Notes on Wick Rotation and Chiral Field Theories

Peter Woit
Department of Mathematics, Columbia University
woit@math.columbia.edu

Preliminary version
July 9, 2026

Abstract

The usual notion of Wick rotation as analytic continuation in complexified spacetime is incompatible with the simplest description of examples of chiral QFTs in two and four dimensions. Instead one can start with a Euclidean spacetime description, together with a conjugation which allows reconstruction of the Minkowski theory.

Contents

1 Introduction

Our best rigorous foundations for relativistic quantum field theory are generally taken to be based on relating a physical Minkowski spacetime theory to a Euclidean spacetime one by analytic continuation in complexified spacetime (Wick rotation). The Euclidean theory is defined in terms of a rigorous version of the path integral formalism based on methods of measure theory, probability theory and statistical mechanics.

These foundations suffer from two critical problems:

- In four dimensions they do not give non-trivial interacting theories.
- The methods do not apply to the quantum field theories that provide our best fundamental description of nature.

The main purpose of these notes is to point to a fundamental incompatibility between the usual understanding of Wick rotation and the chiral nature of our best theories of matter. In two dimensions chiral theories are already of a holomorphic nature in Euclidean signature. The same is true in four dimensions when using twistor methods.

Complexifying an already complex spacetime in order to invert Wick rotation and recover a Minkowski signature theory requires introducing new degrees of freedom of opposite chirality. One can avoid doing this and instead recover a Minkowski signature theory not by analytic continuation, but by an appropriate notion of conjugation, such as that of Osterwalder-Schrader which involves a reflection in imaginary time.

The first part of these notes will review the conventional Osterwalder-Schrader formalism, with a detailed discussion of the simplest possible case, that of a harmonic oscillator degree of freedom treated in field theoretic language.

Later sections will discuss the nature of chiral quantum field theories in two and four dimensions, emphasizing that their Euclidean and Minkowski spacetime versions are related not by the usual analytic continuation in complexified spacetime, but something quite different. The Euclidean theory is already holomorphic, no complexification is needed. A state space and Minkowski version comes from choosing a conjugation map on this holomorphic theory. Twistor theory provides the natural geometrical context for this in the four dimensional case.

2 Wick rotation

The usual discussion of Wick rotation begins with showing that in a relativistic field theory, vacuum expectation values of field operators on Minkowski spacetime are boundary values of holomorphic functions of points in complexified spacetime. These are called "Wightman functions", although they are distributions. One then goes on to show (a standard reference is [15]), that these holomorphic functions extend uniquely to non-coincident points in Euclidean spacetime. Restricting to these points one gets what are known as "Schwinger functions".

Schwinger was the first to promote the "Euclidean quantum field theory" philosophy that knowing the Schwinger functions provided an independent way of defining the theory, one with better analytical behavior (e.g. functions not distributions) than the real time theory. This philosophy that one should define a QFT in Euclidean spacetime and later worry about analytically continuing back to Minkowski spacetime is now the dominant paradigm. It fits well with the idea that QFTs should be defined as path integrals, since the Euclidean QFT path integrals are much better behaved than the ones defined in Minkowski spacetime.

The argument that one can uniquely continue Wightman distributions to Euclidean points begins with the relativistic analog of the axiom that quantum mechanical states are of non-negative energy (or can be made so by a finite shift). The relativistic version of this axiom is the assumption that (Fourier-transformed) Wightman distributions have support in the positive energy-momentum lightcone

$$E \geq |\mathbf{P}|$$

One then exploits this support condition to show that Wightman functions are holomorphic functions on a "forward tube" in complex spacetime. For the simplest case of how this works, consider the case of no spatial dimensions, and a Wightman function W_2 that depends on a single time variable t, with Fourier transform

$$\widetilde{W_2}(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itE} W_2(t)dt$$

Then the inverse Fourier transform formula, extended to complex time z is

$$W_2(z) = \int_{-\infty}^{\infty} e^{-izE}\widetilde{W_2}(E)dE \tag{1}$$

Defining

$$z = t - i\tau$$

one finds that $\widetilde{W_2}(E)$ supported on $E > 0$ implies $W_2(z)$ is holomorphic in the open upper half z plane. This is one version of the Paley-Wiener theorem which follows from the fact that the factor

$$e^{-izE} = e^{-itE}e^{-\tau E}$$

is exponentially damped for $\tau > 0$. Note that the Schwinger function

$$S_2(\tau) = W_2(-i\tau) = \int_{0}^{\infty} e^{-\tau E}\widetilde{W_2}(E)dE \tag{2}$$

is the Laplace transform of $\widetilde{W_2}(E)$ (which we'll write $\mathcal{L}\widetilde{W_2}$).

