this problem by doubling the number of degrees of freedom. Schwinger's argument was that this was necessary in order to have Euclidean transformation properties that did not distinguish a time direction. The problem also appears when one tries to find a generalization of the time-reflection operator Θ that allows reconstruction of the Minkowski theory from the Euclidean theory. The conventional wisdom has been to follow Osterwalder-Schrader, who deal with this by doubling the degrees of freedom, using a Θ which interchanges the two sorts of fields[27]. A fairly complete bibliography of attempts to deal with the Euclidean quantum spinor field is included at the end of this article.

2.4 Physical states and $SO(4)$ symmetry breaking

It appears to be a fundamental feature of Euclidean quantum field theory that, although Schwinger functions are $SO(4)$ invariant, recovering a connection to the physical theory in Minkowski space-time requires breaking $SO(4)$ invariance by a choice of time direction. In Minkowski space-time there is a Lorentz-invariant distinction between positive and negative energy, while in Euclidean space-time the corresponding distinction between positive and negative imaginary time is not $SO(4)$ invariant.

While the Euclidean Fock space has an $SO(4)$ action, states in it correspond not to physical states, but to paths in the space of physical states. A choice of imaginary time direction is needed to get physical states, either by restriction to a constant imaginary time subspace or by restriction to a positive imaginary time subspace together with use of reflection in imaginary time. The path integral formalism has the same feature: one can write Schwinger functions as an $SO(4)$ invariant path integral, but to get states one must choose a hypersurface and then define states using path integrals with fixed data on the hypersurface.

Needing to double spinor degrees of freedom and not being able to write down a free chiral spinor theory have always been disconcerting aspects of Euclidean quantum field theory. An alternate interpretation of the problems with quantizing spinor fields in Euclidean space-time would be that they are a more severe version of the problem with scalars, and again the quantization of such theories requires introducing a new degree of freedom that picks out an imaginary time direction.

This breaking of $SO(4)$ symmetry is a sort of spontaneous symmetry breaking with not the lowest energy state, but the distinction between positive and negative energy being responsible for the symmetry breaking. In section 4 we will argue that this spontaneous symmetry breaking is exactly the observed spontaneous symmetry breaking of the electroweak symmetry of the standard model. We will first though turn to a geometric framework in which one can naturally understand the analytic continuation between Euclidean and Minkowski spinor fields.

3 Twistor geometry

Twistor geometry is a 1967 proposal [28] due to Roger Penrose for a very different way of formulating four-dimensional space-time geometry. For a detailed expository treatment of the subject, see [31] (for a version aimed at physicists and applications in amplitude calculations, see [2]). Fundamental to twistor geometry is the twistor space $T = \mathbf{C}^4$, as well as its projective version, the space $PT = \mathbf{C}P^3$ of complex lines in T.

3.1 Compactified and complexified space-time

The relation of twistor space to conventional space-time is that complexified and compactified space-time is identified with the Grassmanian $M = G_{2,4}(\mathbf{C})$ of complex two-dimensional linear subspaces in T. A space-time point is thus a $\mathbf{C}^2$ in $\mathbf{C}^4$ which tautologically provides the spinor degree of freedom at that point. The spinor bundle S is the tautological two-dimensional complex vector bundle over M whose fiber S_m at a point $m \in M$ is the $\mathbf{C}^2$ that defines the point.

The group $SL(4, \mathbf{C})$ acts on T and transitively on the spaces PT and M of its complex subspaces. Points in the Grassmanian M can be represented as elements

$$\omega = (v_1 \otimes v_2 - v_2 \otimes v_1) \in \Lambda^2(\mathbf{C}^4)$$

by taking two vectors v_1, v_2 spanning the subspace. $\Lambda^2(\mathbf{C}^4)$ is six conplex dimensional and scalar multiples of ω gives the same point in M, so ω identifies M with a subspace of $P(\Lambda^2(\mathbf{C}^4)) = \mathbf{C}P^5$. Such ω satisfy the equation

$$\omega \wedge \omega = 0 \tag{3.1}$$

which identifies (the "Klein correspondence") M with a submanifold of $\mathbf{C}P^5$ given by a non-degenerate quadratic form. Twistors are spinors in six dimensions, with the action of $SL(4, \mathbf{C})$ on $\Lambda^2(\mathbf{C}^4) = \mathbf{C}^6$ preserving the quadratic form 3.1, and giving the spin double cover homomorphism

$$SL(4, \mathbf{C}) = Spin(6, \mathbf{C}) \to SO(6, \mathbf{C})$$

To get the tangent bundle of M, one needs not just the spinor bundle S, but also another two complex-dimensional vector bundle, the quotient bundle $S^\perp$ with fiber $S_m^\perp = \mathbf{C}^4/S_m$. Then the tangent bundle is

$$TM = Hom(S, S^\perp) = S^* \otimes S^\perp$$

with the tangent space $T_m M$ a four complex dimensional vector space given by the $Hom(S_m, S_m^\perp)$, the linear maps from S_m to $S_m^\perp$.

A choice of coordinate chart on M is given by picking a point $m \in M$ and identifying $S_m^\perp$ with a complex two plane transverse to S_m. The point m will be the origin of our coordinate system, so we will denote S_m by S_0 and $S_m^\perp$ by $S_0^\perp$. Now $T = S_0 \oplus S_0^\perp$ and one can choose basis elements $\mathbf{e}_1, \mathbf{e}_2 \in S_0$, $\mathbf{e}_3, \mathbf{e}_4 \in S_0^\perp$ for T. The coordinate of the two-plane spanned by the columns of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_{01} & z_{01} \\ z_{10} & z_{11} \end{pmatrix}$$

will be the 2 by 2 complex matrix

$$Z = \begin{pmatrix} z_{01} & z_{01} \\ z_{10} & z_{11} \end{pmatrix}$$

This coordinate chart does not include all of M, since it misses those points in M corresponding to complex two-planes that are not transverse to $S_0^\perp$. Our interest however will ultimately be not in the global structure of M, but in its local structure near the chosen point m, which we will study using the 2 by 2 complex matrix Z as coordinates. When we discuss M we will sometimes not distinguish between M and its local version as a complex four-dimensional vector space with origin of coordinates at m.

Writing elements of T as

$$\begin{pmatrix} s_1 \\ s_2 \\ s_1^\perp \\ s_2^\perp \end{pmatrix}$$

an element of T will be in the complex two plane with coordinate Z when

$$\begin{pmatrix} s_1^\perp \\ s_2^\perp \end{pmatrix} = Z \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \tag{3.2}$$

This incidence equation characterizes in coordinates the relation between lines (elements of PT) and planes (elements of M) in twistor space T. We'll sometimes also write this as

$$s^\perp = Zs$$

An $SL(4, \mathbf{C})$ determinant 1 matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

acts on T by

$$\begin{pmatrix} s \\ s^\perp \end{pmatrix} \to \begin{pmatrix} As + Bs^\perp \\ Cs + Ds^\perp \end{pmatrix}$$

On lines in the plane Z this is

$$\begin{bmatrix} s \\ Zs \end{bmatrix} \to \begin{bmatrix} As + BZs \\ Cs + DZs \end{bmatrix} = \begin{bmatrix} (A + BZ)s \\ (C + DZ)(A + BZ)^{-1}(A + BZ)s \end{bmatrix}$$

so the corresponding action on M will be given by

$$Z \to (C + DZ)(A + BZ)^{-1}$$

Since $\Lambda^2(S_0) = \Lambda^2(S_0^\perp) = \mathbf{C}$, S_0 and $S_0^\perp$ have (up to scalars) unique choices ϵ_{S_0} and $\epsilon_{S_0^\perp}$ of non-degenerate antisymmetric bilinear form, and corresponding choices of $SL(2, \mathbf{C}) \subset GL(2, \mathbf{C})$ acting on S_0 and $S_0^\perp$. These give (again, up to scalars), a unique choice of a non-degenerate symmetric form on $Hom(S_0, S_0^\perp)$, such that

$$\langle Z, Z \rangle = \det Z$$

The subgroup

$$Spin(4, \mathbf{C}) = SL(2, \mathbf{C}) \times SL(2, \mathbf{C}) \subset SL(4, \mathbf{C})$$

of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with

$$\det A = \det D = 1$$

acts on M in coordinates by

$$Z \to DZA^{-1}$$

preserving $\langle Z, Z \rangle$.

Besides the spaces PT and M of complex lines and planes in T, it is also useful to consider the correspondence space whose elements are complex lines inside a complex plane in T. This space can also be thought of as $P(S)$, the projective spinor bundle over M. There is a diagram of maps

$$\begin{array}{ccc} & P(S) & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ PT & & M \end{array}$$

where ν is the projection map for the bundle $P(S)$ and μ is the identification of a complex line in S as a complex line in T. μ and ν give a correspondence between geometric objects in PT and M. One can easily see that $\mu(\nu^{-1}(m))$ is the complex projective line in PT corresponding to a point $m \in M$ (a complex two plane in T is a complex projective line in PT). In the other direction, $\nu(\mu^{-1})$ takes a point p in PT to $\alpha(p)$, a copy of $\mathbf{C}P^2$ in M, called the "α-plane" corresponding to p.

