Notes on the Twistor $\mathbf{P}^1$

Peter Woit

Department of Mathematics, Columbia University
woit@math.columbia.edu

February 8, 2022

Abstract

Remarkably, the twistor $\mathbf{P}^1$ occurs as a fundamental object in both four-dimensional space-time geometry and in number theory. In Euclidean signature twistor theory it is how one describes space-time points. In recent work by Fargues and Scholze on the local Langlands conjecture using geometric Langlands on the Fargues-Fontaine curve, the twistor $\mathbf{P}^1$ appears as the analog of this curve at the infinite prime.

These notes are purely expository, written with the goal of explaining, in a form accessible to both mathematicians and physicists, various different ways in which the twistor $\mathbf{P}^1$ makes an appearance, often as a geometric avatar of the quaternions.

1   Introduction

In the Euclidean signature twistor description of space-time, a point in space-time is described by a "twistor $\mathbf{P}^1$": the Riemann sphere, together with the real structure given by the antipodal involution. The physical interpretation of this sphere is that it corresponds to the sphere of light-rays through the corresponding point. Remarkably, the same twistor $\mathbf{P}^1$ occurs not just in this physical context, but in two related mathematical contexts far removed from physics:

- In Simpson's approach to Hodge theory, Hodge structures are given by equivariant vector bundles on the twistor $\mathbf{P}^1$.

- Recent work in arithmetic geometry and the Langlands program has made clear that a useful tool for understanding behavior at a prime p is a new sort of geometrical object, the Fargues-Fontaine curve. The twistor $\mathbf{P}^1$ is the analog of the Fargues-Fontaine curve at the "infinite prime" (the archimedian place of the number field $\mathbf{Q}$).

Euclidean twistor geometry provides a compelling basis for unification of the fundamental internal and space-time symmetries of physics (see [19]). That the same twistor $\mathbf{P}^1$ describing points of physical space-time there also fits into a

description of the points of number theory is a remarkable relation between these two very different subjects. The goal of these notes is to gather together some more detail about this observation, the significance of which remains unclear. Along the way, we'll also point to other occurrences of the same twistor $\mathbf{P}^1$ in such different contexts as hyperkähler geometry and the metaplectic central extension of the symplectic symmetry group of canonical quantization.

2 The twistor $\mathbf{P}^1$

The two-dimensional Riemann sphere can be identified with $\mathbf{CP}^1$, the space of complex lines in $\mathbf{C}^2$. Homogeneous coordinates

$$[z_1, z_2]$$

describe points of this space as equivalence classes of pairs of complex numbers. Taking the quotient

$$z = z_1/z_2$$

gives a coordinate chart identifying $\mathbf{C}$ with $\mathbf{CP}^1$ minus the point $[z_1, 0]$.

A real structure on $\mathbf{CP}^1$ is given by an antiholomorphic map

$$\rho : \mathbf{CP}^1 \to \mathbf{CP}^1$$

such that

$$\rho^2 = 1$$

One such real structure is the usual complex conjugation. In homogeneous coordinates

$$\rho([z_1, z_2]) = [\overline{z}_1, \overline{z}_2]$$

or in the z coordinate

$$\rho(z) = \overline{z}$$

Real points for this real structure on $\mathbf{CP}^1$ are those satisfying

$$\rho([z_1, z_2]) = [z_1, z_2]$$

so are on a circle in $\mathbf{CP}^1$, or the real number line in the z coordinate.

The twistor $\mathbf{P}^1$ (which we'll often denote $\mathbf{P}^1_{tw}$) is $\mathbf{CP}^1$ with a different real structure, given in homogeneous coordinates by

$$\rho_{tw}([z_1, z_2]) = [-\overline{z}_2, \overline{z}_1]$$

or in the z coordinate by

$$\rho_{tw}(z) = -1/\overline{z}$$

As a map on coordinates (z_1, z_2) of $\mathbf{C}^2$ one has $\rho_{tw}^2 = -1$, but $\rho_{tw}^2 = 1$ as a map on the space of complex lines. On the sphere, ρ_{tw} is the antipodal map taking a point to its antipode. Note that there are no points on $\mathbf{CP}^1$ such that

$$\rho_{tw}([z_1, z_2]) = [z_1, z_2]$$

since

$$z = -1/\overline{z} \implies |z|^2 = -1$$

As an object in algebraic geometry, one way the twistor $\mathbf{P}^1$ can be described is as above, as a complex projective variety ($\mathbf{CP}^1$) together with an action of the Galois group $Gal(\mathbf{C}/\mathbf{R})$. It can also be defined in terms of the equation $x^2 + y^2 + z^2 = 0$, which describes a curve in projective two-space. This is a projective algebraic curve defined over the real numbers, with complex points but no real points.