If one tries to invert Wick rotation by inverting the Laplace transform formula 2 one runs into a problem. The standard prescription for computing an inverse Laplace transform is essentially to analytically continue the function and then use the Fourier transform, but this requires knowing what one is trying to find.

Instead of taking as fundamental either functions of real or imaginary time, one could simply use functions of complex time, holomorphic on the upper half plane. Then functions of real time are boundary values of holomorphic functions, which can be thought of as a generalized sort of distribution known as a hyperfunction (see [3] or [2]) for which test functions are real analytic functions. For more about hyperfunctions from a physical point of view, see chapter 9 of Penrose's *Road to Reality* [10].

Showing that W_2 has a unique analytic continuation to Euclidean spacetime points requires using two other axioms for Wightman functions:

- Wightman functions have specified covariance properties under the Lorentz group $SO(3,1)$.
- Wightman functions are symmetric under interchange of spacelike separated points. This is a locality axiom, reflecting the commutativity of field operators at spacelike points.

The first axiom and holomorphicity in the forward tube are used to show that one has a holomorphic action of the complexification $SO(4,\mathbf{C})$ of the Lorentz group on the Wightman functions, which provides analytic continuation to a region called the "extended tube". The second further extends the analytic continuations to the "permuted extended tube", which includes the Euclidean points.

2.1 The harmonic oscillator

The argument outlined above is given in detail in [15]. Since it involves holomorphic functions of multiple complex variables, it is rather intricate and can be hard to follow. In addition, the assumption is almost always made that one is dealing with real scalar fields, while our interest is going to be in fields that are holomorphic in Euclidean spacetime coordinates. In this section we'll work out what happens for the simplest possible case, that of a single harmonic oscillator, described in field theoretic language.

This is the theory of a pair of adjoint field operators $\widehat{\psi}(t), \widehat{\psi}^\dagger(t)$ with Hamiltonian

$$H = \omega\widehat{\psi}(t)\widehat{\psi}^\dagger(t)$$

with $\omega > 0$. These are Heisenberg picture operators, which satisfy the equation of motion

$$\left(i\frac{d}{dt} - \omega\right)\widehat{\psi} = 0 \tag{3}$$

This gives the quantum system describing a single oscillator degree of freedom of energy $\omega > 0$. For $\omega = |\mathbf{p}|^2/2m$ this is the quantum system describing a single momentum $\mathbf{p}$ mode of the free non-relativistic quantum field theory of a charged particle.

Solutions to this equation can be written as

$$\widehat{\psi}(t) = ae^{-i\omega t}$$

with adjoint operator

$$\widehat{\psi}^\dagger(t) = a^\dagger e^{i\omega t}$$

where $a, a^\dagger$ are adjoint operators satisfying $[a, a^\dagger] = 1$. The theory has Wightman function

$$W_2(t_1, t_2) = \langle 0|\widehat{\psi}(t_1)\widehat{\psi}^\dagger(t_2)|0\rangle = e^{-i\omega(t_1 - t_2)}$$

and all other non-zero vacuum expectation values of products of operators can be expressed in terms of this function. Because of translation invariance, W_2 is a function of a single variable $t = t_1 - t_2$. It will have Fourier transform the distribution.

$$\widetilde{W}_2(E) = \delta(E - \omega)$$

This distribution has analytic continuation

$$W_2(z) = e^{-i\omega z}$$

and Schwinger function

$$S_2(\tau) = e^{-\omega\tau}$$

The field operators are actually operator-valued distributions, which for each f in a space of test functions (usually taken to be $\mathcal{S}(\mathbf{R})$, the Schwartz functions) provide adjoint operators

$$\widehat{\psi}^\dagger(f) = \int_{-\infty}^{\infty} f(t)\widehat{\psi}^\dagger(t)dt$$

and

$$\widehat{\psi}(f) = \int_{-\infty}^{\infty} \overline{f(t)}\widehat{\psi}(t)dt$$

The inner product between states

$$\widehat{\psi}^\dagger(f)|0\rangle \quad \text{and} \quad \widehat{\psi}^\dagger(g)|0\rangle$$

is

$$\begin{aligned}\langle 0|\widehat{\psi}(f)\widehat{\psi}^\dagger(g)|0\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(t_1)}W(t_1, t_2)g(t_2)dt_1 dt_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(t_1)}e^{-i\omega t_1}g(t_2)e^{i\omega t_2}dt_1 dt_2 \\ &= 4\pi^2\overline{\widetilde{f}(\omega)}\widetilde{g}(\omega)\end{aligned}$$