In our chosen coordinate chart, this diagram of maps is given by

$$\begin{array}{ccc} & (Z, s) \in P(S) & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ \begin{bmatrix} s \\ Zs \end{bmatrix} \in PT & & Z \in M \end{array}$$

The incidence equation 3.2 relating PT and M implies that an α-plane is a null plane in the metric discussed above. Given two points Z_1, Z_2 in M corresponding to the same point in PT, their difference satisfies

$$s^{\perp} = (Z_1 - Z_2)s = 0$$

$Z_1 - Z_2$ is not an invertible matrix, so has determinant 0 and is a null vector.

3.2 The Penrose-Ward transform

The Penrose transform relates solutions of conformally-invariant wave equations on M to sheaf cohomology groups, identifying

- Solutions to a helicity $\frac{k}{2}$ holomorphic massless wave equation on U.
- The sheaf cohomology group

$$H^1(\widehat{U}, \mathcal{O}(-k-2))$$

Here $U \subset M$ and $\widehat{U} \subset PT$ are open sets related by the twistor correspondence, i.e.

$$\widehat{U} = \mu(\nu^{-1}(U))$$

We will be interested in cases where U and $\widehat{U}$ are orbits in M and PT for a real form of $SL(4, \mathbf{C})$. Here $\mathcal{O}(-k-2)$ is the sheaf of holomorphic sections of the line bundle $L^{\otimes(-k-2)}$ where L is the tautological line bundle over PT. For a detailed discussion, see for instance chapter 7 of [31].

The Penrose-Ward transform is a generalization of the above, introducing a coupling to gauge fields. One aspect of this is the Ward correspondence, an isomorphism between

- Holomorphic anti-self-dual $GL(n, \mathbf{C})$ connections A on $U \subset M$.
- Holomorphic rank n vector bundles E over $\widehat{U} \subset PT$.

Here "anti-self-dual" means the curvature of the connection satisfies

$$*F_A = -F_A$$

where $*$ is the Hodge dual. There are some restrictions on the open set U, and E needs to be trivial on the complex projective lines corresponding to points $m \in U$.

In one direction, the above isomorphism is due to the fact that the curvature F_A is anti-self-dual exactly when the connection A is integrable on the intersection of an α-plane with U. One can then construct the fiber E_p of E at p as the covariantly constant sections of the bundle with connection on the corresponding α-plane in M. In the other direction, one can construct a vector bundle $\widetilde{E}$ on U by taking as fiber at $m \in U$ the holomorphic sections of E on the corresponding complex projective line in PT. Parallel transport in this vector bundle can be defined using the fact that two points m_1, m_2 in U on the same α-plane correspond to intersecting projective lines in PT. For details, see chapter 8 of [31] and chapter 10 of [24].

Given an anti-self-dual gauge field as above, the Penrose transform can be generalized to a Penrose-Ward transform, relating

- Solutions to a helicity k holomorphic massless wave equation on U, coupled to a vector bundle $\widetilde{E}$ with anti-self-dual connection A.
- The sheaf cohomology group

$$H^1(\widehat{U}, \mathcal{O}(E)(-k-2))$$

For more about this generalization, see [14].

3.3 Twistor geometry and real forms

So far we have only considered complex twistor geometry, in which the relation to space-time geometry is that M is a complexified version of a four real dimensional space-time. From the point of view of group symmetry, the Lie algebra of $SL(4, \mathbf{C})$ is the complexification

$$\mathfrak{sl}(4, \mathbf{C}) = \mathfrak{g} \otimes \mathbf{C}$$

for several different real Lie algebras $\mathfrak{g}$, which are the real forms of $\mathfrak{sl}(4, \mathbf{C})$. To organize the possibilities, recall that $SL(4, \mathbf{C})$ is $Spin(6, \mathbf{C})$, the spin group for orthogonal linear transformations in six complex dimensions, so $\mathfrak{sl}(4, \mathbf{C}) = \mathfrak{so}(6, \mathbf{C})$. If one instead considers orthogonal linear transformations in six real dimensions, there are different possible signatures of the inner product to consider, all of which become equivalent after complexification. This corresponds to the possible real forms

$$\mathfrak{g} = \mathfrak{so}(3, 3), \mathfrak{so}(4, 2), \mathfrak{so}(5, 1), \text{and } \mathfrak{so}(6)$$

which we will discuss (there's another real form, $\mathfrak{su}(3, 1)$, which we won't consider). For more about real methods in twistor theory, see [38].

3.3.1 $Spin(3, 3) = SL(4, \mathbf{R})$

The simplest way to get a real version of twistor geometry is to take the discussion of section 3 and replace complex numbers by real numbers. Equivalently, one can look at subspaces invariant under the usual conjugation, given by the map σ

$$\sigma \begin{pmatrix} s_1 \\ s_2 \\ s_1^\perp \\ s_2^\perp \end{pmatrix} = \begin{pmatrix} \overline{s_1} \\ \overline{s_2} \\ \overline{s_1^\perp} \\ \overline{s_2^\perp} \end{pmatrix}$$

which acts not just on T but on PT and M. The fixed point set of the action on M is $M^{2,2} = G_{2,4}(\mathbf{R})$, the Grassmanian of real two-planes in $\mathbf{R}^4$. As a manifold, $G_{2,4}(\mathbf{R})$ is $S^2 \times S^2$, quotiented by a $\mathbf{Z}_2$. $M^{2,2}$ is acted on by the group $Spin(3, 3) = SL(4, \mathbf{R})$ of conformal transformations. σ acting on PT acts on the $\mathbf{C}P^1$ corresponding to a point in $M^{2,2}$ with an action whose fixed points form an equatorial circle.

Coordinates can be chosen as in the complex case, but with everything real. A point in $M^{2,2}$ is given by a real 2 by 2 matrix, which can be written in the form

$$Z = \begin{pmatrix} x_0 + x_3 & x_1 - x_2 \\ x_1 + x_2 & x_0 - x_3 \end{pmatrix}$$

for real numbers x_0, x_1, x_2, x_3. $M^{2,2}$ is acted on by the group $Spin(3, 3) = SL(4, \mathbf{R})$ of conformal transformations as in the complex case by

$$Z \to (C + DZ)(A + BZ)^{-1}$$

with the subgroup of rotations

$$Z \to DZA^{-1}$$

for $A, D \in SL(2, \mathbf{R})$ given by

$$Spin(2,2) = SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$$

This subgroup preserves

$$\langle Z, Z\rangle = \det Z = x_0^2 - x_3^2 - x_1^2 + x_2^2$$

For the Penrose transform in this case, see Atiyah's account in section 6.5 of [5]. For the Ward correspondence, see section 10.5 of [24].

3.3.2 $Spin(4,2) = SU(2,2)$

The real case of twistor geometry most often studied (a good reference is [31]) is that where the real space-time is the physical Minkowski space of special relativity. The conformal compactification of Minkowski space is a real submanifold of M, denoted here by $M^{3,1}$. It is acted upon transitively by the conformal group $Spin(4,2) = SU(2,2)$. This conformal group action on $M^{3,1}$ is most naturally understood using twistor space, as the action on complex planes in T coming from the action of the real form $SU(2,2) \subset SL(4,\mathbf{C})$ on T.

$SU(2,2)$ is the subgroup of $SL(4,\mathbf{C})$ preserving a real Hermitian form Φ of signature $(2,2)$ on $T = \mathbf{C}^4$. In our coordinates for T, a standard choice for Φ is given by

$$\Phi\left(\begin{pmatrix} s \\ s^\perp \end{pmatrix}, \begin{pmatrix} s' \\ (s^\perp)' \end{pmatrix}\right) = \begin{pmatrix} \overline{s} & \overline{s^\perp} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s' \\ (s^\perp)' \end{pmatrix} = s^\dagger (s^\perp)' + (s^\perp)^\dagger s' \quad (3.3)$$

Minkowski space is given by complex planes on which $\Phi = 0$, so

$$\Phi\left(\begin{pmatrix} s \\ Zs \end{pmatrix}, \begin{pmatrix} s \\ Zs \end{pmatrix}\right) = s^\dagger (Z + Z^\dagger) s = 0$$

Thus coordinates of points on Minkowski space are anti-Hermitian matrices Z, which can be written in the form

$$Z = -i \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = -i(x_0 \mathbf{1} + \mathbf{x} \cdot \sigma)$$

where σ_j are the Pauli matrices. The metric is the usual Minkowski metric, since

$$\langle Z, Z\rangle = \det Z = -x_0^2 + x_1^2 + x_2^2 + x_3^2$$

One can identify compactified Minkowski space $M^{3,1}$ as a manifold with the Lie group $U(2)$ which is diffeomorphic to $(S^3 \times S^1)/\mathbf{Z}_2$. The identification of the tangent space with anti-Hermitian matrices reflects the usual identification of the tangent space of $U(2)$ at the identity with the Lie algebra of anti-Hermitian matrices.