One way to understand where ρ_{tw} comes from is in terms of the quaternions $\mathbf{H}$. A conventional identification of $\mathbf{H}$ with $\mathbf{C}^2$ is given by

$$x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbf{H} \leftrightarrow (x_0 + x_1 i, x_2 + x_3 i) \in \mathbf{C}^2$$

allowing one to write quaternions as (identifying $i \in \mathbf{C}^2$ and $\mathbf{i} \in \mathbf{H}$)

$$q = z_1 + z_2\mathbf{j}$$

for complex numbers $z_1 = x_0 + x_1 i$ and $z_2 = x_2 + x_3 i$. ρ_{tw} is then multiplication by $\mathbf{j}$, since

$$\begin{aligned}
\mathbf{j}(z_1 + z_2\mathbf{j}) =& \mathbf{j}(x_0 + x_1\mathbf{i}) + \mathbf{j}(x_2 + x_3\mathbf{i})\mathbf{j} \\
=& -(x_2 - x_3\mathbf{i}) + (x_0 - x_1\mathbf{i})\mathbf{j} \\
=& -\overline{z}_2 + \overline{z}_1\mathbf{j}
\end{aligned}$$

3   Twistor theory in (compactified) Euclidean space-time

Penrose's twistor theory generalizes the use of $\mathbf{C}^2$ and $\mathbf{H}$ above to the case of $\mathbf{C}^4$ and $\mathbf{H}^2$. $T = \mathbf{C}^4$ is Penrose's twistor space, and instead of $\mathbf{CP}^1$ one considers the projective twistor space $\mathbf{CP}^3 = PT$ of complex lines in T. In twistor theory, a point in (complexified, compactified) space-time is a $\mathbf{C}^2 \subset T$, so a point in the Grassmannian $Gr_{2,4}(\mathbf{C})$. This provides a tautological definition of the (Weyl) spinor bundle: the Weyl spinors at a point are the point. In terms of PT, a space-time point is a $\mathbf{CP}^1 \subset PT$. This $\mathbf{CP}^1$ is the sphere of light rays through the corresponding point, exactly the sphere one sees when one opens an eye.

Most discussions of twistor theory and physics focus on a real four-dimensional Minkowski signature subspace of $Gr_{2,4}(\mathbf{C})$, but one can relate physics there by analytic continuation to what happens on a real four-dimensional Euclidean signature subspace. To define such a subspace, one identifies $\mathbf{C}^4$ and $\mathbf{H}^2$ using a pair of the earlier identifications of $\mathbf{C}^2$ with $\mathbf{H}$ by $z_1 + z_2\mathbf{j}$. As in the $\mathbf{CP}^1$ case, one has two different real structures on $\mathbf{CP}^3$, given by the usual conjugation

$$\rho([z_1, z_2, z_3, z_4]) = [\overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4]$$

and by multiplication by $\mathbf{j}$

$$\rho_{tw}([z_1, z_2, z_3, z_4]) = [-\overline{z}_2, \overline{z}_1, -\overline{z}_4, \overline{z}_3]$$

$\mathbf{CP}^3$ with the real structure ρ has a three-real-dimensional space $\mathbf{RP}^3$ of real points, but the twistor real structure ρ_{tw} has no real points. It does however have real lines, i.e. $\mathbf{CP}^1$s in $\mathbf{CP}^3$ that are fixed by ρ_{tw}. Each such $\mathbf{CP}^1$ is a twistor $\mathbf{P}^1$, with ρ_{tw} acting on it in the manner described in the previous section. There will be a four-real-dimensional family of such $\mathbf{CP}^1$s, parametrized by the four-sphere S^4. To see this, consider the quaternionic analog $\mathbf{HP}^1$ of $\mathbf{CP}^1$, with homogeneous coordinates

$$[q_1, q_2]$$

which describe points on this space as equivalence classes of pairs of quaternions. Taking the quotient

$$q = (q_2)^{-1} q_1$$

gives a coordinate chart identifying $\mathbf{H}$ with $\mathbf{HP}^1 = S^4$ minus the point $[q_1, 0]$.

Our identification of $\mathbf{C}^4$ and $\mathbf{H}^2$ gives a mapping

$$\pi : [z_1, z_2, z_3, z_4] \in \mathbf{CP}^3 \to [z_1 + z_2\mathbf{j}, z_3 + z_4\mathbf{j}] \in \mathbf{HP}^1$$

that takes a complex line in $\mathbf{C}^4$ to the quaternionic line in $\mathbf{H}^2$ that it generates. This mapping fibers $\mathbf{CP}^3$ over S^4, with fibers the $\mathbf{CP}^1$s that are real lines for the real structure ρ_{tw}.

For the case of $\mathbf{C}^2$, from the linear action of $SL(2, \mathbf{C})$ on $\mathbf{C}^2$ one gets an action of $SL(2, \mathbf{C})$ on $\mathbf{CP}^1$ by conformal transformations, an action which does not commute with the real structure. In the case of $\mathbf{C}^4 = \mathbf{H}^2$, one gets an action of the conformal group $SL(2, \mathbf{H}) = Spin(5, 1)$, again not commuting with the real structure.

There are only the two inequivalent real structures ρ and ρ_{tw} on projective twistor space $\mathbf{CP}^3$. To pick out (compactified) Minkowski space-time points in the complexification $Gr_{2,4}(\mathbf{C})$ one needs to use a real structure that does not act on $PT = \mathbf{CP}^3$ but instead takes PT to a dual projective space PT^*. In terms of spinors, this Minkowski space real structure takes one chirality of spinors to the other (unlike the Euclidean case).