One can reconstruct the state space and field operators from the Wightman functions ("Wightman reconstruction"). The single quantum state space is given by

$$\mathcal{H}_1 = \frac{\{f \in \mathcal{S}(\mathbf{R})\}}{\{f : W_2(f, f) = 0\}}$$

which in this case is one-dimensional. The full state space is the symmetric tensor product space $\mathcal{H} = S^*(\mathcal{H}_1)$.

In the case of a relativistic complex scalar field, a single momentum mode is described by operators $\widehat{\psi}(t)$ that satisfy the equation of motion

$$(-\frac{d^2}{dt^2} - \omega^2)\widehat{\psi}(t) = (i\frac{d}{dt} + \omega)(i\frac{d}{dt} - \omega)\widehat{\psi}(t) = 0 \tag{4}$$

with $\omega^2 = \sqrt{|\mathbf{p}|^2 + m^2}$. This has two kinds of solutions, positive and negative energy. The negative energy solutions must be quantized with a second set of annihilation and creation operators $b, b^\dagger$ which annihilate and create anti-quanta. Most rigorous discussions of quantum field theory focus on the case of real scalar fields, which corresponds to setting $b = a$ and identifying particle and antiparticle states. When trying to understand Wick rotation and analytic continuation, this restriction to real fields makes things not simpler, but more confusing. I am not aware of a reference that discusses the relativistic complex case, giving details of how to treat negative-energy solutions.

We do not use path integral methods here, since they are quite awkward, even for this very simple case. For the special case mentioned above of real scalar fields and a second-order equation of motion, in imaginary time one can write down a path integral formalism that can be made rigorous. The case of complex fields satisfying a first order equation is quite different, since now the fields take values in a phase space, not a configuration space. There is a formalism for such path integrals (see chapter 6 of [20]) but they have a very formal nature, since integrating over paths in phase space is highly problematic.

3 Euclidean quantum field theory and Osterwalder-Schrader reconstruction

Schwinger early on recognized that finding an inverse Wick rotation (i.e. going from a set of Schwinger functions to a set of Wightman functions and a physical Minkowski spacetime theory) was a non-trivial problem.[1]

Osterwalder and Schrader [7] gave an answer to this problem by proving a reconstruction theorem showing that, given a set of Schwinger functions satisfying certain properties, one could reconstruct the Wightman functions, state space and field operators of a Minkowski spacetime quantum field theory. This naively appears to provide an equivalence between Minkowski and Euclidean formulations of quantum field theory, but the situation is more complicated. The two formulations have different properties and there are subtleties in the problem of inverting the Wick rotation map taking Minkowski to Euclidean.[2]

Some reasons why the inverse Wick rotation is challenging are:

[1] In the discussion at the end of [13] Schwinger says "The question of to what extent you can go backwards, remains unanswered, i.e. if one begins with an arbitrary Euclidean theory and one asks: when do you get a sensible Lorentz theory? This I do not know. The development has been in one direction only: the possibility of future progress comes from the examination of the reverse direction, and that is completely open."

[2] Slava Rychkov comments in a talk [12] "these papers appeared in *Communications in Mathematical Physics*. If you start reading these papers you immediately get a headache. The first ten pages are just notation. You have to go through then another theorem, lemma,

- As mentioned earlier, one cannot invert the Laplace transform formula used to do Wick rotation.

- If one sticks to the conventional formalism, one needs to find a way to characterize Schwinger functions that are the analytic continuation by a Laplace transform of tempered distributions. The initial Osterwalder-Schrader paper [7] had a mistake, missing a subtle aspect of this problem. Fixing the mistake [8] leaves one needing to assume properties of the Schwinger functions that are hard to check (for a detailed discussion, see section 9 of [5]). One can try to work with hyperfunctions (see for instance [6]), but that comes with its own set of challenges.

- A fundamental complicating aspect of the problem is that Hermitian inner products are not preserved under analytic continuation. One wants the Euclidean theory to come with some additional structure ensuring that inverse Wick rotation provides a physical state space with a Hermitian inner product.