$SL(4,C)$ matrices are in $SU(2,2)$ when they satisfy

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Poincaré subgroup P of $SU(2,2)$ is given by elements of $SU(2,2)$ of the form

$$\begin{pmatrix} A & 0 \\ C & (A^\dagger)^{-1} \end{pmatrix}$$

where $A \in SL(2,\mathbf{C})$ and $A^\dagger C = -C^\dagger A$. These act on Minkowski space by

$$Z \to (C + (A^\dagger)^{-1} Z) A^{-1} = (A^\dagger)^{-1} Z A^{-1} + C A^{-1}$$

One can show that CA^{-1} is anti-Hermitian and gives arbitrary translations on Minkowski space. The Lorentz subroup is $Spin(3,1) = SL(2,\mathbf{C})$ acting by

$$Z \to (A^\dagger)^{-1} Z A^{-1}$$

Here $SL(2,C)$ is acting by the standard representation on S_0, and by the conjugate-dual representation on $S_0^\perp$.

Note that, for the action of the Lorentz $SL(2,\mathbf{C})$ subgroup, twistors written as elements of $S_0 \oplus S_0^\perp$ behave like usual Dirac spinors (direct sums of a standard $SL(2,\mathbf{C})$ spinor and one in the conjugate-dual representation), with the usual Dirac adjoint, in which the $SL(2,\mathbf{C})$-invariant inner product is given by the signature $(2,2)$ Hermitian form

$$\langle \psi_1, \psi_2 \rangle = \psi_1^\dagger \gamma_0 \psi_2$$

Twistors, with their $SU(2,2)$ conformal group action and incidence relation to space-time points, are however something different than Dirac spinors.

The $SU(2,2)$ action on M has six orbits: $M_{++}, M_{--}, M_{+0}, M_{-0}, M_{00}$, where the subscript indicates the signature of Φ restricted to planes corresponding to points in the orbit. The last of these is a closed orbit $M^{3,1}$, compactified Minkowski space. Acting on projective twistor space PT, there are three orbits: PT_+, PT_-, PT_0, where the subscript indicates the sign of Φ restricted to the line in T corresponding to a point in the orbit. The first two are open orbits with six real dimensions, the last a closed orbit with five real dimensions. The points in compactified Minkowski space $M_{00} = M^{3,1}$ correspond to projective lines in PT that lie in the five dimensional space PT_0. Points in M_{++} and M_{--} correspond to projective lines in PT_+ or PT_- respectively.

One can construct infinite dimensional irreducible unitary representations of $SU(2,2)$ using holomorphic geometry on PT_+ or M_{++}, with the Penrose transform relating the two constructions [13]. For $\overline{PT_+}$ the closure of the orbit PT_+, the Penrose transform identifies the sheaf cohomology groups $H^1(\overline{PT_+}, \mathcal{O}(-k-2))$ for $k > 0$ with holomorphic solutions to the helicity $\frac{k}{2}$ wave equation on

M_{++}. Taking boundary values on $M^{3,1}$, these will be real-analytic solutions to the helicity $\frac{k}{2}$ wave equation on compactified Minkowski space. If one instead considers the sheaf cohomology $H^1(PT_+, \mathcal{O}(-k-2))$ for the open orbit PT_+ and takes boundary values on $M^{3,1}$ of solutions on M_{++}, the solutions will be hyperfunctions, see [32].

The Ward correspondence relates holomorphic vector bundles on PT_+ with anti-self-dual $GL(n, \mathbf{C})$ gauge fields on M_{++}. However, in this Minkowski signature case, all solutions to the anti-self-duality equations as boundary values of such gauge fields are complex, so one does not get anti-self-dual gauge fields for compact gauge groups like $SU(n)$.

3.3.3 $Spin(5,1) = SL(2, \mathbf{H})$

Changing from Minkowski space-time signature $(3,1)$ to Euclidean space-time signature $(4,0)$, the compactified space-time $M^4 = S^4$ is again a real submanifold of M. To understand the conformal group and how twistors work in this case, it is best to work with quaternions instead of complex numbers, identifying $T = \mathbf{H}^2$. When working with quaternions, one can often instead use corresponding complex 2 by 2 matrices, with a standard choice

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \leftrightarrow q_0 - i(q_1\sigma_1 + q_2\sigma_2 + q_3\sigma_3)$$

For more details of the quaternionic geometry that appears here, see [5] or [30]

The relevant conformal group acting on S^4 is $Spin(5,1) = SL(2, \mathbf{H})$, again best understood in terms of twistors and the linear action of $SL(2, \mathbf{H})$ on $T = \mathbf{H}^2$. The group $SL(2, \mathbf{H})$ is the group of quaternionic 2 by 2 matrices satisfying a single condition that one can think of as setting the determinant to one, although the usual determinant does not make sense in the quaternionic case. Here one can interpret the determinant using the isomorphism with complex matrices, or, at the Lie algebra level, $\mathfrak{sl}(2, \mathbf{H})$ is the Lie algebra of 2 by 2 quaternionic matrices with purely imaginary trace.

While one can continue to think of points in $S^4 \subset M$ as complex two planes, one can also identify these complex two planes as quaternionic lines and S^4 as $\mathbf{H}P^1$, the projective space of quaternionic lines in $\mathbf{H}^2$. The conventional choice of identification between $\mathbf{C}^2$ and $\mathbf{H}$ is

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \leftrightarrow s = s_1 + s_2\mathbf{j}$$

One can then think of the quaternionic structure as providing an alternate notion of conjugation than the usual one, given instead by left multiplying by $j \in \mathbf{H}$. Using $jzj = -\overline{z}$ one can show that

$$\sigma \begin{pmatrix} s_1 \\ s_2 \\ s_1^\perp \\ s_2^\perp \end{pmatrix} = \begin{pmatrix} -\overline{s_2} \\ \overline{s_1} \\ -\overline{s_2^\perp} \\ \overline{s_1^\perp} \end{pmatrix} \tag{3.4}$$

σ satisfies $\sigma^2 = -1$ on T, so $\sigma^2 = 1$ on PT. We will see later that while σ has no fixed points on PT, it does fix complex projective lines.

The same coordinates used in the complex case can be used here, where now $S_0^\perp$ is a quaternionic line transverse to S_0, so coordinates on T are the pair of quaternions

$$\begin{pmatrix} s \\ s^\perp \end{pmatrix}$$

These are also homogeneous coordinates for points on $S^4 = \mathbf{H}P^1$ and our choice of $Z \in \mathbf{H}$ given by

$$\begin{pmatrix} s \\ Zs \end{pmatrix}$$

as the coordinate in a coordinate system with origin the point with homogeneous coordinates

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The point at ∞ will be the one with homogeneous coordinates

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This is the quaternionic version of the usual sort of choice of coordinates in the case of $S^2 = \mathbf{C}P^1$, replacing complex numbers by quaternions. The coordinate of a point on S^4 with homogeneous coordinates

$$\begin{pmatrix} s \\ s^\perp \end{pmatrix}$$

will be

$$s^\perp s^{-1} = \frac{(s_1^\perp + s_2^\perp \mathbf{j})(\overline{s_1} - s_1\mathbf{j})}{|s_1|^2 + |s_2|^2} = \frac{s_1^\perp \overline{s_1} + s_2^\perp \overline{s_2} + (-s_1^\perp s_2 + s_2^\perp s_1)\mathbf{j}}{|s_1|^2 + |s_2|^2} \tag{3.5}$$

A coordinate of a point will now be a quaternion $Z = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ corresponding to the 2 by 2 complex matrix

$$Z = x_0\mathbf{1} - i\mathbf{x}\cdot\sigma = \begin{pmatrix} x_0 - ix_3 & -ix_1 - x_2 \\ -ix_1 + x_2 & x_0 + ix_3 \end{pmatrix}$$

The metric is the usual Euclidean metric, since

$$\langle Z, Z\rangle = \det Z = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

The conformal group $SL(2, \mathbf{H})$ acts on $T = \mathbf{H}^2$ by the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are now quaternions, satisfying together the determinant 1 condition. These act on the coordinate Z as in the complex case, by

$$Z \to (C + DZ)(A + BZ)^{-1}$$

The Euclidean group in four dimensions will be the subgroup of elements of the form

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

such that A and D are independent unit quaternions, thus in the group $Sp(1) = SU(2)$, and C is an arbitrary quaternion. The Euclidean group acts by

$$Z \to DZA^{-1} + CA^{-1}$$

with the spin double cover of the rotational subgroup now $Spin(4) = Sp(1) \times Sp(1)$. Note that spinors behave quite differently than in Minkowski space: there are independent unitary $SU(2)$ actions on S_0 and $S_0^\perp$ rather than a non-unitary $SL(2, \mathbf{C})$ action on S_0 that acts at the same time on $S_0^\perp$ by the conjugate transpose representation.

The projective twistor space PT is fibered over S^4 by complex projective lines

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & PT = \mathbf{C}P^3 \\ & & \big\downarrow \pi \\ & & S^4 = \mathbf{H}P^1 \end{array} \tag{3.6}$$

The projection map π is just the map that takes a complex line in T identified with $\mathbf{H}^2$ to the corresponding quaternionic line it generates (multiplying elements by arbitrary quaternions). In this case the conjugation map σ of 3.4 has no fixed points on PT, but does fix the complex projective line fibers and thus the points in $S^4 \subset M$. The action of σ on a fiber takes a point on the sphere to the opposite point, so has no fixed points.

Note that the Euclidean case of twistor geometry is quite different and much simpler than the Minkowski one. The correspondence space $P(S)$ (here the complex lines in the quaternionic line specifying a point in $M^4 = S^4$) is just PT itself, and the twistor correspondence between PT and S^4 is just the projection π. Unlike the Minkowski case where the real form $SU(2, 2)$ has a non-trivial orbit structure when acting on PT, in the Euclidean case the action of the real form $SL(2, \mathbf{H})$ is transitive on PT.