4 Twistor theory for $\mathbf{R}^4$

The $\mathbf{CP}^1$ fiber in the fibration

$$\mathbf{CP}^1 \longrightarrow PT = \mathbf{CP}^3$$
$$\downarrow \pi$$
$$S^4 = \mathbf{HP}^1$$

of the previous section can be interpreted as parametrizing orthogonal complex structures on the tangent space of the base space, allowing one to study S^4 (which is not a complex manifold) using $\mathbf{CP}^3$, which is. Solutions of the self-dual Yang-Mills equations on S^4 correspond to holomorphic structures on vector bundles on $\mathbf{CP}^3$, and this can be generalized from S^4 with its usual metric to

self-dual solutions to the Einstein equations on other four-manifolds (this is Penrose's non-linear graviton construction[12]).

One can instead pursue the idea of working with all orthogonal complex structures on $\mathbf{R}^4 = \mathbf{H}$ simultaneously by identifying such complex structures with points on the sphere of unit length imaginary quaternions

$$x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is a unit vector. This sphere of complex structures on $\mathbf{R}^4 = \mathbf{H}$ can be identified with $\mathbf{CP}^1$ and given the usual complex structure on that space.

Instead of Penrose's twistor space T and its projective version PT, one can form the product of $\mathbf{R}^4$ and $\mathbf{CP}^1$, with the fibration

$$\mathbf{R}^4 \times \mathbf{CP}^1$$
$$\downarrow \pi$$
$$\mathbf{CP}^1$$

where π is projection on the second factor. In this context, $\mathbf{R}^4 \times \mathbf{CP}^1$ is called the twistor space. It can be given the complex structure that on $\mathbf{R}^4$ is determined by the point on $\mathbf{CP}^1$ and on $\mathbf{CP}^1$ is the usual complex structure. With respect to this complex structure, π is a holomorphic map. Note that this construction comes with a real structure ρ which on $\mathbf{R}^4 = \mathbf{H}$ is the identity, and is the twistor real structure ρ_{tw} on $\mathbf{CP}^1$.

The holomorphic fibration above is isomorphic to the fibration

$$\mathcal{O}(1) \oplus \mathcal{O}(1)$$
$$\downarrow \pi$$
$$\mathbf{CP}^1$$

Instead of using quaternions, one can realize the sphere $\mathbf{CP}^1$ in terms of Pauli matrices, as the set of matrices

$$x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$$

with $\mathbf{x}$ a unit vector. Taking as fiber above such a point the $+1$ eigenspace of the matrix acting on $\mathbf{C}^2$ gives the vector bundle $\mathcal{O}(1)$. Two copies of this construction give $\mathcal{O}(1) \oplus \mathcal{O}(1)$, a vector bundle with total space $\mathbf{R}^4 \times \mathbf{CP}^1$ and fiber $\mathbf{R}^4$ identified with $\mathbf{C} \oplus \mathbf{C} = \mathbf{C}^2$. More generally, if one replaces $\mathbf{R}^4 = \mathbf{H}$ by any quaternionic vector space of dimension d, one gets a holomorphic bundle over $\mathbf{CP}^1$, direct sum of $2d$ copies of $\mathcal{O}(1)$.

5   The twistor $\mathbf{P}^1$ and geometry

In this section we'll see that the twistor $\mathbf{P}^1$ appears in geometry in several different guises.

5.1   Hyperkähler manifolds

One can generalize the notion of a Kähler manifold by considering manifolds M of dimension $4k$, which come not with a single Kähler structure I, but a sphere of such structures, given by

$$aI + bJ + cK$$

where J, K are also Kähler structures, $IJ = K$ and (a, b, c) lie on the unit sphere in $\mathbf{R}^3$. Besides the simple example of $\mathbf{R}^{4k} = \mathbf{H}^k$, other examples of such manifolds include co-adjoint orbits of complex Lie groups, as well as spaces of holomorphic flat $GL(n, \mathbf{C})$ connections or Higgs bundles on a Riemann surface Σ. For such manifolds one can define a corresponding twistor space, in the same manner as was done above for the case of $\mathbf{R}^4$. These twistor spaces were first studied in [6]. For a survey of results about hyperkähler manifolds, see [7].

The authors of [6] showed that one could generalize the notion of symplectic quotient by a group G from the case of Kähler manifolds to the hyperkähler case, allowing one to construct new hyperkähler manifolds by taking hyperkähler quotients. A fundamental source of examples is the space of self-dual Yang-Mills connections on a bundle on $\mathbf{R}^4$, where quotienting by the group of gauge transformations gives a finite-dimensional moduli space of solutions. One can get many more interesting hyperkähler manifolds from these. Imposing an invariance condition in one direction gives the moduli space of monopole solutions to the Bogomolny equations, in two directions the moduli space of Higgs fields and solutions to the Hitchin equations, in three directions complex co-adjoint orbits and solutions to the Nahm equations.

5.2   Hodge theory and $\mathbf{P}^1_{tw}$

In Hodge theory, one studies extra structure on the cohomology groups $H^n(M, \mathbf{C})$ arising from looking at harmonic forms representing de Rham cohomology classes. When M is a smooth compact Kähler manifold, harmonic n-forms decompose into harmonic (p, q)-forms and one has a Hodge decomposition

$$H^n(M, \mathbf{C}) = \oplus_{p+q=n} H^{p,q}(M)$$

This decomposition does not vary holomorphically with changes in the complex structure of M, but the filtration on $V = H^n(M, \mathbf{C})$ using the degrees of holomorphic differential forms does have this property.