While Wick rotation from Minkowski to Euclidean can be done by explicitly constructing an analytic continuation, the way that Osterwalder and Schrader found to go the other direction is something very different. Given a set of Schwinger functions in Euclidean spacetime satisfying various properties, they indirectly show that there must be a set of Wightman distributions for which they are the Wick rotation. They do not however construct these Wightman distributions.

Instead of doing the analytic continuation back to Minkowski spacetime, they found that, given a new structure in Euclidean spacetime that fixes the Hermitian inner product problem, they could reconstruct explicitly the physical state space without doing any analytic continuation at all. The simple idea behind the new structure is to realize that while in real time the appropriate Hermitian inner product involves a conjugation

$$f(t) \to \overline{f(t)}$$

in imaginary time one needs to also complex conjugate time, reflecting $\tau \to -\tau$. This is often called "Osterwalder-Schrader reflection".

The needed new structure is a new notion of conjugation we'll write Θ, given by

$$f(\tau) \to \Theta f(\tau) = \overline{f(-\tau)}$$

We'll describe it as a "conjugation" since it is an antilinear involution, i.e.

$$\Theta(\lambda f) = \overline{\lambda} f, \quad \text{and} \quad \Theta^2 = 1$$

---

lemma, theorem, Hille-Yosida theorem, things like that.

Very few people have read these papers and very few people know what has actually been done there. It's almost irresistible, people love to cite these papers because it's like a feeling of ancient magic books, the scriptures. Many normally very careful people misquote these papers and miscite them by attributing to them results which are not there."

It is the same notion of conjugation that occurs in the theory of hyperfunctions. Hyperfunctions can be thought of as equivalence classes of pairs $[F_+, F_-]$ of functions, one holomorphic on the open upper half plane, the other holomorphic on the lower half plane. The right notion of conjugation takes

$$[F(z), 0] \to [0, \overline{F(\overline{z})}]$$

What Osterwalder and Schrader did was define an Hermitian inner product on the space $\mathcal{S}_+(\mathbf{R})$ of Schwartz functions $f(\tau)$ supported on $\tau > 0$ by

$$\langle f, g\rangle_{OS} = S_2(\Theta f, g)$$

This gives, without any analytic continuation at all, exactly the physical state space $\mathcal{H}_1$ (for the free field case).

To see this, the definition of S_2 as a Laplace transform of W_2 implies

$$S_2(\Theta f, g) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(-\tau_1)}g(\tau_2)\left(\int_{-\infty}^{\infty} e^{-(\tau_2-\tau_1)E}\widetilde{W}_2(E)dE\right)d\tau_1 d\tau_2$$

Changing sign of τ_1 and interchanging order of integration, this is

$$\int_{-\infty}^{\infty} \overline{\mathcal{L}f(E)}\mathcal{L}g(E)\widetilde{W}_2(E)dE$$

This shows that

$$\langle f, g\rangle_{OS} = \langle \mathcal{L}f, \mathcal{L}g\rangle$$

The physical state space $\mathcal{H}_1$ can be taken to be

$$\mathcal{H}_1 = \frac{\{f \in \mathcal{S}(\mathbf{R}_+)\}}{\{f : \langle f, f\rangle_{OS} = 0\}}$$

In [7] Osterwalder and Schrader show that this reconstruction of the Minkowski spacetime theory has the expected properties. One thing to point out is that something is very different than in the Minkowski case: the conjugation operator Θ picks out a direction in Euclidean spacetime, breaking $SO(4)$ invariance. This is what turns $SO(4)$ invariance into $SO(3,1)$ Lorentz invariance. For more about this, see [4] and [1].

4 Chiral conformal field theories in two dimensions

Conformal field theories in two dimensions decompose into chiral sectors labeled L and R, with the simplest examples bosonic or fermionic massless scalar fields satisfying the equations of motion

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\psi_R = 0$$

or

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\psi_L = 0$$

These describe respectively right-moving and left-moving massless particles, as well as anti-particles moving in the opposite direction. For fermions, this is the two dimensional analog of the four dimensional theory of massless Weyl spinor fields, where

$$\frac{\partial}{\partial x} \to \sum_{j=1}^{3} \sigma_j \frac{\partial}{\partial x_j}$$

Changing variables to $x_R = x-t, x_L = x+t$, solutions $\psi_R(x_R, x_L), \psi_L(x_R, x_L)$ are given by arbitrary functions of x_R or x_L respectively. These chiral theories are effectively theories in one spacetime variable, with no equation of motion. Wick rotation now is based on complexifying the spacetime coordinates x_R, x_L to get a complex spacetime $\mathbf{C}^2 = \mathbf{C}_R \oplus \mathbf{C}_L$ with coordinates z_R, z_L. Positivity of the operators $H \pm P$ implies holomorphicity in the upper half z_R and z_L planes and a holomorphic action of the semigroup $\mathbf{C}^+_R \oplus \mathbf{C}^+_L$.