In the Euclidean case, the projective twistor space has another interpretation, as the bundle of orientation preserving orthogonal complex structures on S^4. A complex structure on a real vector space V is a linear map J such that $J^2 = -1$, providing a way to give V the structure of a complex vector space (multiplication by i is multiplication by J). J is orthogonal if it preserves an inner product on V. While on $\mathbf{R}^2$ there is just one orientation-preserving orthogonal complex structure, on $\mathbf{R}^4$ the possibilities can be parametrized by a sphere S^2. The fiber $S^2 = \mathbf{C}P^1$ of 3.6 above a point on S^4 can be interpreted as the space of orientation preserving orthogonal complex structures on the four real dimensional tangent space to S^4 at that point.

One way of exhibiting these complex structures on $\mathbf{R}^4$ is to identify $\mathbf{R}^4 = \mathbf{H}$ and then note that, for any real numbers x_1, x_2, x_3 such that $x_1^2 + x_2^2 + x_3^2 = 1$, one gets an orthogonal complex structure on $\mathbf{R}^4$ by taking

$$J = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

Another way to see this is to note that the rotation group $SO(4)$ acts on orthogonal complex structures, with a $U(2)$ subgroup preserving the complex structure, so the space of these is $SO(4)/U(2)$, which can be identified with S^2.

More explicitly, in our choice of coordinates, the projection map is

$$\pi : \begin{bmatrix} s \\ s^\perp = Zs \end{bmatrix} \rightarrow Z = \begin{pmatrix} x_0 - ix_3 & -ix_1 - x_2 \\ -ix_1 + x_2 & x_0 + ix_3 \end{pmatrix}$$

For any choice of s in the fiber above Z, $s^\perp$ associates to the four real coordinates specifying Z an element of $\mathbf{C}^2$. For instance, if $s = \big(1, 0\big)$, the identification of $\mathbf{R}^4$ with $\mathbf{C}^2$ is

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \leftrightarrow \begin{pmatrix} x_0 - ix_3 \\ -ix_1 + x_2 \end{pmatrix}$$

The complex structure on $\mathbf{R}^4$ one gets is not changed if s gets multiplied by a complex scalar, so it just depends on the point $[s]$ in the $\mathbf{C}P^1$ fiber.

For another point of view on this, one can see that for each point $p \in PT$, the corresponding α-plane $\nu(\mu^{-1}(p))$ in M intersects its conjugate $\sigma(\nu(\mu^{-1}(p)))$ in exactly one real point, $\pi(p) \in M^4$. The corresponding line in PT is the line determined by the two points p and $\sigma(p)$. At the same time, this α-plane provides an identification of the tangent space to M^4 at $\pi(p)$ with a complex two plane, the α-plane itself. The $\mathbf{C}P^1$ of α -planes corresponding to a point in S^4 are the different possible ways of identifying the tangent space at that point with a complex vector space. The situation in the Minkowski space case is quite different: there if $\mathbf{C}P^1 \subset PT_0$ corresponds to a point $Z \in M^{3,1}$, each point p in that $\mathbf{C}P^1$ gives an α-plane intersecting $M^{3,1}$ in a null line, and the $\mathbf{C}P^1$ can be identified with the "celestial sphere" of null lines through Z.

In the Euclidean case , the Penrose transform will identify the sheaf cohomology group $H^1(\pi^{-1}(U), \mathcal{O}(-k-2))$ for $k > 0$ with solutions of helicity $\frac{k}{2}$ linear field equations on an open set $U \subset S^4$. Unlike in the Minkowski space case, in Euclidean space there are $U(n)$ bundles $\widetilde{E}$ with connections having non-trivial anti-self-dual curvature. The Ward correspondence between such connections and holomorphic bundles E on PT for $U = S^4$ has been the object of intensive study, see for example Atiyah's survey [5]. The Penrose-Ward transform identifies

- Solutions to a field equation on U for sections $\Gamma(S^k \otimes \widetilde{E})$, with covariant derivative given by an anti-self-dual connection A, where S^k is the k'th symmetric power of the spinor bundle.

- The sheaf cohomology group

$$H^1(\widehat{U}, \mathcal{O}(E)(-k-2))$$

where $\widehat{U} = \pi^{-1}U$.

For the details of the Penrose-Ward transform in this case, see [18].

3.3.4 $Spin(6) = SU(4)$

If one picks a positive definite Hermitian inner product on T, this determines a subgroup $SU(4) = Spin(6)$ that acts on T, and thus on PT, M and $P(S)$. One has

$$PT = \frac{SU(4)}{U(3)}, \quad M = \frac{SU(4)}{S(U(2) \times U(2))}, \quad P(S) = \frac{SU(4)}{S(U(1) \times U(2))}$$

and the $SU(4)$ action is transitive on these three spaces. There is no four real dimensional orbit in M that could be interpreted as a real space-time that would give M after complexification.

In this case the Borel-Weil-Bott theorem relates sheaf-cohomology groups of equivariant holomorphic vector-bundles on PT, M and $P(S)$, giving them explicitly as certain finite dimensional irreducible representations of $SU(4)$. For more details of the relation between the Penrose transform and Borel-Weil-Bott, see [6]. The Borel-Weil-Bott theorem [8] can be recast in terms of index theory, replacing the use of sheaf-cohomology with the Dirac equation [9]. For a more general discussion of the relation of representation theory and the Dirac operator, see [19].

4 Twistor geometry and unification

Conventional attempts to relate twistor geometry to fundamental physics have concentrated on the Minkowski signature real form, providing a fundamental role for Weyl spinors and a different perspective on space-time symmetries, naturally incorporating conformal invariance. The Penrose transform gives an alternative treatment of conformally invariant massless linear field equations on Minkowski space-time in terms of sheaf cohomology of powers of the tautological bundle on PT. Since there are no non-trivial solutions of the Minkowski space-time $SU(n)$ anti-self-duality equations, there is no role for the Penrose-Ward transform to play. Twistor geometry appears to have little to say about either the internal symmetries of the Standard Model or the origin of the Einstein equations of general relativity.

Taking Euclidean space-time as fundamental, the situation is quite different, indicating a possible new unified way of understanding the basic degrees of freedom of the Standard Model and gravity. Weyl spinors still play a fundamental role, and usual space-time symmetries are recovered after analytic continuation. One also gets the Standard Model internal symmetries, from $U(2)$ and

$SU(3)$ internal symmetries at each point on the projective twistor space PT, with $SU(2) \subset U(2)$ spontaneously broken. The fundamental degrees of freedom governing gravitational forces are now chiral, related to the usual story of Ashtekar variables. The Penrose-Ward transform relates a holomorphic version of gauge-field dynamics on projective twistor space to a chiral one on Euclidean space-time. A generation of Standard Model fermions naturally fit into spinor fields on PT.

In this section we'll examine this proposal in more detail, in the next turn to the new problems it raises.

4.1 $U(2)$ electroweak symmetry and its spontaneous symmetry breaking

As discussed in section 3.3.3, the fibration 3.6 of PT over M^4 can be identified with the projective spinor bundle $P(S)$. The fiber above each point of M^4 is the space of orthogonal complex structures on the tangent space at the point, so a copy of $SO(4)/U(2)$. To each element s of the fiber S_0, one gets an identification of the real tangent space at 0 with maps from s to elements of $S_0^\perp$, which has a complex vector space structure. The corresponding complex structure this puts on the real tangent space at 0 only depends on the complex line generated by s, so the point it determines in $P(S_0)$.

One thus has for each point in $PT = P(S)$ a complex structure on the tangent space at $\pi(s)$ and a $U(2) \subset SO(4)$ group that leaves that complex structure invariant. These together give a principal bundle

$$\begin{array}{ccc} U(2) & \longrightarrow & Sp(2) \\ & & \downarrow \\ & & PT = Sp(2)/U(2) \end{array}$$

over PT where the choice of $Sp(2)$ (acting in the usual way on $\mathbf{H}^2$) depends on how twistor space $\mathbf{C}^4$ is identified with $\mathbf{H}^2$.

As discussed in section 2, given a quantum field theory defined in Euclidean space time, the four-dimensional rotational symmetry needs to be broken by a choice of (imaginary) time direction in order to define the states of the theory. The choice of an (imaginary) time direction is given by the choice of vector in the tangent space of M^4. For each point in the fiber of 3.6 this tangent space gets identified with $\mathbf{C}^2$ and a tangent vector in the (imaginary) time direction transforms under $U(2)$ as the usual representation on $\mathbf{C}^2$. Note that this is the way the Higgs field in the Standard Model transforms under the electroweak $U(2)$. This indicates that the Higgs field of the Standard Model has a Euclidean space-time geometrical significance, as a vector pointing in the imaginary time direction, with the necessary breaking of symmetry needed to define the space of states corresponding to electroweak symmetry breaking.

A choice of (imaginary) time direction has been made by our choice of coordinates on M^4 (equation 3.5): the real direction in the quaternionic coordinate.

