# TOPOLOGICAL QUANTUM THEORIES AND REPRESENTATION THEORY

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ABSTRACT: We discuss the relationship between path integrals, geometric quantization and representation theory for a simple quantum theory whose Hilbert space is a group representation. The path integrals involved have interesting cohomological significance and can be evaluated in terms of fixed point formulas to give the Kirillov and Weyl character formulas. The relation to recent work of Witten on Chern-Simons gauge theory is also discussed.

## INTRODUCTION

In recent years certain quantum theories have been discovered that have an essentially geometrical or topological nature. These theories have deep connections to index theory and to other areas of active interest in geometry and topology. In this talk we will begin by discussing a simple class of such theories. The physical content of the simplest of these theories is a description of a quantum spin. The mathematical content is that of the representation theory of compact Lie groups. These rather simple quantum systems have a very rich geometrical structure and a proper understanding of this is essential for understanding both the quantization of spin and the more complicated topological quantum theories[1] that have excited recent interest. In the final part of this talk we will see to what extent the simple quantum mechanical systems discussed earlier shed light on Witten's[2] Chern-Simons quantum gauge theory.

The quantum theories that we will be considering are topological quantum theories in the sense that with appropriate choice of boundary conditions their partition functions are indices of elliptic operators. However we wish to correct the wide-spread belief that such quantum theories contain only topological information and no physical degrees of freedom. One of these theories is the supersymmetric quantum mechanics of a Dirac particle coupled to a background electromagnetic field. For a particular choice of boundary conditions all contributions to the partition function except those from the zero modes of the Dirac operator cancel. This does not change the fact that this is a non-trivial theory of great physical importance.

Even when one restricts one's attention to the zero modes of the Dirac operator



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one may find that they carry more structure than just a dimension. In particular they may transform under a group and it is this aspect that will interest us in this paper. The path integral quantization of a quantum spin has often been considered in the physics literature, for references see [3] and the paper [4]. Recently, Stone[5] has considered the quantization of spin from a point of view similar to ours.

## QUANTUM MECHANICS AND GROUP REPRESENTATION THEORY

Let us consider what is perhaps the simplest mathematical structure that deserves to be called a quantum theory. The Hilbert space $\mathcal{H} = V_R$ will be the finite dimensional complex vector space corresponding to a unitary representation R of a compact, connected Lie group G. A state of the quantum system will be a vector in $\mathcal{H}$ written

$$|\Psi(\tau)>$$

that depends on the parameter $\tau$, which will have the physical interpretation of time.

The simplest example that we will consider will be for G=SU(2), which has irreducible unitary representations of dimension $n+1$ for every non-negative integer n. Such a representation is said to have "spin" $\frac{n}{2}$ and will describe the dynamics of the spin degrees of freedom of a particle of that spin coupled to a time-varying magnetic field.

The Hamiltonian for this system will be a time dependent Lie algebra element $H(\tau)$ describing the magnetic field acting on the particle and the dynamics of the theory is described by the Schrödinger equation

$$\frac{d}{d\tau}|\Psi(\tau)>= iH(\tau)|\Psi(\tau)>$$

This equation describes the unitary time evolution of a vector in $\mathcal{H}$ and given an initial condition $|\Psi(0)>$ its solution can be written

$$|\Psi(\tau)>= U(\tau)|\Psi(0)>$$

where $U(\tau)$ is, for each value of $\tau$, an element of G acting in the representation $\mathcal{H}$. $U(\tau)$ is often written as the "path-ordered exponential of $H(\tau)$"

$$U(\tau) = Pe^{i\int_0^\tau H(s)ds}$$

If we consider this theory on a fixed time interval T we can define the "partition function" of the theory to be

$$Z = Tr_R(U(T))$$

Z ia a character of G and is the simplest physically relevant quantity in the theory since it is independent of the choice of basis of $\mathcal{H}$.

Solving the quantum theory just requires finding $U(\tau)$, we will try and do this by expressing $U(\tau)$ as a Feynman path integral. This theory is so simple that path integral techniques are clearly not the most efficient way of solving the theory, but the apparatus we will develop generalizes easily to more interesting theories where the Schrödinger formulation is not very useful. Furthermore this is the simplest system in which the notion of geometric quantization works nicely and we will thus be able to explore the relationship between path integral and geometric quantization.