Groups $SL(2, \mathbf{C})_{R,L}$ act conformally on $\mathbf{C}_{R,L}$ by fractional linear transformations, with subgroups $SL(2, \mathbf{R})_{R,L}$ acting with three orbits (the upper and lower complex half planes, and the real line). The actions of $SL(2, \mathbf{R})_{R,L}$ on the upper half planes extends to actions of semi-groups $SL^+(2, \mathbf{C})_{R,L} \subset SL(2, \mathbf{C})_{R,L}$.

One usually conformally compactifes by making the Möbius transformation

$$z_{R,L} \to \frac{z_{R,L} - i}{z_{R,L} + i}$$

which takes the real line to a circle and the upper half plane to the unit disk. $SL(2, \mathbf{R})$ becomes the isomorphic group $SU(1, 1)$. Thinking of this new coordinate as a coordinate on the Riemann sphere, the real line is taken to an equator, upper and lower half-planes to upper and lower hemispheres. Compactified, complexified spacetime is the product

$$S^2_R \times S^2_L$$

of two Riemann spheres with coordinates (z_R, z_L). Wick rotation is analytic continuation between two real forms

- Minkowski: $S^1_R \times S^1_L$, fixed points of the conjugation

  $$(z_R, z_L) \to (\overline{z}_R, \overline{z}_L)$$

  acted on by the conformal group $SL(2, \mathbf{R})_R \times SL(2, \mathbf{R})_L$.

- Euclidean: S^2, fixed points of the conjugation

  $$(z_R, z_L) \to (-\overline{z}_L, -\overline{z}_R)$$

  acted on by the conformal group $SL(2, \mathbf{C})$.

This is rarely spelled out in the conformal field theory literature, but can be found in section 3 of Graeme Segal's notes *On the definition of conformal field theory* [14]. In those notes, Segal also explains how the actions of the conformal groups $SL(2,\mathbf{R})_{R,L}$ can be extended to actions of the infinite-dimensional diffeomorphism groups $Diff(S^1)_{R,L}$, with semigroups $\mathcal{A}_{R,L}$ of conformal transformations of an annulus providing analogs of the semigroups $SL^+(2,\mathbf{C})_{R,L}$.

Segal and other authors then go on to restrict attention to just one chirality, considering a holomorphic theory on the upper hemisphere S^2_R with boundary S^1_R and taking a chiral CFT to be a representation of $\mathcal{A}_R$. This theory is often described as the Wick rotation to Euclidean spacetime of the chiral Minkowski spacetime theory, but that's not really what it is. The Euclidean real form of the complexified story uses as real spacetime not S^2_R, but an S^2 embedded anti-diagonally in its complexification $S^2_R \times S^2_L$. Wick rotation, as analytic continuation in the complexification between Minkowski and Euclidean, inherently uses both chiralities and does not have a purely chiral formulation.

If one thinks of a chiral Euclidean theory as a holomorphic theory on Euclidean spacetime S^2_R, with global conformal symmetry group $SL(2,\mathbf{C})_R$, then one gets a corresponding chiral Minkowski spacetime theory not by Wick rotation, but by a choice of a conjugation map on S^2_R. Such a map picks out a Minkowski conformal group $SL(2,\mathbf{R})_R$ and has fixed points S^1_R.

In this version of Minkowski spacetime, physics has no dependence on S^1_L of $SL(2,\mathbf{R})_L$ and the theory is effectively one-dimensional and "right-handed", depending only on x_R, a coordinate on S^1_R. As in the case of a quantum system depending only on t, it is useful to complexify x_R, the analog of t. But now the "Euclidean" theory is the holomorphic theory on a complex half-plane, not the theory on the imaginary x_R axis.

In [17] Witten gives a formulation of the Euclidean holomorphic free chiral fermion theory on a disk in terms of an infinite-dimensional grassmannian. There's no discussion there of Wick rotation to Minkowski spacetime, but his formulation implements the point of view argued for above. While the theory on a sphere is purely Euclidean, the choice of an equator and a decomposition into hemispheres that it bounds corresponds to the passage to (a one chirality only) choice of Minkowski spacetime.