This choice could be changing by changing coordinates, for instance by an action of $Spin(4)$,

$$s^{\perp}s^{-1} \rightarrow q_1 s^{\perp}s^{-1}q_2^{-1}$$

for (q_1, q_2) a pair of unit quaternions. The subgroup $q_1 = q_2$ is a $Spin(3)$ subgroup which changes the coordinates while leaving the (imaginary) time direction direction invariant. This will correspond to spatial rotations, and these transformations will act on the states of the theory.

One way to characterize the single-particle state space $\mathcal{H}_1$ for a spinor field is in terms of the initial data at $t = 0$ for a solution to a Dirac equation. This has the disadvantage of obscuring the Poincaré group action on $\mathcal{H}_1$, but the advantage that one can identify the spacelike $t = 0$ subspace of Minkowski space $M^{3,1}$ (which will be a 3-sphere denoted M^3) with the $\tau = 0$ equator M^4_0 in Euclidean space $M^4 = S^4$ that divides the space into upper ($\tau > 0$) and lower ($\tau < 0$) hemispheres M^4_+ and M^4_-.

Taking the Euclidean point of view as starting point, recall from section 3.3.3 that, after choosing an identification of $\mathbf{H}^2$ with $\mathbf{C}^4$, one has a fibration of PT over $S^4 = M^4$. In the coordinates for S^4 of equation 3.5, setting $\tau = x_0 = 0$ corresponds to the condition that the real part of the numerator vanish, so

$$s_1^{\perp}\overline{s_1} + s_2^{\perp}\overline{s_2} + \overline{s_1^{\perp}}s_1 + \overline{s_2^{\perp}}s_2 = 0$$

Note that (by equation 3.3), this is exactly the condition

$$\Phi(s, s) = 0$$

that describes the five-dimensional subspace $N = PT_0$ of PT which contains the complex lines corresponding to Minkowski space $M^{3,1}$. One has the fibration

$$\begin{array}{ccccc}
\mathbf{C}P^1 & \longrightarrow & N = PT_0 & \hookrightarrow & PT = \mathbf{C}P^3 \\
 & & \downarrow & & \downarrow \pi \\
 & & M^3 & \hookrightarrow & S^4 = \mathbf{H}P^1
\end{array}$$

as well as

$$\begin{array}{ccccc}
\mathbf{C}P^1 & \longrightarrow & PT_{\pm} & \longrightarrow & PT = \mathbf{C}P^3 \\
 & & \downarrow & & \downarrow \pi \\
 & & M^4_{\pm} & \longrightarrow & S^4 = \mathbf{H}P^1
\end{array}$$

Instead of relating Euclidean and Minkowski space spinor fields by analytic continuation of solutions of the massless Dirac equation between M^4 and $M^{3,1}$, one can instead use the Euclidean and Minkowski Penrose transforms to relate both to holomorphic objects on PT. The single-particle state space then will be given by holomorphic sections of a bundle over PT_+, the part of PT that projects to the upper hemisphere of M^4.

4.2 Chirality and gravitational degrees of freedom

Recall that the tangent bundle to M is

$$TM = Hom(S, S^{\perp}) = S^* \otimes S^{\perp}$$

and tangent vectors in T_mM identify the fibers S_m and $S_m^{\perp}$. This implies that cotangent vectors also give such an identification, and one finds that the bundle of two-forms satisfies

$$\Lambda^2 M \subset Hom(S, S) \oplus Hom(S^{\perp}, S^{\perp})$$

We'll call two-forms in $\Lambda^2_- \subset Hom(S, S)$ anti-self dual and those in $\Lambda^2_+ \subset Hom(S^{\perp}, S^{\perp})$ self-dual. Here we're following the conventions of [4] which should be consulted for more details about this and what follows. An alternate definition of $\Lambda^2_{\pm} M$ is as the ± 1 eigenspaces of the Hodge star operator.

If one gauges the $Spin(4)$ symmetry acting on the tangent space to M, the gauge fields are the spin-connection, taking values in the Lie algebra of $Spin(4)$, $\mathfrak{su}(2) + \mathfrak{su}(2)$. These two factors separately act on the S and $S^{\perp}$ bundles and can be identified with the anti-self dual and self dual fibers of $\Lambda^2(M)$. The curvature of the spin connection takes values in linear maps from $\Lambda^2(M)$ to itself, and with respect to the $\Lambda^2_{\pm}$ decomposition has the block-diagonal form:

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

The corresponding metric will be Einstein when $B = 0$.

There is a long history of "chiral" formulations of the theory of general relativity which use the spin connection and take advantage of the above decomposition into self-dual and anti-self-dual pieces. For an extensive discussion of this topic, see [22] (this book includes a final chapter discussing twistor space versions of this, see also [17]). Besides the spin connection, such a theory also needs to incorporate the tetrad fields giving the vectors in orthonormal frames.

Note that the proposal here is that the electroweak $SU(2)$ gauge fields are the spin connection fields above corresponding to the second $\mathfrak{su}(2)$ factor, the one acting on the bundle $S^{\perp}$. There is also a long history of proposals for "graviweak unification", for two examples see [26] and [3]. The version of this idea proposed here is rather different due to its use of Euclidean space-time and identification of the imaginary time component of a tetrad as responsible for spontaneous symmetry breaking. Further work is needed to see if this gives a viable version of quantized gravity, one which would be naturally unified with the electroweak theory.

4.3 Spinors on PT

Taking as fundamental the space PT with its fibration to M^4, one can ask what holomorphic vector bundle on PT corresponds to the Standard Model matter fields. It turns out that the spinor bundle on PT has the correct properties to describe a generation of leptons. At a point $p \in PT$, the complex tangent space splits into a sum

$$T_p = V_p \oplus H_p$$

of

- a complex one-dimensional vertical subspace V_p, tangent to the $\mathbf{C}P^1$ fiber.
- a complex two-dimensional horizontal subspace H_p, which is the real four-dimensional tangent space to M^4 at $\pi(p)$, given the complex structure corresponding to the point p in the fiber above $\pi(p)$.

For details about the relation between spinors and the complex exterior algebra, see chapter 31 of [37], in particular section 31.5 about the case of spinors in four dimensions.

Spinors for the sum $V_p \oplus H_p$ will be given by a tensor product of spinors for V_p and those for H_p. Spinors for V_p give the usual spinor fiber $S_{\pi(p)}$, those for H_p are given by $\Lambda^*(H_p) \otimes \mathbf{C}_p$, where $\mathbf{C}_p$ is the complex line in the fiber $S_\pi(p)$ corresponding to the point p. Elements of $\Lambda^*(H_p)$ transform $U(2)$ like a generation of leptons:

- $\Lambda^1(H_p)$ is complex two-dimensional, has the correct transformation properties to describe a left-handed neutrino and electron.
- $\Lambda^2(H_p)$ is complex one-dimensional, has the correct transformation property (weak hypercharge -2) to describe a right-handed electron.
- $\Lambda^0(H_p)$ is complex one-dimensional, has the correct transformation properties (zero electroweak charges) to describe a conjectural right-handed neutrino.

4.4 $SU(3)$ symmetry

So far we have just been using aspects of twistor geometry that at a point $p \in PT$ involve the fiber $L_p \subset \mathbf{C}^4$ of the tautological line bundle L over PT, as well as the fibration 3.6 to M^4. Just as in the case of the Grassmanian M, where one could define not just a tautological bundle S, but also a quotient bundle $S^\perp$, over PT one has not just L, but also a quotient bundle $L^\perp$. This quotient bundle will have a complex 3-dimensional fiber at p given by $L_p^\perp = \mathbf{C}^4/L_p$. One can think of PT as

$$PT = \frac{U(4)}{U(1)\times U(3)} = \frac{SU(4)}{S(U(1)\times U(3))} = \frac{SU(4)}{U(3)}$$

where the $U(1)$ factor acts as unitary transformations on the fiber L_p, while the $U(3)$ acts as unitary transformations on the fiber $\mathbf{C}^4/L_p$. The $SU(3) \subset U(3)$ subgroup provides a possible origin for the color gauge group of the Standard Model, with fermion fields taking values in $L_P^\perp$ giving the quarks.

In the case of the $U(2)$ electroweak symmetry, to a point $p \in PT$ we associated not just the line L_p, but also the spinor space $S_{\pi(p)}$, with $L_p \subset S_{\pi(p)}$.

The internal electroweak $SU(2)$ acts on $S^{\perp}_{\pi(p)}$, while the color $SU(3)$ acts on $L^{\perp}_{p}$. One needs to avoid defining these spaces as subspaces of the same $\mathbf{C}^4$ in order to ensure that the two group actions commute as needed by the Standard Model.

5 Open problems and speculative remarks

The unification proposal discussed here is still missing some crucial aspects. Exactly how to formulate a generation of fermions in terms of objects on PT in such a way that the $SU(3)$ internal symmetry acts as expected remains to be worked out. Note that the well-known problems of defining spinor fields in lattice gauge theory may take on a new aspect when lifted to PT. In particular, a well-known problem is that Kähler-Dirac (or Kogut-Susskind) fermions carry far too many degrees of freedom, taking values in $\Lambda^*(\mathbf{R}^4)$ rather than the spinors. Identifying these degrees of freedom as internal degrees of freedom when lifted to $PT = P(S)$ (with the spinor degree of freedom coming from the fiber) may be possible. The well-known problems with putting chiral gauge symmetry on the lattice may also have some solution when working on PT instead of the four-dimensional base space.

