

Euclidean Twistor Unification

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- 2 Four dimensional geometry and spinors
- 3 Gravi-weak unification in Euclidean space-time
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Note: These slides at

<https://www.math.columbia.edu/~woit/twistorunification/brown9-23-21.pdf>

Old version of paper at arXiv:2104.05099 (to be updated soon)

Why Euclidean QFT?

Euclidean QFT goes back to Schwinger (1958), who realized that QFT has inherent definitional problems in Minkowski space-time.

Non-perturbative (path integral) problems

Looking at the path integrals

$$\int F[\phi] e^{iS_M(\phi)} d\phi \text{ versus } \int F[\phi] e^{-S_E(\phi)} d\phi$$

If you do rigorous mathematics you can't make sense of the first, can make sense of the second (ask a mathematical physicist).

If you do numerical calculations, same thing (ask a lattice gauge theorist).

Philosophy: fundamental theory should be defined in Euclidean space-time, our observed physical space time is an analytic continuation.

Euclidean QFT and the two-point function: momentum space

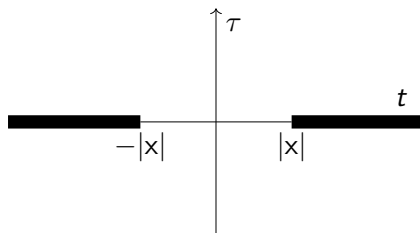
Every quantum field theory textbook explains that there's a problem even in free field QFT. Computing the two-point function involves taking the Fourier transform of

$$\frac{i}{\omega_p^2 - E^2}$$

To do this you have to decide what to do about at the poles $E = \pm\omega_p$. The physically sensible answer corresponds to analytically continuing the calculation from Euclidean space-time.

Euclidean QFT and the two-point function: position space

In position space, as a function of complex time, the two-point function is well-defined except along the real axis for time-like (t, x) . There it needs to be defined as a distribution given by a boundary value of a holomorphic function (a “hyperfunction”), as one takes a limit approaching the real axis.



Minkowski and Euclidean QFT are very different I

So, one difference between Minkowski and Euclidean is

Minkowski

2-point functions are hyperfunctions.

Euclidean

2-point functions are functions.

A fundamental physical difference is

Minkowski

Positive energy
 $\hat{f}(E)$ supported on $E > 0$

Euclidean

$f(t) = \int_{-\infty}^{\infty} \hat{f}(E) e^{iEt} dE$
 is holomorphic on the upper
 half complex time plane
 $(\tau > 0)$.

The fundamental asymmetry in energy corresponds to a fundamental asymmetry in imaginary time τ .

Minkowski and Euclidean QFT are very different II

More reasons:

Minkowski

Field operators satisfy a wave equation.

Field operators in general don't commute.

Physical state space can be defined Lorentz covariantly (can specify $E > 0$ covariantly).

Euclidean

Field operators satisfy no equation of motion (always off-shell).

Field operators always commute.

Defining physical state space requires breaking 4d rotational invariance (can't specify $\tau > 0$ without breaking $SO(4)$).

Minkowski and Euclidean QFT are very different III

Fundamental difference between Minkowski and Euclidean space-time symmetries:

Minkowski

The Lorentz group $SO(3, 1)$ acts on physical states and operators.

Euclidean

The rotation group $SO(4)$ acts on Euclidean Fock space states and operators, but these are not physical states or operators.

To get from the $SO(4)$ covariant Euclidean Fock space theory to a physical Fock space theory with $SO(3, 1)$ covariance, you need to break $SO(4)$ covariance by choosing a $\tau = 0$ hyperplane and using (Osterwalder-Schrader) reflection in that hyperplane.

Four-dimensional geometry and 2×2 complex matrices

It is very useful to think of four-dimensional geometry in terms of 2×2 complex matrices. This works especially well for complex four-dimensional geometry, where one identifies \mathbb{C}^4 with matrices by

$$(z_0, z_1, z_2, z_3) \leftrightarrow z = z_0 1 - i(z_1 \sigma_1 + z_2 \sigma_2 + z_3 \sigma_3)$$

and defines

$$|z|^2 = \det z$$

Pairs $g_L, g_R \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ act by

$$z \rightarrow g_L z g_R^{-1}$$

preserving $|z|$.