The classical phase space corresponding to this quantum theory will be the orbit under the action of G of the ray corresponding to a highest weight vector in the complex projective space (V). The highest weight that defines V will be denoted $\lambda_V$. The state vectors corresponding to rays in this orbit are often called "coherent states". Any element of V can be written as a linear combination of these state vectors. This orbit in $\mathbf{P}(V)$ is diffeomorphic to $G/G_{\lambda_V}$ where $G_{\lambda_V}$ is the subgroup of G that acts on the highest weight vector by a phase. For a "generic" representation this subgroup will be the maximal torus, denoted T. In what follow we will refer to these orbits as G/T, although for certain representations V what we actually mean is $G/G_{\lambda_V}$.

These orbit spaces, which also are often called flag manifolds or co-adjoint orbits (corresponding to two alternate ways of defining them) are Kähler manifolds. Even better, they are projective algebraic varieties with an explicitly given embedding in $\mathbf{P}(V)$. The tautological line bundle L over $\mathbf{P}(V)$ is the complex line bundle whose fiber above a point p in $\mathbf{P}(V)$ is the corresponding complex line. Its restriction to G/T will be denoted $L_{\lambda_V}$, it is a holomorphic line bundle and will be of great importance in what follows.

## COHERENT STATE PATH INTEGRALS

The most common Feynman path integral is an integral over paths in a configuration space X and is used to construct a quantum theory with Hilbert space $L^2(X)$. This corresponds to a theory with classical mechanical phase space $T^*X$. In the quantum theory that interests us, the classical phase space M=G/T does not have the structure of a cotangent bundle so the standard sort of path integral does not apply (however see [6] for a discussion of the path integral quantization of spin using a real polarization as in the $T^*X$ case). Various efforts have been made to construct path integrals as integrals over the phase space, and in this case such path integrals go under the name of coherent state path integrals.

Defining a path integral over paths in M seems bound to lead to trouble with the Heisenberg uncertainty principle since one is attempting to specify at each value of time values of both conjugate variables. In this section we will review the standard formalism of coherent state path integrals and see what problems arise. We will deal with the simplest coherent state path integral, that corresponding to the case $H(\tau) = 0$.

The standard treatment of the coherent state path integral is based upon the so-called "resolution of the identity" which expresses the identity operator on the representation space V as

$$1 = \frac{1}{\Gamma}\int_G |g \cdot z_0 >< g \cdot z_0|$$

where $|z_0>$ is a highest weight vector, $\int_G$ denotes Haar measure on G and

$$\Gamma = \int_G |< g \cdot z_0|z_0 >|^2$$

is a normalization constant.

This can also be thought of as an integral over M of projection operators

$$1 = \frac{1}{\Gamma'}\int_M |z >< z|$$

Here $z$ labels points in M, there is a phase ambiguity in the definition of $|z>$ but this cancels the phase ambiguity in $<z|$. $\Gamma'$ is a normalization constant, and the integral over M is defined using the symplectic measure on M.

This identity is used to express the inner product between two highest weight vectors $|z'>$ and $|z''>$ in V as

$$< z''|\prod_{\tau}\frac{1}{\Gamma}(\int_G |g_\tau \cdot z_0 >< g_\tau \cdot z_0|)|z'>$$

or

$$< z''|\prod_{\tau}\frac{1}{\Gamma'}(\int_M |z_\tau >< z_\tau|)|z'>$$

where $\tau$ is a variable parametrizing the projection operators which takes a finite number of values. One can for instance define $\tau$ to be the finite set

$$\tau = \{0, \Delta\tau, 2\Delta\tau, 3\Delta\tau, \ldots, 1\}$$

where

$$\Delta = 1/N$$

for some integer N. The coherent state path integral representation of this inner product is formally the limit as $N \to \infty$ of this expression.