More speculatively, it is possible that the fundamental theory involves not just the usual twistor geometry of PT, but should be formulated on the seven-sphere S^7, which is a circle bundle over PT. S^7 is a remarkably unusual geometric structure, exhibiting a wide range of different symmetry groups, since one has

$$S^7 = Spin(8)/Spin(7) = Spin(7)/G_2 = Spin(6)/SU(3) = Spin(5)/Sp(1)$$

as well as algebraic structures arising from identifying S^7 with the unit octonions. Our discussion has exploited the last two geometries on S^7, not the first two.

Most critically, it is unclear what the origin of generations might be. This issue is crucial for any hope of understanding where fermion masses and mixing angles come from.

The proposal to think about the Standard Model on PT rather than on Minkowski space involves a significant reconfiguration of the degrees of freedom and symmetry principles governing the theory. In the conventional definition of the Standard Model, internal symmetry groups are attached to each point in space-time, giving a gauge symmetry when treated independently at each point in space-time. In the twistor space setting described here, internal symmetry groups are attached instead to each point in PT. Recall that, from the Minkowski space point of view, such a point corresponds to a null-line, a light ray, so gauge degrees of freedom live not at points but on light rays. From the Euclidean point of view, each point in PT projects to a single point in M^4, but this is true for an entire sphere of points in PT. So, for a Euclidean space-time point one has not a single gauge degree of freedom, but a sphere's worth of them.

Much remains to be done in order to realize a fundamental theory based on PT, since is not clear how the quantum field theory formalism should be implemented there. From the point of view of geometric quantization and representation theory, the relevant case here of the orbits of $SU(2,2)$ on PT is an exceptionally challenging one. It is a fundamental example of a "minimal" orbit, for which geometric quantization runs into difficult technical problems due to the lack of an appropriate invariant polarization. For a 1982 history of work on this specific case, see appendix A of [29]. These problems have been studied from the point of view of Minkowski space-time conformal symmetry ($SU(2,2)$). It may be that the analytic continuation to the Euclidean space-time perspective will give new insight into these problems. For some discussion of the relation of the Dirac operator on a manifold such as M^4 to the Dolbeault operator on the the projective twistor space, see [12].

The idea of studying quantum Yang-Mills theory on twistor space has attracted attention over the years, going back for instance to work of Nair [25] in 1988. This works best in a formalism based on expanding about a chiral version of Yang-Mills, as studied by Chalmers-Siegel [10]. In 2003 Witten [34] made major advances in calculating Yang-Mills amplitudes using twistor space, and this led to an active ongoing program of studying such amplitudes that exploits twistor space ideas. For more of the literature relating supersymmetric Yang-Mills theories and quantum field theories on projective twistor space, see for instance [7], [11] and the review article [1].

The Standard Model internal symmetries on projective twistor space considered here are part of the twistor space geometry, so somewhat different than the usual purely internal symmetries studied when relating twistor space and Yang-Mills theory. Together with the extra feature of a degree of freedom breaking Euclidean rotational invariance, this does not appear to correspond to previously studied theories.

For another possible point of view on the anti-self-duality equations, note that they can be formulated as the vanishing of a moment map, and see an old speculative discussion of the significance of this in [36]. $N = 2$ and $N = 4$ super Yang-Mills give topological quantum field theories (see [33]), with the feature that a "twisting" of the space-time symmetry into an internal symmetry plays a crucial role.

6 Conclusions

The main conclusion of this work is that twistor geometry provides a compelling picture of fundamental physics, integrating internal and space-time symmetries, as long as one treats together its Euclidean and Minkowski aspects, related through the projective twistor space PT. The Euclidean aspect is crucial for understanding the origin of the Standard Model internal symmetries and the breaking of electroweak symmetry, which is inherent in the Euclidean space-time definition of physical states.

Much work remains to be done to explicitly construct and understand a full theory defined on PT that would correspond to the Standard Model and general relativity, with the expected three generations of matter fields. Such a theory might allow understanding of currently unexplained features of the Standard Model, as well as possibly making testable predictions that differ from those of the Standard Model. In particular, the framework proposed is fundamentally chiral as a theory of gravity, not just in the electroweak sector, and this may have observable implications.

An argument from beauty can be made, as twistor unification provides a strikingly elegant way of understanding both space-time geometry and known internal symmetries. In addition, the geometric Langlands program in recently years has given evidence for a dramatic unified perspective relating number theory, geometry and representation theory. The quantum field theory version of geometric Langlands [20] is based on the same sort of $N = 4$ super Yang-Mills theory that may be related to twistor unification (for a recent twistor version, see [15]), raising the possibility of a remarkable unification of mathematics and physics at a fundamental level.

A Hyperfunctions

Wightman functions are conventionally described as tempered distributions on a Schwarz space of test functions. Such distributions occur as boundary values of holomorphic functions, and one can instead work with hyperfunctions, which are spaces of such boundary values. Like distributions, they can be thought of a elements of a dual space to a space of well-behaved test functions, which will be real analytic, not just infinitely differentiable. For an enlightening discussion of hyperfunctions in this context, a good source is chapter 9 of Roger Penrose's *The Road to Reality* [41].

A.1 Hyperfunctions on the circle

In the case of the unit circle, one can generalize the notion of functions by considering boundary values of holomorphic functions on the open unit disk. Taking the circle to be the equator of a Riemann sphere, a hyperfunction on the circle can be defined as a pair of functions, one holomorphic on the open upper hemisphere, the other holomorphic on the open lower hemisphere, with pairs equivalent when they differ by a globally holomorphic function.

Boundary values of functions holomorphic on the upper hemisphere correspond to Fourier series with Fourier coefficients satisfying $a_n = 0$ for $n < 0$, those with $a_n = 0$ for $n > 0$ correspond to boundary values of functions holomorphic on the lower hemisphere. The global holomorphic functions on the sphere are just the constants, those with only a_0 non-zero. Hyperfunctions allow one to make sense of a very large class of Fourier series (those with coefficients growing at less than exponential rate as $n \to \pm\infty$) as linear functionals on real analytic test functions (whose coefficients a_n fall off faster than $e^{-c|n|}$ for some $c > 0$).

The discrete series representations of the non-compact Lie group $SL(2,\mathbf{R})$ can naturally be constructed using such hyperfunctions on the circle. The group $SL(2,\mathbf{R})$ acts on the Riemann sphere, with orbits the upper hemisphere, the lower hemisphere, and the equator. The discrete series representations are hyperfunctions on the equator, boundary values of holomorphic sections of a line bundle on either the upper or lower hemisphere. For more about this, see section 10.1 of [6]. For a more general discussion of hyperfunctions on the circle and their relation to hyperfunctions on $\mathbf{R}$, see the previously mentioned chapter 9 of [41].

A.2 Hyperfunctions on R

Solutions to wave equations are conventionally discussed using the theory of distributions, since even the simplest plane-wave solutions are delta-functions in energy-momentum space. Distributions are generalizations of functions that can be defined as elements of the dual space (linear functionals) of some well-behaved set of functions, for instance smooth functions of rapid decrease (Schwartz functions) for the case of tempered distributions. The theory of hyperfunctions gives a further generalization, providing a dual of an even more restricted set of functions, analytic functions. Two references which contain extensive discussions of the theory of hyperfunctions with applications are [40] and [39].

To motivate the definition of a hyperfunction on $\mathbf{R}$, consider the boundary values of a holomorphic function Φ_+ on the open upper half plane. These give a generalization of the usual notion of distribution, by considering the linear functional on analytic functions (satisfying an appropriate growth condition) on $\mathbf{R}$

$$f \rightarrow \lim_{\epsilon\rightarrow 0^+}\int_{-\infty}^{\infty}\Phi_+(t+i\epsilon)f(t)dt$$

Usual distributions are often written with a formal integral symbol denoting the linear functional. In the case of hyperfunctions, this is no longer formal, but becomes (a limit of) a conventional integral of a holomorphic function in the complex plane, so contour deformation and residue theorem techniques can be applied to its evaluation.

It is sometimes more convenient to have a definition involving symmetrically the upper and lower complex half-planes. The space $\mathcal{B}(\mathbf{R})$ of hyperfunctions on $\mathbf{R}$ can be defined as equivalence classes of pairs of functions (Φ_+,Φ_-), where Φ_+ is a holomorphic function on the open upper half-plane, Φ_- is a holomorphic function on the open lower upper half-plane. Pairs $(\Phi_{1,+},\Phi_{1,-})$ and $(\Phi_{2,+},\Phi_{2,-})$ are equivalent if

$$\Phi_{2,+}=\Phi_{1,+}+\psi,\quad \Phi_{2,-}=\Phi_{1,-}+\psi$$

for some globally holomorphic function ψ. We'll then write a hyperfunction as

$$\phi=[\Phi_+,\Phi_-]$$

The derivative ϕ' of a hyperfunction ϕ is given by taking the complex derivatives of the pair of holomorphic functions representing it

$$\phi' = [\Phi'_+, \Phi'_-]$$

As a linear functional on analytic functions, the hyperfunction ϕ is given by

$$f \to \oint_{-\infty}^{\infty} \phi(t)f(t)dt \equiv \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} (\Phi_+(t+i\epsilon) - \Phi_-(t-i\epsilon))f(t)dt$$

We'll use coordinates t on $\mathbf{R}$, $z = t+i\tau$ on $\mathbf{C}$ since our interest will be in physical applications involving functions of time t, as well as their analytic continuations to imaginary time τ.