Euclidean geometry

To do Euclidean geometry, use the “real form” (x_j real) of the above

$$(x_0, x_1, x_2, x_3) \leftrightarrow x = x_0 1 - i(x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3)$$

$|x|$ is preserved by pairs $g_L, g_R \in SU(2)$, and $Spin(4) = SU(2) \times SU(2)$ is a double-cover of $SO(4)$

Instead of complex matrices one can use quaternions here

$$(x_0, x_1, x_2, x_3) \leftrightarrow x = x_0 1 + x_1 i + x_2 j + x_3 k$$

with $|x|^2 = x\bar{x}$ and rotations given by pairs of unit length quaternions.

Minkowski geometry

To do Minkowski space-time geometry, instead use the real form

$$(x_0, x_1, x_2, x_3) \leftrightarrow x = -i(x_0 \mathbf{1} + x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3)$$

$|x|$ is preserved by pairs $g_L, g_R \in SL(2, \mathbb{C})$ satisfying

$$g_L = (g_R^{-1})^\dagger$$

$Spin(3, 1) = SL(2, \mathbb{C})$ is a double-cover of the Lorentz group $SO(3, 1)$.

Spinor geometry

If one expresses four-dimensional vectors as 2×2 complex matrices, one can think of vectors as linear maps from one \mathbb{C}^2 (called the (half)-spinor space S_R) to another \mathbb{C}^2 (called the (half)-spinor space S_L). Corresponding to the action on vectors by

$$x \rightarrow g_L x g_R^{-1}$$

we have actions on S_R, S_L by

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_R \in S_R \rightarrow g_R \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_R \in S_R$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_L \in S_L \rightarrow g_L \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_L \in S_L$$

Note that

- In Euclidean space, g_R and g_L are independent $SU(2)$ matrices.
- In Minkowski space, $g_R \in SL(2, \mathbb{C})$ and g_L is determined by g_R ($= (g_R^{-1})^\dagger$).

Problems with Euclidean spinor QFT

There is a long history of attempts to understand how spinor QFT in Euclidean space-time is related to spinor QFT in Minkowski space-time. The usual conclusion is that the way to do this is to

- Examine the Euclidean 2-point function, analytic continuation of the Minkowski 2-point function. It transforms under $Spin(4)$ as expected, but does not have the right Hermiticity properties to correspond to an interpretation in terms of field operators.
- Double the number of fields in order to be able to construct Euclidean 2-point functions with correct Hermiticity properties to have an interpretation in terms of field operators.

Apparently one has twice as many degrees of freedom as expected, making the reconstruction of a Minkowski space-time theory more complicated. Lattice gauge theory: using Kogut-Susskind fermions get ever more copies.

Yang-Mills theory

There's a long history of attempts to treat Einstein's general relativity as a gauge theory, trying to emulate the success of the Yang-Mills gauge theory. In the Yang-Mills case one has

- A principal G -bundle over M , for G some Lie group.
- A connection A with curvature F_A on this bundle, given by $Lie(G) = \mathfrak{g}$ -valued 1-forms and 2-forms respectively.
- In the Lagrangian formalism, an action given by integrating the norm-squared of F_A , or in the Hamiltonian formalism a Hamiltonian given by adding the norm-squareds of the electric and magnetic components of the curvature.

General relativity as a gauge theory

One can formulate GR as a gauge theory, taking

- $G = SO(3, 1)$ and the principal G -bundle of orthonormal frames on spacetime M .
- A connection ω (the spin-connection) with curvature Ω on this bundle.
- A frame bundle comes with an \mathbb{R}^4 -valued canonical 1-form e (the vierbeins).
- The Palatini action is

$$\int_M \epsilon_{ABCD} e^A \wedge e^B \wedge \Omega^{CD}(\omega)$$

Equations of motion: from varying ω , ω is torsion-free (Levi-Civita connection), from varying e , get the Einstein equations.

Euclidean Ashtekar variables

If we work in Euclidean signature spacetime, ω takes values in $\mathfrak{spin}(4) = \mathfrak{su}(2)_R \oplus \mathfrak{su}(2)_L$. We can just use the $\mathfrak{su}(2)_R$ component ω_R , and its curvature Ω_R and still get the Einstein equations. One way to do this is to just replace Ω in the Palatini action by Ω_R . Both ω_R and Ω_R act on S_R spinors, not on S_L spinors. Remarkably, one can recover the Einstein equations just using ω_R, Ω_R .