The added structure that occurs in Hodge theory is described by defining a pure real Hodge structure of weight w on a complex vector space V with a real structure to be a filtration

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^2 = V$$

such that the filtration and its conjugate satisfy

$$F^p \oplus \overline{F^{n-p+1}} = V$$

There are more general "mixed" Hodge structures, which we will not consider here. Taking $V = H^n(M, \mathbf{C})$ with its usual real structure and $F^p = \oplus_{i>p} H^q(M, \Omega^i)$, one has $H^{p,q}(M) = F^p \cap \overline{F^q}$.

In [14] and elsewhere, Simpson has shown that one can replace this definition of a Hodge structure by a definition of a "twistor structure", which uses bundles on $\mathbf{P}^1_{tw}$. The classification of holomorphic vector bundles on $\mathbf{CP}^1$ is well-known: they are direct sums of the $\mathcal{O}(w)$ which are degree w rank 1 line bundles. The corresponding objects in the case of $\mathbf{P}^1_{tw}$ (which we'll continue to refer to as holomorphic bundles) are rank two bundles when the degree is odd, rank one when the degree is even. They can be denoted

$$\mathcal{O}_{\mathbf{P}^1_{tw}}\left(\frac{w}{2}\right)$$

for w an integer. Here $w/2$ is the slope (degree/rank) of the vector bundle.

Simpson identifies a real Hodge structure, pure of weight w, with the twistor stucture given by the holomorphic vector bundle $\mathcal{E}$ on $\mathbf{P}^1_{tw}$ such that

$$\mathcal{E} = \mathcal{O}_{\mathbf{P}^1_{tw}}\left(\frac{w}{2}\right) \oplus \mathcal{O}_{\mathbf{P}^1_{tw}}\left(\frac{w}{2}\right) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1_{tw}}\left(\frac{w}{2}\right)$$

where the number of terms in the sum is the dimension of the underlying vector space V. The filtration part of the Hodge structure corresponds to an action of the group $U(1)$ on the bundle $\mathcal{E}$.

To go back and forth between Hodge structures and twistor stuctures, one uses the coordinate z on $\mathbf{P}^1_{tw}$, which picks out a point $\infty \in \mathbf{P}^1_{tw}$, about which one can use the coordinate $\lambda = 1/z$. A $U(1)$ group now acts on the $\mathbf{P}^1_{tw}$, preserving ∞. Given a Hodge structure on V, one constructs a twistor structure by taking the trivial vector bundle

$$\mathcal{E}_1 = V \otimes_{\mathbf{R}} \mathcal{O}_{\mathbf{P}^1_{tw}}$$

away from $\lambda = 0$ and gluing it to a bundle $\mathcal{E}_2$ on the formal neighborhood of $\lambda = 0$ using the filtration.

Given a $U(1)$ equivariant twistor structure $\mathcal{E}$, one can recover the Hodge structure by taking V to be the fiber of E at a point away from $\lambda = 0$. The action of $U(1)$ at $\lambda = 0$ will decompose the fiber there into $U(1)$-weight spaces which will be the associated graded spaces of the filtration. One can make the following table:

Hodge structures	twistor structures
Real Hodge structure	$U(1)$ equivariant holomorphic vector bundle $\mathcal{E}$ on $\mathbf{P}^1_{tw}$
pure of weight w	sum of $\mathcal{O}_{\mathbf{P}^1_{tw}}\left(\frac{w}{2}\right)$
underlying vector space V	trivial bundle $\mathcal{E}_1 = V \otimes \mathcal{O}_{\mathbf{P}^1_{tw}}$ away from ∞
Hodge filtration	modification of $\mathcal{E}_1$ at ∞ to give $\mathcal{E}$.

5.3   Non-abelian Hodge theory

There is a non-abelian version of Hodge theory due to Simpson (see for instance
[15]), which is based on considering for a space X the moduli space of represen-
tations of the fundamental group $\pi_1(X)$. One takes the space of representations
of $\pi(X)$ into $GL(n, \mathbf{C})$ up to equivalence, and denotes it

$$M = H^1(X, GL(n, \mathbf{C}))$$

thinking of it as non-abelian cohomology in degree one. This is a set, not a
vector space, and the non-abelian nature of the coefficients implies it does not
extend to higher-degree cohomology.

For the case $X = \Sigma$ a Riemann surface, this moduli space is hyperkähler,
and one can study it by using the corresponding twistor space. For different
points on $\mathbf{CP}^1$ one gets different definitions of the space M of quite different
natures, relating moduli spaces of Higgs bundles on Σ and moduli spaces of
holomorphic flat connections on Σ to moduli spaces of representations. One
method for constructing this twistor space uses Deligne's λ-connections, with λ
a complex parameter corresponding to a coordinate on $\mathbf{P}^1$. For $\lambda = 1$ one gets
usual connections, at $\lambda = 0$ Higgs bundles.

6   The twistor $\mathbf{P}^1$ and arithmetic

The story of the twistor $\mathbf{P}^1$ has an arithmetic generalization, with a different
but very similar structure occuring for each prime number p. For each such p
one can define a norm on the field $\mathbf{Q}$ of rational numbers by taking

$$|x|_p = p^{-a}$$

where a is the power of p that occurs when one factors $x = n/m \in \mathbf{Q}$. In this
norm two rational numbers are close together when their difference has a large
positive factor of p. Just as one defines the field $\mathbf{R}$ as the completion of $\mathbf{Q}$ with
respect to the usual norm $|\cdot|$, one can define new fields $\mathbf{Q}_p$ as the completions
of $\mathbf{Q}$ with respect to the norms $|\cdot|_p$. The usual norm the corresponds to an
"infinite prime" and is denoted by $|\cdot|_\infty$.