One way to get hyperfunctions is by choosing a function $\Phi(z)$ on $\mathbf{C}$, holomorphic away from the real axis $\mathbf{R}$, and taking

$$\phi = [\Phi_{|UHP}, \Phi_{|LHP}]$$

For example, consider the function

$$\Phi = \frac{i}{2\pi}\frac{1}{z-\omega}$$

where $\omega \in \mathbf{R}$. As a distribution, corresponding hyperfunction will be given by the limit

$$\phi(t) = \lim_{\epsilon\to 0^+} \frac{i}{2\pi}\left(\frac{1}{t+i\epsilon-\omega} - \frac{1}{t-i\epsilon-\omega}\right) = \lim_{\epsilon\to 0^+} \frac{1}{\pi}\frac{1}{(t-\omega)^2+\epsilon^2}$$

The limit on the right-hand side is well-known as a way to describe the delta function distribution $\delta(t-\omega)$ as a limit of functions. Using contour integration methods one finds that the hyperfunction version of the delta function behaves as expected since

$$\oint_{-\infty}^{\infty} \frac{i}{2\pi}\frac{1}{t-\omega}f(t)dt = f(\omega)$$

One would like to define a Fourier transform for hyperfunctions, with the same sort of definition as an integral in the usual case, so

$$\mathcal{F}(\phi)(E) = \widetilde{\phi}(E) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iEt}\phi(t)dt \qquad \text{(A.1)}$$

with the inverse Fourier transform defined by

$$\mathcal{F}^{-1}(\widetilde{\phi})(t) = \phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iEt}\widetilde{\phi}(E)dE$$

The problem with this though is that the Fourier transform and its inverse don't take functions holomorphic on the upper or lower half plane to functions with the same property.

One can however define a Fourier transform for hyperfunctions (satisfying a growth condition, called "Fourier hyperfunctions") by taking advantage of the fact that for a class of functions $f(E)$ supported on $E > 0$ (respectively $E < 0$)

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iEz} f(E)dE$$

is holomorphic in the lower half (respectively upper half) z plane (since the exponential falls off there). The decomposition of a hyperfunction $\phi(t)$ into limits of holomorphic functions Φ_+, Φ_- on the upper and lower half planes corresponds to decomposition of $\widetilde{\phi}(E)$ into hyperfunctions $\widetilde{\phi}_-(E), \widetilde{\phi}_+(E)$ supported for negative and positive E respectively. This is similar to what happened for hyperfunctions on the circle, with Φ_+, Φ_- analogous to functions holomorphic on the upper or lower hemispheres, $\widetilde{\phi}_-(E), \widetilde{\phi}_+(E)$ analogous to the Fourier coefficients for positive or negative n.

For an example, consider the hyperfunction version of a delta function supported at $E = \omega, \omega > 0$:

$$\widetilde{\phi}(E) = \widetilde{\phi}_+(E) = \frac{i}{2\pi}\frac{1}{E-\omega} \equiv \frac{i}{2\pi}\lim_{\epsilon\to 0^+}\left(\frac{1}{E+i\epsilon-\omega} - \frac{1}{E-i\epsilon-\omega}\right)$$

This has as inverse Fourier transform the hyperfunction

$$\phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{i}{2\pi}\frac{1}{E-\omega}e^{-iEt}dE = \frac{1}{\sqrt{2\pi}}e^{-i\omega t}$$

which has a representation as

$$\phi(t) = [0, -\frac{1}{\sqrt{2\pi}}e^{-i\omega z}]$$

The Fourier transform of this will be

$$\begin{aligned}\widetilde{\phi}(E) =&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iEt}\phi(t)dt\\ =&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iEt}\frac{1}{\sqrt{2\pi}}e^{-i\omega t}dt\end{aligned}$$

but this needs to be interpreted as a sum of integrals for t negative and t positive

$$\begin{aligned}=&\frac{1}{2\pi}\lim_{\epsilon\to 0^+}(\int_{-\infty}^{0} e^{i(E-i\epsilon-\omega)t}dt + \int_{0}^{\infty} e^{i(E+i\epsilon-\omega)t}dt)\\ =&\frac{i}{2\pi}\lim_{\epsilon\to 0^+}\left(\frac{1}{E+i\epsilon-\omega} - \frac{1}{E-i\epsilon-\omega}\right)\end{aligned}$$

An example that is relevant to the case of the harmonic oscillator is that of

$$\widetilde{\phi}(E) = \frac{1}{E^2-\omega^2} = \frac{1}{2\omega}\left(\frac{1}{E-\omega} - \frac{1}{E+\omega}\right)$$

where the first term is a hyperfunction with support only at $\omega > 0$, the second only at $-\omega < 0$. The inverse Fourier transform is

$$\phi(t) = \frac{i\pi}{\omega}\frac{1}{\sqrt{2\pi}}(e^{i\omega t} - e^{-i\omega t})$$

where the first term should be interpreted as the equivalence class

$$\frac{i\pi}{\omega}\frac{1}{\sqrt{2\pi}}[e^{i\omega z}, 0]$$

and the second as the equivalence class

$$\frac{i\pi}{\omega}\frac{1}{\sqrt{2\pi}}[0, e^{-i\omega z}]$$

References

[1] Tim Adamo. *Twistor actions for gauge theory and gravity.* 2013. arXiv: 1308.2820 [hep-th].

[2] Timothy Adamo. "Lectures on twistor theory". In: *Proceedings of the XIII Modave Summer School in Mathematical Physics. 10-16 September 2017 Modave.* 2017, p. 3. arXiv: 1712.02196 [hep-th].

[3] Stephon Alexander, Antonino Marcianò, and Lee Smolin. "Gravitational origin of the weak interaction's chirality". In: *Physical Review D* 89 (2014), p. 065017. arXiv: 1212.5246 [hep-th].

[4] Michael F. Atiyah, Nigel J. Hitchin, and Isidore M. Singer. "Self-duality in four-dimensional Riemannian geometry". In: *Proc. R. Soc. Lond. A* 362 (1978), pp. 425–461.

[5] Michael F. Atiyah. *Geometry of Yang-Mills fields.* Lezioni Fermiane. Scuola Normale Superiore, Pisa, 1979.

[6] Robert J. Baston and Michael G. Eastwood. *The Penrose Transform: Its Interaction With Representation Theory.* Oxford University Press, 1989.

[7] Rutger Boels, Lionel Mason, and David Skinner. "Supersymmetric gauge theories in twistor space". In: *Journal of High Energy Physics* 2007 (2007). arXiv: hep-th/0604040 [hep-th].

[8] Raoul Bott. "Homogeneous Vector Bundles". In: *Annals of Mathematics* 66 (1957), pp. 203–248.

[9] Raoul Bott. "The Index Theorem for Homogeneous Differential Operators". In: *Differential and Combianatorial Topology.* Ed. by Stewart Scott Cairns. Princeton University Press, 1965, pp. 167–186.

[10] G. Chalmers and W. Siegel. "Self-dual sector of QCD amplitudes". In: *Physical Review D* 54 (1996), 7628–7633. ISSN: 1089-4918. arXiv: hep-th/9606061 [hep-th].

[11] Kevin J. Costello. *Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4*. 2013. arXiv: 1111.4234 [math.QA].

[12] Michel Dubois-Violette. "Structures Complexes Au-dessus des Variétés, Applications". In: *Mathématique et Physique*. Ed. by L. Boutet de Montvel, A. Douady, and J.L. Verdier. Vol. 37. Progress in Mathematics. Birkhaüser, 1983, pp. 1–42.

[13] Michael G. Eastwood, Roger Penrose, and Raymond O. Wells Jr. "Cohomology and Massless Fields". In: *Communications in Mathematical Physics* 78 (1981), pp. 305–351.

[14] Michael G. Eastwood. "The generalized Penrose-Ward transform". In: *Proc. R. Soc. Lond. A* 370 (1980), pp. 173–191.

[15] Chris Elliott and Philsang Yoo. *Quantum Geometric Langlands Categories from N = 4 Super Yang-Mills Theory*. 2020. arXiv: 2008.10988 [math-ph].

[16] James Glimm and Arthur Jaffe. *Quantum Physics: A Functional Integral Point of View*. Springer-Verlag, 1981.

[17] Yannick Herfray. "Pure connection formulation, twistors, and the chase for a twistor action for general relativity". In: *Journal of Mathematical Physics* 58.11 (2017), p. 112505. arXiv: 1610.02343 [hep-th].

[18] N. J. Hitchin. "Linear field equations on self-dual spaces". In: *Math. Proc. Camb. Phil. Soc.* 97 (1984), pp. 165–187.

[19] Jing-Song Huang and Pavle Pandžić. *Dirac Operators in Representation Theory*. Birkhäuser, 2006.

[20] Anton Kapustin and Edward Witten. "Electric-Magnetic Duality And The Geometric Langlands Program". In: *Commun. Num. Theor. Phys.* 1 (2007), pp. 1–236. arXiv: hep-th/0604151.