In the Hamiltonian formalism a la Ashtekar, one notes that if one uses ω_R on a space-like hypersurface as configuration variable, the phase space is the same as in $SU(2)$ Yang-Mills theory, with the same sort of Gauss-law constraint from time-independent gauge transformations.

Instead of dynamics being determined by the Yang-Mills Hamiltonian, it is given by the constraints coming from diffeomorphism invariance of the Palatini action.

One usually tries to do this for $\mathfrak{so}(3,1)$ rather than $\mathfrak{spin}(4)$, this requires working with complexified variables.

Gravi-weak unification

There have been attempts to unify the weak interactions with gravity, using the chiral decomposition of the spin connection as above, with $SU(2)_R$ a space-time symmetry giving a gravity theory, and $SU(2)_L$ the internal symmetry of a Yang-Mills theory of the weak interactions. Our proposal is of this nature, but with the following different features:

- Take the Euclidean signature QFT theory as fundamental, with Minkowski signature physics to be found later by analytic continuation.
- Note that in Euclidean QFT one component of the vierbein is distinguished (the imaginary time direction).
- Use twistor geometry to get not just an $SU(2)_L$ internal symmetry but the full electroweak $SU(2)_L \times U(1)$ electroweak internal symmetry, with the imaginary time component of the vierbein behaving like a Higgs field.

Twistor theory

Twistor geometry is a different way of thinking about the geometry of space-time, first proposed in 1967 by Penrose. It naturally provides a joint complexification of Minkowski and Euclidean space-time and a way to look at analytic continuation between them.

Most discussions for physicists focus on the Minkowski version, we'll instead start by focusing on the Euclidean version, later explain the relation to the Minkowski version. In both Euclidean and Minkowski versions we'll focus on conformally compactified space-time, so S^4 rather than \mathbb{R}^4 in the Euclidean version.

Suggested references:

- *Twistor Geometry and Field Theory* by Ward and Wells.
- Any expository article about twistors by Penrose, or The Road to Reality chapter.

Twistor space

In twistor theory one takes as fundamental twistor space $T = \mathbb{C}^4$ (or its projective version $PT = \mathbb{C}P^3$, the complex lines in T). Conformal invariance is built-in, with the conformal groups given by real forms $(SO(5, 1), SO(4, 2))$ of the group $SL(4, \mathbb{C})$ which acts linearly on T . Points of space-time will correspond to $\mathbb{C}^2 \subset T$, tautologically giving the fiber of the half-spinor bundle. Such \mathbb{C}^2 correspond to $\mathbb{C}P^1$ s in PT . Looking at all $\mathbb{C}^2 \subset T$, one gets the Grassmanian $Gr_{2,4}(\mathbb{C})$ which is 4-complex dimensional. When one looks at Minkowski or Euclidean points, one is looking at a 4-dimensional real subspace.

Euclidean twistor theory

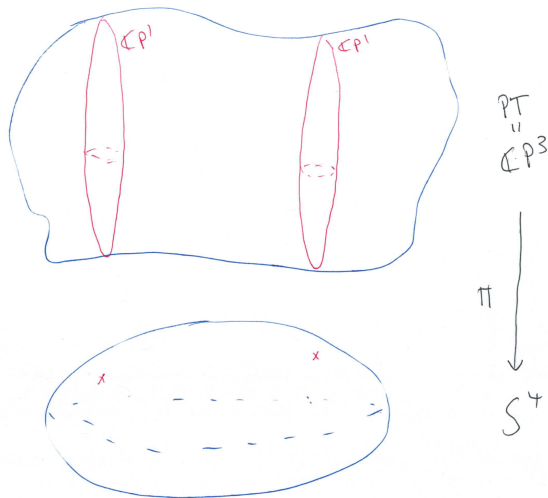
We'll concentrate on Euclidean space-time twistors, which are best understood using quaternions. One can identify $T = \mathbb{C}^4 = \mathbb{H}^2$ and use the fact that $S^4 = \mathbb{H}P^1$, quaternionic projective space. One has a fibration with fibers $\mathbb{C}P^1$

$$\begin{array}{ccc} \mathbb{C}P^1 & \longrightarrow & PT = \mathbb{C}P^3 \\ & & \downarrow \pi \\ & & S^4 \end{array}$$

where the map π takes a complex line in \mathbb{C}^4 to the quaternionic line it generates.