One can then study elements of $\mathbf{Q}$ as elements of the completions $\mathbf{Q}_p$ and
$\mathbf{R}$, and field extensions of $\mathbf{Q}$ in terms of field extensions of $\mathbf{Q}_p$ and of $\mathbf{R}$. Field
extensions of $\mathbf{R}$ are very simple to understand, since the Galois group $Gal(\mathbf{C}/\mathbf{R})$
of the algebraic closure is just $\mathbf{Z}/2\mathbf{Z}$. Field extensions of $\mathbf{Q}_p$ are a much more
complicated subject, with $Gal(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ having a rather intricate structure.

6.1   Analogs at p of H and the twistor $\mathbf{P}^1$

For each prime p, part of the structure of $\mathbf{Q}_p$ and its extensions can be under-
stood as closely analogous to what happens at the infinite prime, in terms of
generalizations of the algebra $\mathbf{H}$ over $\mathbf{R}$ to more general quaternion algebras for

each $\mathbf{Q}_p$. To get this generalization, for an arbitrary field F, consider the algebra over F with basis elements $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying the usual quaternion relations

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

as well as the more general

$$\mathbf{i}^2 = a, \quad \mathbf{j}^2 = b$$

where a, b are invertible elements in F. We'll use the notation

$$\left(\frac{a, b}{F} \right)$$

to denote this algebra.

In this notation the usual quaternion algebra is

$$\mathbf{H} = \left(\frac{-1, -1}{\mathbf{R}} \right)$$

For different values of a, b, $\left(\frac{a,b}{\mathbf{R}} \right)$ will be isomorphic to either $\mathbf{H}$ or the algebra $\mathbf{M}(2, \mathbf{R})$ of two-by-two real matrices. These two algebras are not isomorphic, but become so after allowing complex coefficients, i.e.

$$\mathbf{H} \otimes_{\mathbf{R}} \mathbf{C} = M(2, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = M(2, \mathbf{C})$$

Something very similar happens for $F = \mathbf{Q}_p$: the $\left(\frac{a,b}{\mathbf{Q}_p} \right)$ fall into two isomophism classes, one of which includes

$$M(2, \mathbf{Q}_p)$$

and the other of which includes

$$\left(\frac{p, u}{\mathbf{Q}_p} \right)$$

where u is a unit and not a square. The second of these is a division algebra and is the analog at the prime p of the usual quaternion algebra $\mathbf{H}$ at the infinite prime. Allowing coefficients in a quadratic extension K of $\mathbf{Q}_p$ (for $p \neq 2$ there are three of these: $\mathbf{Q}_p[\sqrt{p}], \mathbf{Q}_p[\sqrt{u}], \mathbf{Q}_p[\sqrt{up}]$) one finds that the two non-isomorphic algebras become isomorphic

$$\left(\frac{p, u}{\mathbf{Q}_p} \right) \otimes_{\mathbf{Q}_p} K = M(2, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} K = M(2, K)$$

The Hilbert symbol distinguishes the two isomorphism classes, and is defined by

$$(a, b)_F = \begin{cases} 1 \quad \text{if} \quad \left(\frac{a,b}{F} \right) \simeq M(2, F) \\ -1 \quad \text{if} \quad \left(\frac{a,b}{F} \right) \not\simeq M(2, F) \end{cases}$$

9

One can associate to each quaternion algebra the equation of a conic

$$\left(\frac{a,b}{F}\right) \leftrightarrow C = \{(x,y,z) : -ax^2 - by^2 + abz^2 = 0\}$$

This will have solutions in F when $(a,b)_F = 1$, but not when $(a,b)_F = -1$ For the case of the quaternions $\mathbf{H}$ the equation for C is

$$x^2 + y^2 + z^2 = 0$$

which has no non-zero solutions over the real numbers, but over the complex numbers describes $\mathbf{CP}^1 \subset \mathbf{CP}^2$ in homogeneous coordinates. At a prime p, the analog of the twistor $\mathbf{P}^1$ will be the conic C with equation

$$-px^2 - uy^2 + upz^2 = 0$$

which has no solutions in $\mathbf{Q}_p$, but does have solutions in the quadratic extensions K of $\mathbf{Q}_p$. The Galois group

$$Gal(K/\mathbf{Q}_p) = \mathbf{Z}/2\mathbf{Z}$$

acts on these solutions with no fixed points, providing an analog of the antipodal map on the $\mathbf{CP}^1$.

A generalization of the class of quaternion algebras over a field F is the class of finite dimensional simple algebras over F with center F. One can put an equivalence relation

$$A \sim B \leftrightarrow A \otimes M_n(F) \simeq B \otimes M_m(F)$$

on such algebras, and a product

$$[A] \cdot [B] = [A \otimes B]$$

on the equivalence classes, giving a group called the Brauer group $Br(F)$. For $F = \mathbf{R}$ one can show that $Br(\mathbf{R}) = \mathbf{Z}/2\mathbf{Z}$, with generator $[\mathbf{H}]$.