Witten's formulation of the theory is very much algebro-geometric, working not just for a disk, but for arbitrary algebraic curves. In such a holomorphic theory version of the chiral fermion theory, one can introduce interactions through coupling to a holomorphic principal bundle, providing a two dimensional analog of much of the structure of our best four dimensional fundamental theory.

5 Twistors and chiral conformal field theories in four dimensions

While in two spacetime dimensions chirality corresponds to the decomposition of one-forms into positive and negative eigenspaces of the Hodge star operator

$*$, in four spacetime dimensions it is instead two-forms that decompose in this way. It is not complexified spacetime that decomposes into right and left pieces, but complexified infinitesimal rotations. The complexified conformal group in two dimensions

$$Spin(4, \mathbf{C}) = SL(2, \mathbf{C})_R \times SL(2, \mathbf{C})_L$$

reappears in four dimensions in a different role, as the complexified group of rotations.

Wick rotation in four dimensions is supposed to be analytic continuation in complexified spacetime between a real Euclidean spacetime with spin rotation group the real form $SU(2)_R \times SU(2)_L$ and Minkowski spacetime where the real form is $SL(2, \mathbf{C})$. If one tries to Wick rotate chiral spinors, one finds much the same problem as in two dimensions. The Minkowski $SL(2, \mathbf{C})$ is embedded anti-diagonally in $SL(2, \mathbf{C})_R \times SL(2, \mathbf{C})_L$ so one cannot formulate an analytic continuation involving just one chirality.

In the usual discussion of problems with Wick rotating spinor fields in four dimensions, the above fatal problem of chiral spinors is not even mentioned (although see page 226 of [11]). Instead the assumption is made that one is starting with a theory with both chiralities. One then finds an additional problem (with Hermiticity), which gets resolved [9] by doubling the number of spinor fields. So, if one starts with a Weyl spinor field, for the usual Wick rotation to work one needs to first introduce fields with the opposite chirality, then further double the number of degrees of freedom.

Since Wick rotation of a chiral theory fails in four dimensions in much the same way as in two dimensions, one should try to find something like what works in two dimensions. Instead of analytically continuing in complexified spacetime, one should start with a Euclidean holomorphic theory involving just one chirality, say the right-handed one, then choose a conjugation that provides the extra structure needed to get a Minkowski signature theory.

Twistor theory naturally provides a way to do this, while at the same time building into the theory the action not just of rotations but of conformal transformations. For details of the twistor theory story, one place to start is [16]. Things are simplest when one complexifies, in which case the complexified four-dimensional conformal group is the group $Spin(6, \mathbf{C}) = SL(4, \mathbf{C})$. Twistor space $T = \mathbf{C}^4$ is one of the half-spinor representations of $Spin(6, \mathbf{C})$, or equivalently, the fundamental representation of $SL(4, \mathbf{C})$. It is often convenient to mod out by the action of $\mathbf{C}^*$ and consider the projective space $PT = \mathbf{C}P^3$ instead of T.

The relation of twistor space to spacetime is that points in complexified, conformally compactified spacetime correspond to complex two-planes in T. The relation to spinors is tautological: the $\mathbf{C}^2 \subset T$ corresponding to a point is the space of right-handed spinors at the point. This makes the twistor geometry inherently chiral: points are right-handed spinors, left-handed spinors are something else.

In terms of PT instead of T, a point is a $\mathbf{C}P^1 \subset \mathbf{C}P^3$. Complexified, compactified spacetime is the Grassmannian $Gr(2, 4, \mathbf{C})$ or equivalently the space parametrizing linearly embedded $\mathbf{C}P^1 \subset \mathbf{C}P^3$.

5.1 Twistors and Minkowski spacetime

The conformal group for compactified Minkowski spacetime is the real form $Spin(4,2) = SU(2,2)$ of $Spin(6,\mathbf{C}) = SL(4,\mathbf{C})$. It is the subgroup of $SL(4,\mathbf{C})$ preserving a non-degenerate indefinite Hermitian form Φ on T of signature $(2,2)$. This is a close analog of the chiral right-handed part of the two-dimensional story. There one had $SL(2,\mathbf{C})$ acting on $\mathbf{C}P^1$ with real subgroup $SL(2,\mathbf{R}) = SU(1,1)$ preserving a signature $(1,1)$ indefinite Hermitian form. The $SU(1,1)$ acts on $\mathbf{C}P^1$ with three orbits (the equator and two hemispheres).