This deserves a picture:

Euclidean twistor fibration: a picture



Two interpretations of PT

PT is the projective spin bundle $P(S_R)$

The fiber at a point is the \mathbb{CP}^1 of projective S_R space.

PT is the bundle of complex structures on S^4

The $\mathbb{CP}^1 = S^2$ fiber above a point on S^4 can be identified with the possible choices of complex structure on the tangent space at the point.

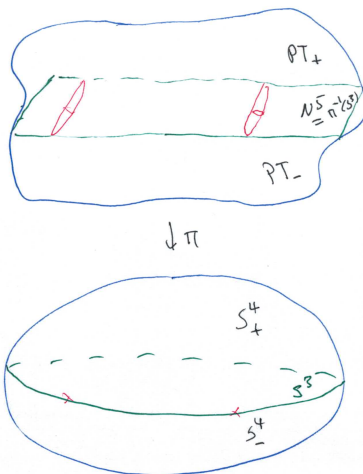
These definitions generalize PT to give a twistor space for any Riemannian manifold in $d = 4$. If the metric is ASD, this twistor space is a complex manifold and allows study of the Riemannian geometry using holomorphic methods.

Euclidean twistors and distinguished imaginary time

We have argued that our Euclidean space-time has to have an extra structure, a distinguished imaginary time direction, to allow for analytic continuation to Minkowski space-time. This gives a new structure on PT , a 5-dimensional hypersurface N which splits it into two pieces.

Another picture:

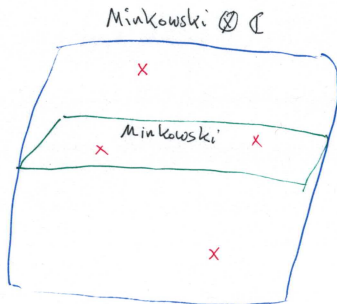
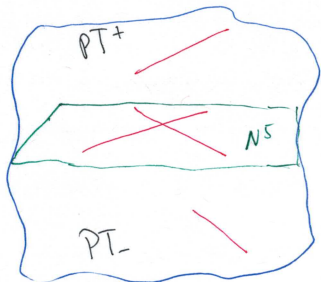
Euclidean twistor fibration: distinguished imaginary time



Minkowski space twistors

The subspace N determines the Minkowski space-time geometry as follows. N is the zero-set of a nondegenerate signature $(2, 2)$ Hermitian form Φ on \mathbb{C}^4 . Minkowski space-time is the subspace of $\mathbb{C}^2 \subset \mathbb{C}^4$ on which $\Phi = 0$. $\Phi = 0$ determines a real form $SU(2, 2) = Spin(4, 2)$ of $SL(4, \mathbb{C})$ that acts transitively on (compactified) Minkowski space. This is the conformal group, it also acts on solutions to massless wave equations. In the Euclidean case the conformal group real form was $SL(2, \mathbb{H})$. The $\mathbb{C}P^1 = S^2$ in PT corresponding to a point in Minkowski space can be identified with the “celestial sphere” of light rays through that point. This provides a very intuitive interpretation of the twistor point of view: your point in space-time is the sphere you see when you open your eyes. Another picture:

Minkowski space-time twistors: a picture



Twistor unification: gravi-weak

If one works on the projective twistor space PT , one can get the idea of gravi-weak unification to work (in its Euclidean form):

- There is not just an $SU(2)$ internal symmetry, but also a $U(1)$, given by the complex structure specified by the point in the fiber. This complex structure picks out a $U(2) \subset SO(4)$, the complex structure preserving orthogonal transformations of the tangent space to the point on the base S^4 . This is the electroweak $U(2)$ symmetry, to be gauged to get the standard electroweak gauge theory.
- If one lifts the choice of vector in the imaginary time direction up to PT , it transforms like the Higgs field: it is a vector in \mathbb{C}^2 (using the complex structure on the tangent space given by the point in the fiber). The $U(2)$ act on this \mathbb{C}^2 in the usual way. Each choice of Higgs field breaks the $U(2)$ down to a $U(1)$ subgroup, which will be the unbroken gauge symmetry of electromagnetism.

Twistor unification: QCD

Besides specifying a point on S^4 and a complex structure on its tangent space, a point in PT specifies a complex line $\mathbb{C} \subset \mathbb{C}^4$. The $U(1)$ discussed above is the group of phase transformations of that complex line. At the same time, the point in PT specifies a three-complex dimensional space, the quotient space \mathbb{C}^4/\mathbb{C} . Using the standard Hermitian form on \mathbb{C}^4 , the group $SU(4)$ acts on \mathbb{C}^4 preserving this form.