For the case $F = \mathbf{Q}_p$, the class $[\left(\frac{p,u}{\mathbf{Q}_p}\right)]$ is of order two in $Br(\mathbf{Q}_p)$, but in this case the Brauer group is

$$Br(\mathbf{Q}_p) = \mathbf{Q}/\mathbf{Z}$$

with generators of order n subgroups given by higher dimensional algebras. One has higher dimensional analogs of the twistor $\mathbf{P}^1$, given by Brauer-Severi varieties, which are varieties over $\mathbf{Q}_p$ which become isomorphic to projective space when one extends scalars to the algebraic closure of $\mathbf{Q}_p$.

One way of understanding why one gets the same classification for quaternion algebras and these conics is that they both represent elements in the same Galois cohomology group

$$H^1(Gal(\overline{F}/F), PGL_2(\overline{F}))$$

which has two elements. In the case $F = \mathbf{R}$ one gets nothing new by going to higher degree and considering

$$H^1(Gal(\overline{F}/F), PGL_n(\overline{F}))$$

but for $F = \mathbf{Q}_p$ one gets something new at each n.

In the classification of quadratic forms over a field, the Brauer group of the field (or at least its 2-torsion) makes an appearance as an invariant of quadratic forms. The Witt ring of equivalence classes of quadratic forms up to hyperbolic planes has a $\mathbf{Z}/2\mathbf{Z}$-valued rank map, with kernel I. The two-torsion of the Brauer group is isomorphic to I^2/I^3.

Generalizing from the case of $Br(\mathbf{R}) = \mathbf{Z}/2\mathbf{Z}$ in another direction, one can consider $\mathbf{Z}/2\mathbf{Z}$-graded algebras and define in the same way as the Brauer group instead the Brauer-Wall or super-Brauer group $SBr(F)$ of a field. Clifford algebras provide representatives of elements of the super-Brauer group. In particular, for $F = \mathbf{R}$ one finds that $SBr(\mathbf{R}) = \mathbf{Z}/8\mathbf{Z}$, with representatives given by the real Clifford algebras $\mathrm{Cliff}(r, s, \mathbf{R})$ for real vector spaces with a non-degenerate quadratic form of signature (r, s).

The structure of $SBr(\mathbf{R})$ explains the 8-fold periodicity found in the structure of real Clifford algebras (and in the KO version of K-theory). The real Clifford algebras are given by a somewhat complicated pattern of different matrix algebras over $\mathbf{R}, \mathbf{C}, \mathbf{H}$, which become isomorphic after complexification:

$$\mathrm{Cliff}(r, s, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = \begin{cases} M(2^n, \mathbf{C}) & \text{for } r + s = 2n \\ M(2^n, \mathbf{C}) \oplus M(2^n, \mathbf{C}) & \text{for } r + s = 2n + 1 \end{cases}$$

For details of this story, see [3].

6.2   Canonical quantization, quaternions and arithmetic

For one degree of freedom, canonical quantization starts with a position space $\mathbf{R}$ with coordinate q and a dual momentum space $\mathbf{R}^*$ with coordinate p. It gives operators Q, P on a space of functions of q, which we can take to be the Schwarz space $\mathcal{S}(\mathbf{R})$. These operators satisfy the Heisenberg commutation relations

$$[Q, P] = i\mathbf{1}$$

(we are setting $\hbar = 1$) so provide an irreducible unitary representation of the Heisenberg Lie algebra $\mathfrak{h}_3 = \mathbf{R} \oplus \mathbf{R}^* \oplus \mathbf{R}$ on the state space $\mathcal{H} = \mathcal{S}(\mathbf{R})$. This exponentiates to a representation of the Heisenberg Lie group H_3, which is unique up to unitary equivalence (by the Stone-von Neumann theorem). The symplectic group $Sp(2, \mathbf{R}) = SL(2, \mathbf{R})$ acts by automorphisms on the Heisenberg group, and this implies that the irreducible representation of H_3 provides a projective unitary representation of $SL(2, \mathbf{R})$. This is known by various names, for concision we'll refer to it as the Weil representation. For a detailed explanation of all this, including the generalization to any finite number of degrees of freedom, see [20].

One can replace the field $F = \mathbf{R}$ used here by more general local fields, for instance $F = \mathbf{Q}_p$. The Weil representation in all these cases becomes a true (not just projective) representation of a non-trivial double cover of the symplectic group, called the metaplectic group. In [16], Weil studied these representations and the ± 1-valued cocycle which determines the double cover. He also noted that, given a vector space V over F of dimension d with a quadratic form Q, one could replace the symplectic vector space $F \oplus F^*$ by the symplectic vector space $W = (F \oplus F^*) \otimes V$ and again construct a Weil representation, which now will be an irreducible projective representation of $Sp(2d, F)$. If one takes V to be a quaternion algebra $\left(\frac{a,b}{F}\right)$ and writes

$$W_{a,b} = (F \oplus F^*) \otimes \left(\frac{a,b}{F}\right)$$

then Weil showed that the Brauer class

$$\left[\left(\frac{a,b}{F}\right)\right] \in \mathbf{Z}/2\mathbf{Z}$$

determines whether the Weil representation constructed using $W_{a,b}$ is a true or projective (up to a factor ± 1) representation of the symplectic group.