In the four dimensional analog, $SU(2,2)$ acts on $PT = \mathbf{C}P^3$ with three orbits, the subsets PN, PT^+, PT^- on which Φ is zero, positive or negative respectively. Minkowski spacetime is the four dimensional space of $\mathbf{C}P^1$s linearly embedded in N. These have a very direct physical interpretation: the $\mathbf{C}P^1$ corresponding to a point in spacetime is the sphere of light-rays through the point, the sphere one is looking at when one opens an eye.

The space PN can also be interpreted as the space of all light rays in (compactified) Minkowski spacetime, and one point of view on twistor theory is that it replaces spacetime points with light-rays as fundamental objects. The space PN can be identified with $S^3 \times S^2$: a light ray is determined by a choice of space (S^3) and a direction in space (S^2).

5.2 Twistors and Euclidean spacetime

To understand the Euclidean real form of spacetime in twistor theory, it is best to use quaternions, picking an identification $T = \mathbf{C}^4 = \mathbf{H}^2$. Then (compactified) Euclidean spacetime is the space of $\mathbf{H} \subset \mathbf{H}^2$, which is the projective space $\mathbf{H}P^1$ and can be identified with S^4. Again this is closely analogous to the two-dimensional case. There the right-chiral version of Euclidean spacetime was $S^2 = \mathbf{C}P^1$, the set of $\mathbf{C} \subset \mathbf{C}^2$, acted on transitively by the conformal group $SL(2,\mathbf{C})$. For the four-dimensional analog, juts replace $\mathbf{C}$ by $\mathbf{H}$.

In two dimensions, up to orientation there is just one way of identifying $\mathbf{R}^2$ with $\mathbf{C}$ (i.e. identifying i as a rotation of $\mathbf{R}^2$), so just one notion of "holomorphic". Equivalently, $SO(2) = U(1)$. Passing to four dimensions, the analogous choices of an identification of $\mathbf{R}^4$ with $\mathbf{C}^2$ are parametrized by $SO(4)/U(2) = \mathbf{C}P^1$.

In the study of four-dimensional Riemannian manifolds, a different notion of twistor space occurs. One can define a six-dimensional bundle $Z(M)$ over M with fiber $SO(4)/U(2)$ over a point $m \in M$, the space of complex structures on the tangent space T_mM. When M is a spin manifold, $Z(M) = P(S_R)$, the projectivization of the right-handed spin bundle. When M is a hyperkähler manifold, then $Z(M) = M \times \mathbf{C}P^1$.

In the special case $M = S^4$, $Z(S^4) = \mathbf{C}P^3$ and one recovers the Euclidean version of Penrose's twistor theory. While S^4 does not have a complex structure and there is no notion of holomorphic functions on S^4, its twistor space $\mathbf{C}P^3$ is a complex manifold and one can exploit holomorphicity there. This phenomenon is more general than S^4. Whenever M has anti-self-dual curvature, its twistor space $Z(M)$ is a complex manifold.

5.3 Relating Euclidean and Minkowski in twistor space

In twistor theory, complexified spacetime is $Gr(2, 4, \mathbf{C})$ and one can do Wick rotation by analytically continuing between the real Minkowski subspace and the real Euclidean subspace. This will come with the usual problem that it won't work for chiral spinors. While in Minkowski spacetime, tangent vectors can be described as the tensor product of one chirality of spinor and its complex conjugate, in complexified spacetime one must take the tensor product of the two chiralities of spinors.

Instead of using holomorphicity in the space of $\mathbf{C}P^1$s in $PT = \mathbf{C}P^3$, one can more simply just use holomorphicity in PT itself. Twistor theory allows one to formulate a purely chiral, holomorphic Euclidean theory in four dimensions, much like the one in two dimensions. One can define the holomorphic theory on PT instead of S^4, and pass to Minkowski spacetime using the choice of a Φ which splits PT into $SU(2, 2)$ orbits PN, PT^+, PT^-. On PT^+ one has a Euclidean holomorphic theory, which has boundary values on PN giving the chiral Minkowski spacetime theory.

The Ward-Penrose transform relates holomorphic vector bundles on $\mathbf{C}P^3$ to solutions of massless field equations on complexified spacetime. In particular, for certain holomorphic bundles on PT^+ one gets solutions to the right-handed Weyl equation coupled to a self-dual connection, both on a Euclidean S^4 hemisphere and (as boundary values of holomorphic sections) on Minkowski spacetime.