Looking at this action as an action on the space of lines $PT = \mathbb{C}P^3$, the stabilizer of a point is the group $U(3)$. This includes the $U(1)$ which acts on the line, but also an $SU(3)$ that acts on the quotient.

Using the quaternion picture we've found that a choice of a point on S^4 gives a decomposition $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$ and picks out an $Sp(1) \times Sp(1)$ subgroup of $Sp(2)$. Using the complex picture, a point on PT gives a decomposition $\mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C}^3$ and picks out a $U(3)$ subgroup of $SU(4)$. We thus have the right internal and spin rotation symmetries to gauge and get a unified theory.

A generation of matter fields

A generation of SM matter fields has exactly the transformation properties under the SM gauge groups as maps from \mathbb{C}^4 to itself, or

$$\text{Hom}(\mathbb{C} \oplus \mathbb{C}^3, S_R \oplus S_L) = (\mathbb{C} \oplus \mathbb{C}^3)^* \otimes (S_L \oplus S_R)$$

One could write this space as

$$(\mathbb{C}_{-1} \otimes \mathbb{C}_{\frac{1}{3}}^3) \otimes (\mathbb{C}_0^2 \oplus \mathbb{C}_{-1} \oplus \mathbb{C}_{+1})$$

which is

$$\mathbb{C}_{-1}^2 \oplus \mathbb{C}_{-2} \oplus \mathbb{C}_0 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^2)_{\frac{1}{3}} + \mathbb{C}_{-\frac{2}{3}}^3 + \mathbb{C}_{\frac{4}{3}}^3$$

Here the subscripts are $U(1)$ weights (weak hypercharge), the \mathbb{C}^2 are the fundamental representation of $SU(2)_L$ and the \mathbb{C}^3 are the fundamental representation of $SU(3)$. For the first generation, the terms above correspond respectively to the fundamental particles

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, e_R, (\nu_e)_R, \begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R$$

Why three generations?

While one generation fits into a very simple construction, why three?

S^7 instead of $\mathbb{C}P^3$

Using quaternions and complex numbers, one has not fully exploited all the possible structures on the real eight-dimensional space T . In terms of unit vectors, S^7 carries several different kinds of geometry

$$S^7 = Spin(8)/Spin(7) = Spin(7)/G_2 = Spin(6)/SU(3) = Spin(5)/Sp(1)$$

In particular, we have used complex ($Spin(6) = SU(4)$) and quaternionic ($Spin(5) = Sp(2)$) aspects of the geometry, but not the octonionic aspects that appear in $S^7 = Spin(7)/G_2$.

Gauge theory on PT

In this proposal, there's a profound reorganization of fundamental degrees of freedom. They now live on points of PT which one can think of as light-rays, rather than on points of space-time. Mathematically, one needs to find a formalism on PT that corresponds to the usual Yang-Mills formalism on the base S^4 . Need to use holomorphicity on the \mathbb{CP}^1 fibers to match degrees of freedom on S^4 and on PT . The Penrose-Ward correspondence does this for anti-self-dual connections.

Similarly need to match the Dirac equation on S^4 and equations on PT . For bundles on the base with ASD connections, this is done by the Penrose-Ward correspondence, but the $U(1)$ and $SU(3)$ bundles are on PT , vary on a fiber.

True gravi-weak unification?

Have mostly just rewritten the usual electroweak and GR theory. One difference though is that one component of the vierbein is now the Higgs, which has the electroweak dynamics. Does this change the usual problems about renormalizability, higher order terms in the curvature, etc?

Conclusions

Attractive aspects of this picture of fundamental physics:

- Spinors are tautological objects (a point in space-time is a space of Weyl spinors), rather than complicated objects that must be separately introduced in the usual geometrical formalism.
- Analytic continuation between Minkowski and Euclidean space-time can be naturally performed, since twistor geometry provides their joint complexification.
- Exactly the internal symmetries of the Standard Model occur.
- The intricate transformation properties of a generation of Standard Model fermions correspond to a simple construction.
- One gets a new chiral formulation of gravity, unified with the Standard Model.
- Conformal symmetry is built into the picture in a fundamental way.