[21] Abel Klein and Lawrence Landau. "From the Euclidean Group to the Poincaré Group via Osterwalder-Schrader Positivity". In: *Commun. Math. Phys.* 87 (1983), pp. 469–484.

[22] Kirill Krasnov. *Formulations of General Relativity*. Cambridge University Press, 2020.

[23] H. Blaine Lawson Jr. and Marie-Louise Michelsohn. *Spin geometry*. Vol. 38. Princeton Mathematical Series. Princeton University Press, 1989.

[24] L. J. Mason and N.M.J. Woodhouse. *Integrability, Self-duality, and Twistor Theory*. London Mathematical Society Mongraphs. Oxford University Press, 1996.

[25] V.P. Nair. "A current algebra for some gauge theory amplitudes". In: *Physics Letters B* 214 (1988), pp. 215–218.

[26] Fabrizio Nesti and Roberto Percacci. "Gravi-weak unification". In: *Journal of Physics A: Mathematical and Theoretical* 41 (2008), p. 075405. arXiv: 0706.3307 [hep-th].

[27] Konrad Osterwalder and Robert Schrader. “Euclidean Fermi Fields and a Feynman-Kac formula for Boson-Fermion models”. In: *Helvetica Physica Acta* 46 (1973), pp. 277–302.

[28] Roger Penrose. “Twistor Algebra”. In: *Journal of Mathematical Physics* 8 (1967), pp. 345–366.

[29] John Rawnsley, Wilfried Schmid, and Joseph A. Wolf. “Singular Unitary Representations and Indefinite Harmonic Theory”. In: *Journal of Functional Analysis* 51 (1983), pp. 1–114.

[30] Simon Salamon. “Topics in Four-dimensional Riemannian geometry”. In: *Geometry Seminar Luigi Bianchi*. Ed. by E. Vesentini. Vol. 1022. Lecture Notes in Mathematics. Springer, 1983, pp. 33–124.

[31] R. S. Ward and Raymond O. Wells Jr. *Twistor geometry and field theory*. Cambridge University Press, 1990.

[32] R. O. Wells Jr. “Hyperfunction Solutions of the Zero-Rest-Mass Field Equations”. In: *Communications in Mathematical Physics* 78 (1981), pp. 567–600.

[33] Edward Witten. “Topological quantum field theory”. In: *Communications in Mathematical Physics* 117.3 (1988), pp. 353–386.

[34] Edward Witten. “Perturbative Gauge Theory as a String Theory in Twistor Space”. In: *Communications in Mathematical Physics* 252.1-3 (2004), pp. 189–258. arXiv: `hep-th/0312171 [hep-th]`.

[35] Peter Woit. “Supersymmetric quantum mechanics, spinors and the Standard Model”. In: *Nuclear Physics* B303 (1988), pp. 329–342.

[36] Peter Woit. *Quantum Field Theory and Representation Theory: A Sketch*. 2002. arXiv: `hep-th/0206135 [hep-th]`.

[37] Peter Woit. *Quantum theory, groups and representations: An introduction*. Springer, 2017.

[38] N.M.J. Woodhouse. “Real Methods in Twistor Theory”. In: *Classical and Quantum Gravity* 2 (1985), pp. 257–291.

[39] Urs Graf. *Introduction to hyperfunctions and their integral transforms*. Birkhäuser, 2010.

[40] Isao Imai. *Applied hyperfunction theory*. Vol. 8. Mathematics and its applications. Kluwer Academic Publishers, 1992.

[41] Roger Penrose. *The Road to Reality*. Jonathan Cape, 2004.

Euclidean Spinor References

[42] Julian Schwinger. “On the Euclidean structure of relativistic field theory”. In: *Proceedings of the National Academy of Sciences* 44.9 (1958), pp. 956–965.

[43] Julian Schwinger. "Four-dimensional Euclidean formulation of quantum field theory". In: *8th International Annual Conference on High Energy Physics*. 1958, pp. 134–140.

[44] Julian Schwinger. "Euclidean Quantum Electrodynamics". In: *Phys. Rev.* 115 (3 1959), pp. 721–731.

[45] Tadao Nakano. "Quantum Field Theory in Terms of Euclidean Parameters". In: *Progress of Theoretical Physics* 21.2 (1959), pp. 241–259.

[46] K. Osterwalder. "Euclidean fermi fields". In: *International School of Mathematical Physics, Ettore Majorana: 1st course: Constructive Quantum Field Theory.* 1973, pp. 326–331.

[47] K. Osterwalder and R. Schrader. "Euclidean fermi fields and a Feynman-Kac formula for boson-fermion models". In: *Helv. Phys. Acta* 46 (1973), pp. 277–302.

[48] David N. Williams. "Euclidean Fermi fields with a Hermitean Feynman-Kac-Nelson formula, I". In: *Comm. Math. Phys.* 38.1 (1974), pp. 65–80.

[49] David N. Williams. "Euclidean Fermi fields with a Hermitean Feynman-Kac-Nelson formula, II". 1974. URL: http://www-personal.umich.edu/~williams/papers/eufermi.pdf.

[50] Edward Nelson. "Markov fields". In: *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 2.* 1975, pp. 395–398.

[51] Jurg Frohlich and Konrad Osterwalder. "Is There a Euclidean Field Theory for Fermions?" In: *Helv. Phys. Acta* 47 (1975), p. 781.

[52] Leonard Gross. "On the formula of Mathews and Salam". In: *Journal of Functional Analysis* 25.2 (1977), pp. 162 –209.

[53] H. Nicolai. "A possible constructive approach to (super $-\ \phi^3)_4$: (I). Euclidean formulation of the model". In: *Nuclear Physics B* 140.2 (1978), pp. 294 –300. ISSN: 0550-3213.

[54] Diego de Falco and Francesco Guerra. "On the local structure of the Euclidean Dirac field". In: *Journal of Mathematical Physics* 21.5 (1980), pp. 1111–1114.

[55] John Palmer. "Euclidean Fermi fields". In: *Journal of Functional Analysis* 36.3 (1980), pp. 287 –312.

[56] Pierre Ramond. *Field Theory: A Modern Primer*. Benjamin/Cummings, 1981.

[57] Shigeaki Nagamachi and Nobumichi Mugibayashi. "Covariance of Euclidean Fermi Fields". In: *Progress of Theoretical Physics* 66.3 (1981), pp. 1061–1077.

[58] Bengt Ek. "Parity Conservation and the Euclidean Field Formulation of Relativistic Quantum Theoriesj". In: *Letters in Mathematical Physics* 5 (1981), pp. 149–154.

[59] Bengt Ek. "Euclidean Quantum Fields with Spin on an Indefinite Inner Product State Space". In: *Publ. RIMS, Kyoto Univ.* 18 (1982), pp. 251–274.

[60] J. Frohlich, K. Osterwalder, and E. Seiler. "On Virtual Representations of Symmetric Spaces and Their Analytic Continuation". In: *Annals of Mathematics* 118.3 (1983), pp. 461–489.

[61] J. Kupsch. "Measures for Fermionic Integration". In: *Fortschritte der Physik/Progress of Physics* 35.5 (1987), pp. 415–436.

[62] J. Kupsch. "Functional integration for Euclidean Dirac fields". In: *Annales de l'I.H.P. Physique théorique* 50.2 (1989), pp. 143–160.

[63] J. Kupsch and W. D. Thacker. "Euclidean Majorana and Weyl Spinors". In: *Fortschritte der Physik/Progress of Physics* 38.1 (1990), pp. 35–62.

[64] Mayank R. Mehta. "Euclidean continuation of the Dirac fermion". In: *Phys. Rev. Lett.* 65 (16 1990), pp. 1983–1986.

[65] David Borthwick. "Euclidean Majorana fermions, fermionic integrals, and relative Pfaffians". In: *Journal of Mathematical Physics* 34.7 (July 1993), pp. 2691–2712.

[66] Peter van Nieuwenhuizen and Andrew Waldron. "On Euclidean spinors and Wick rotations". In: *Physics Letters B* 389.1 (1996), 29–36.

[67] P. van Nieuwenhuizen and A. Waldron. "A continuous Wick rotation for spinor fields and supersymmetry in Euclidean space". In: *Gauge Theories, Applied Supersymmetry and Quantum Gravity II* (1997). eprint: hep-th/9608174.

[68] Andrew Waldron. "A Wick rotation for spinor fields: the canonical approach". In: *Physics Letters B* 433.3-4 (1998), 369–376. eprint: hep-th/9702057.

[69] Arthur Mountain. "Wick rotation and supersymmetry". In: *Quantum aspects of gauge theories, supersymmetry and unification* (Sept. 1–7, 1999). Paris, 2000. DOI: 10.22323/1.004.0036.

[70] V. Wessels and W. N. Polyzou. "Stability of Covariant Relativistic Quantum Theory". In: *Few-Body-Systems* 35.1-2 (2004).

[71] Arthur Jaffe and Gordon Ritter. "Reflection positivity and monotonicity". In: *Journal of Mathematical Physics* 49.5 (2008), p. 052301. arXiv: 0705.0712 [math-ph].

[72] C. Wetterich. "Spinors in euclidean field theory, complex structures and discrete symmetries". In: *Nuclear Physics B* 852.1 (2011), pp. 174 –234. arXiv: 1002.3556 [hep-th].