The construction of the metaplectic extension of the symplectic group is a rather intricate story, even for the local field $\mathbf{R}$, see for example [10]. The extension can be constructed using the Maslov index, which takes values in the Witt group of a quadratic form. The metaplectic extension just depends on I^2/I^3, the piece of this given by two-torsion in the Brauer group.

It is rather striking that one gets in this way a generalization to the arithmetic context of a theory which for $F = \mathbf{R}$ is canonical quantization for the case of configuration space $\mathbf{H}$, which is just the case of quantization relevant to wave-functions on the usual four-dimensional space-time. The author is not aware of this being previously considered from any physical viewpoint, but it clearly deserves further study.

So far we have just been considering the local fields $\mathbf{Q}_p$ and $\mathbf{R}$, but one can use these to study the global field $\mathbf{Q}$. If one takes a, b invertible elements in $\mathbf{Q}$, one finds that the Hilbert symbols of the corresponding a, b for the various primes satisfy

$$\prod_p (a, b)_p = 1$$

where the product is over the finite primes (for which $(a, b)_p = (a, b)_{\mathbf{Q}_p}$) and the infinite prime (for which $(a, b)_\infty = (a, b)_{\mathbf{R}}$). This is known as the Hilbert reciprocity law, and from it one can easily derive Gauss's quadratic reciprocity law.

In [16] Weil shows that one can construct Heisenberg and metaplectic groups, as well as the Weil representation, not just for the local fields $F = \mathbf{Q}_p, \mathbf{R}$, but also for the adèle group $\mathbf{A}_{\mathbf{Q}}$ which puts all the local fields together. If one does this for $W_{a,b}$, replacing F by $\mathbf{A}_{\mathbf{Q}}$, one can show that this adèlic Weil

representation is a true (rather than projective) representation of the diagonal $Sp(W_{a,b}, \mathbf{Q})$ subgroup of the corresponding adelic symplectic group and that this fact is equivalent to Hilbert reciprocity.

For more about this, besides [16], see [8] and [5]. Howe showed that in general one can get a lot of mileage out of forming the symplectic vector space $W = U \otimes V$ for U a symplectic vector space and V an orthogonal vector space, taking the Weil representation for W, then restricting this to the subgroup $Sp(U) \times O(V)$. One gets in this way ("Howe duality") a matching between irreducible representations of $Sp(U)$ and of $O(V)$ and in the adèlic context a "theta correspondence" between automorphic forms. Jacquet and Langlands used the case considered above of $U = F \oplus F^*$ and V a quaternion algebra in their demonstration of an example of Langlands functoriality.

Witten in [17] and [18] (see also [1]) discussed a formulation of holomorphic conformal field theories such as the theory of a free fermion on a Riemann surface in terms of a function field (rather than number field) analog of the above. Instead of the Hilbert symbol, one has an analogous symbol due to Weil, and there's an analogous reciprocity law.

6.3 The Fargues-Fontaine curve and the twistor $\mathbf{P}^1$

The twistor $\mathbf{P}^1$ turns out to be the analog at the infinite prime of a more sophisticated arithmetic geometry version of a curve (called the "Fargues-Fontaine curve") which can be defined for each prime p. The Fargues-Fontaine curve provides a geometrical interpretation of some of the structure of p-adic Hodge theory. It has recently been used by Fargues and Scholze[4] to recast the arithmetic local Langlands conjecture in terms of a geometric Langlands conjecture on the curve.

Warning:*this section is about an analogy between two different things, but I only actually understand one of them. To limit my embarassment I'm just giving a comparison table. The reader should pay minimal attention to what is here, and instead consult the following sources explaining this material, written by people who understand both sides of the analogy.*

- Overview of the Fargues-Fontaine curve by Jacob Lurie:

    `https://www.math.ias.edu/~lurie/ffcurve/Lecture1-Overview.pdf`

- Lectures by Laurent Fargues at Salt Lake City

    `https://webusers.imj-prg.fr/~laurent.fargues/SaltLake.pdf`

    and Beijing

    `https://webusers.imj-prg.fr/~laurent.fargues/Course%20Shenxing.pdf`

    as well as seminars at Columbia

    `https://webusers.imj-prg.fr/~laurent.fargues/padic_Twistors.pdf`

    and Jussieu

- Peter Scholze's 2018 ICM talk on p-adic geometry [13]

  https://arxiv.org/abs/1712.03708

  See in particular section 6 and the top of Figure 3.

finite p	infinite p
$\mathbf{Q}_p$	$\mathbf{R}$
$\mathbf{C}_p$, completion of $\overline{\mathbf{Q}}_p$	$\mathbf{C}$
Fargues-Fontaine curve FF_p	$\mathbf{P}^1_{tw}$
pt. at ∞ given by $i : Spec\ \mathbf{C}_p \to FF_p$	$0, \infty \in \mathbf{CP}^1$, $\infty \in \mathbf{P}^1_{tw}$
$Gal(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ action on FF_p	$U(1)$ action on $\mathbf{P}^1_{tw}$
vector bundles on FF_p	vector bundles on $\mathbf{P}^1_{tw}$
vector bundles classified by fractions	vector bundles classified by half-integers
$Gal(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-equivariant vector bundles on FF_p	$U(1)$-equivariant vector bundles on $\mathbf{P}^1_{tw}$
p-adic Hodge structure	Hodge structure
Fontaine ring $B^+_{dR} = \widehat{\mathcal{O}}_{FF_p,\infty}$	Power series ring $\mathbf{C}[[\lambda]] = \widehat{\mathcal{O}}_{\mathbf{P}^1_{tw},\infty}$
$\widehat{\mathbf{Z}}$-cover of FF_p	$\mathbf{CP}^1$ is $\mathbf{Z}/2\mathbf{Z}$-cover of $\mathbf{P}^1_{tw}$