6 Euclidean twistor unification

In earlier work (see [18] and [19]) we've explained how the internal symmetries of the Standard Model appear automatically as part of the twistor geometry and its spacetime symmetries. In particular:

- The spacetime geometry is "right-handed", just depends on the right-handed spinors that define points. Restricting attention to Minkowski spacetime, the Lorentz group is $SL(2, \mathbf{C})_R$. The Euclidean geometry does have left-handed spinors and a group $SU(2)_L$ acting on them, but from the physical Minkowski spacetime point of view this is an internal symmetry. This can provide the $SU(2)$ symmetry of electroweak gauge theory, broken by the choice of Φ, which picks out Minkowski spacetime and allows for a definition of the physical state space.

- PT comes with a tautological holomorphic line bundle L and a holomorphic quotient $\mathbf{C}^3$ bundle. These can potentially provide the $U(1)$ hypercharge and $SU(3)$ gauge symmetries of the Standard Model.

Finally, the new relation described here between Euclidean and Minkowski quite possibly gives something new on the old question of Euclidean quantum gravity and Wick rotation in quantum gravity.

Much remains to be done to construct a complete theory implementing the different version of spacetime and internal symmetries described here. This does however seem to be a much more promising starting point for unification than anything else that has been tried to date.

References

[1] J. Frölich, K. Osterwalder, and E. Seiler. "On virtual representations of symmetric spaces and their analytic continuations". In: *Annals of Mathematics* 118 (1983), pp. 461–489.

[2] Urs Graf. *Introduction to hyperfunctions and their integral transforms.* Birkhäuser, 2010.

[3] Isao Imai. *Applied hyperfunction theory.* Vol. 8. Mathematics and its applications. Kluwer Academic Publishers, 1992.

[4] A. Klein and L.J. Landau. "From the Euclidean Group to the Poincare Group via Osterwalder Positivity". In: *Comm. Math. Phys.* 87 (1983), pp. 469–484.

[5] Petr Kravchuk, Jiaxin Qiao, and Slava Rychkov. "Distributions in CFT. Part II. Minkowski space". In: *JHEP* 08 (2021), p. 094. DOI: `10.1007/JHEP08(2021)094`. arXiv: `2104.02090 [hep-th]`.

[6] S. Nagamachi and N. Mugibayashi. "Hyperfunction Quantum Field Theory II: Euclidean Green's functions". In: *Comm. Math. Phys.* 49 (1976), pp. 257–275.

[7] Konrad Osterwalder and Robert Schrader. "Axioms for Euclidean Green's Functions". In: *Commun. Math. Phys.* 31 (1973), pp. 83–112.

[8] Konrad Osterwalder and Robert Schrader. "Axioms for Euclidean Green's Functions II". In: *Commun. Math. Phys.* 42 (1975), pp. 281–305.

[9] Konrad Osterwalder and Robert Schrader. "Euclidean Fermi Fields and a Feynman-Kac formula for Boson-Fermion models". In: *Helvetica Physica Acta* 46 (1973), pp. 277–302.

[10] Roger Penrose. *The Road to Reality.* Jonathan Cape, 2004.

[11] Pierre Ramond. *Field Theory: A Modern Primer.* Benjamin/Cummings, 1981.

[12] Slava Rychkov. *CFT Osterwalder-Schrader Theorem.* Talk at Simons collaboration meeting, Nov. 7, 2019. URL: `https://vimeo.com/374405947`.

[13] Julian Schwinger. "Four-dimensional Euclidean formulation of quantum field theory". In: *8th International Annual Conference on High Energy Physics.* 1958, pp. 134–140.

[14] Graeme Segal. "The definition of conformal field theory". In: *Topology, Geometry and Quantum Field Theory: Proceedings of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal*. Ed. by Ulrike Tillmann. Cambridge University Press, 2004, 421–422.

[15] Raymond F. Streater and Arthur S. Wightman. *PCT, Spin and Statistics, and All That*. Princeton University Press, 2000.

[16] R. S. Ward and Raymond O. Wells Jr. *Twistor geometry and field theory*. Cambridge University Press, 1990.

[17] Edward Witten. "Quantum field theory, Grassmannians, and algebraic curves". In: *Communications in Mathematical Physics* 113.4 (1988), pp. 529 –600.

[18] Peter Woit. *Euclidean Twistor Unification*. 2021. arXiv: `2104.05099 [hep-th]`.

[19] Peter Woit. *Spacetime is Right-handed*. 2023. arXiv: `2311.00608 [hep-th]`.

[20] Jean Zinn-Justin. *Path integrals in quantum mechanics*. Oxford Graduate Texts. Oxford University Press, 2010.