7 Speculations

The significance of the occurrence of the same twistor $\mathbf{P}^1$ as a fundamental description of a point in space-time physics and in number theory remains obscure. David Ben-Zvi has emphasized (see for instance here [2]) that one can take a four-dimensional topological quantum field theory point of view on number theory, in the context of well-known analogies relating number fields to 3-manifolds and primes to knots [11]. Kapustin-Witten [9] realize the geometric Langlands program duality in terms of electro-magnetic duality in such a 4d quantum field theory. In this quantum field theory, the self-duality equations play a central role, and the Penrose-Ward transform indicates that twistor theory is the geometric framework for understanding the significance of these equations.

It may very well be that the appearance of the twistor $\mathbf{P}^1$ in physics and in number theory is of no fundamental significance, but it also seems quite possible that it is pointing towards some insight relating the two subjects yet to be found.

8 Acknowledgements

Many thanks to Johan de Jong for his help with the material of these notes, to Peter Scholze for pointing out to me that the same twistor $\mathbf{P}^1$ of Euclidean

twistor theory was showing up as an analog of the Fargues-Fontaine curve at the infinite prime, and to David Ben-Zvi and Brian Conrad for helpful comments.

References

[1] A. Beilinson, B. Feigin, and B. Mazur. "Notes on Conformal Field Theory". available at `https://www.math.stonybrook.edu/~kirillov/manuscripts.html`. 1991.

[2] David Ben-Zvi. *Between electric-magnetic duality and the Langlands program*. notes by Jackson van Dyke available at `https://web.ma.utexas.edu/users/vandyke/notes/langlands_sp21/langlands.pdf`. 2021.

[3] Pierre Deligne. "Notes on Spinors". In: *Quantum Fields and Strings: A Course for Mathematicians*. AMS, 1999.

[4] Laurent Fargues and Peter Scholze. *Geometrization of the Local Langlands Correspondence*. 2021. arXiv: `2102.13459 [math.RT]`.

[5] Stephen Gelbart. "Examples of dual reductive pairs". In: *Automorphic Forms, Representations and L-functions*. Ed. by A. Borel and W. Casselman. Vol. 33, part I. Proceedings of Symposia in Pure Mathematics. AMS, 1979.

[6] N. J. Hitchin et al. "Hyper-Kåhler metrics and supersymmetry". In: *Communications in Mathematical Physics* 108.4 (1987), pp. 535 –589.

[7] Nigel Hitchin. "Hyperkähler manifolds". In: *Séminaire Bourbaki : volume 1991/92, exposés 745-759*. Astérisque 206. Société mathématique de France, 1992.

[8] Roger Howe. "θ-series and invariant theory". In: *Automorphic Forms, Representations and L-functions*. Ed. by A. Borel and W. Casselman. Vol. 33, part I. Proceedings of Symposia in Pure Mathematics. AMS, 1979.

[9] Anton Kapustin and Edward Witten. *Electric-Magnetic Duality And The Geometric Langlands Program*. 2007. arXiv: `hep-th/0604151 [hep-th]`.

[10] G. Lion and M. Vergne. *The Weil representation, Maslov index and Theta series*. Vol. 6. Progress in Mathematics. Birkhauser, 1980.

[11] Masanori Morishita. *Knots and Primes: An Introduction to Arithmetic Topology*. Springer-Verlag, 2012.

[12] R. Penrose. "The Nonlinear Graviton". In: *Gen. Rel. Grav.* 7 (1976), pp. 171–176.

[13] Peter Scholze. *p-adic geometry*. 2017. arXiv: `1712.03708 [math.AG]`.

[14] Carlos Simpson. *Mixed twistor structures*. 1997. arXiv: `alg-geom/9705006 [alg-geom]`.

[15]   Carlos Simpson. "Non-abelian Hodge Theory". In: *Proceedings of the International Congress of Mathematicians (Kyoto, 1990)*. Vol. 1. Math. Soc. Japan, 1990, pp. 747–756.

[16]   André Weil. "Sur certains groupes d'opérateurs unitaires". In: *Acta Mathematica* 111 (1964), pp. 143 –211.

[17]   Edward Witten. "Free fermions on an algebraic curve". In: *The Mathematical Heritage of Hermann Weyl*. Ed. by Jr. Raymond O. Wells. Vol. 48. Proceedings of Symposia in Pure Mathematics. AMS, 1988.

[18]   Edward Witten. "Quantum field theory, Grassmanians, and algebraic curves". In: *Comm. Math. Phys* 113 (1988), pp. 529–600.

[19]   Peter Woit. *Euclidean Twistor Unification*. 2021. arXiv: `2104.05099` `[hep-th]`.

[20]   Peter Woit. *Quantum Theory, Groups and Representations*. Springer, 2017.