Clearly this limit exists, since it is independent of N anyway. However the standard interpretation of this product as an integral over paths in M joining the ray defining $|z'>$ to that defining $|z''>$ is problematic. There is no sense in which

$$z_\tau \approx z_{\tau+\Delta\tau}$$

as $\Delta\tau \to 0$ and yet this continuity assumption is often invoked in manipulations of these integrals.

Let us however proceed under the assumption that we are dealing with continuous paths and see how the standard formalism is developed. Also, assume that we are dealing with a representation V such that the orbit of a highest weight vector is the full flag manifold G/T rather than something smaller. Then the "naive" limit as $\Delta\tau \to 0$ of the path integral expression for the inner product between $|z'>=|g_0 \cdot z_0>$ and $|z''>=|g_1 \cdot z_0>$ will be of the form

$$K(g_1, g_0) =< g_1 \cdot z_0|g_0 \cdot z_0 >= \frac{1}{\Gamma''}\int_{g_0 \to g_1} e^{\int_0^1 d\tau(\omega + i\theta(\dot{g}))}$$

This is meant to be interpreted as follows. One is integrating over all paths in G from $g_0$ to $g_1$ but what is relevant is their projection onto the orbit of the highest weight vector $|z_0>$. $\theta$ is a left-invariant 1-form on G, it is the canonical connection 1-form for the tautological line bundle L over the orbit in $\mathbf{P}(V)$. $\omega = \frac{d\theta}{2\pi}$ is the lift to G of the standard symplectic 2-form on $\mathbf{P}(V)$.

The first factor in the integrand is

$$e^{\int_0^1 \omega} = \prod_{\tau} e^{\omega} = \prod_{\tau}\frac{(\omega)^n}{n!}$$

it is meant to be formally interpreted as an infinite product of symplectic volume forms, one for each value of $\tau$, providing a volume form for the loop space. The second factor is

$$e^{i\int_0^1 \theta(\dot{g})}$$

It is just the phase corresponding to parallel transport with respect to the connection $\theta$.

The simplest quantity that one would like to calculate is the partition function

$$Z = Tr_V(1) = \int_G K(g,g) = \text{dim}V = \int_{\Omega G} e^{\oint d\tau[\omega + i\theta(\dot{g}(\tau))]}$$

The integrand is conceptually quite simple, especially when considered in terms of loops on M. Then one is just integrating the holonomy around a loop against the (unfortunately still ill-defined) measure on $\Omega M$ that is just the infinite product of the symplectic measures for each $\tau$. The holonomy is

$$hol(C) = e^{2\pi i \int_{S:\partial S = C} \omega}$$

where the exponent here is $2\pi$ times the "action", which is well-defined up to an integer ambiguity.

This sort of path integral has several related problems that prevent one from giving it any well-defined meaning. The first is that the illegitimate assumption of the continuity of paths prevents one from keeping track of the normalization of the integral. In Feynman's configuration space version of path integration one has a term

$$e^{-\frac{1}{2}\int |\dot{x}|^2}$$

in the integrand which damps out discontinuous paths. This sort of term is absent here.

A second problem is in how one has taken the trace. One can think of this path integral as a normal Feynman-type integral over a configuration space G (without the necessary damping term) with an integrand that acts as a projection from the full Hilbert space of complex-valued functions $L^2(G)$ to a subspace of equivariant functions, those that are sections of the line bundle $L_{\lambda_V}$. One is thus evaluating the trace on the full infinite dimensional induced representation of G on $\Gamma(L_{\lambda_V})$. The representation whose trace we wish to calculate is a finite dimensional subspace of $\Gamma(L_{\lambda_V})$. Thus one is evaluating the wrong path integral and one has to figure out some way of removing the unwanted representations from the trace. This is generally done by trying to push the unwanted representations to infinite energy, for instance by adding a factor

$$e^{-\alpha\int |\dot{x}|^2}$$

and taking the limit $\alpha \to \infty$.

We will see that index theory provides a natural way of cancelling contributions of all but the correct finite dimensional subspace of $\Gamma(L_{\lambda_V})$. Thus we will be looking for a supersymmetric quantum mechanical model with a fermionic path integral that will give the necessary cancellations.

## GEOMETRIC QUANTIZATION, BOREL-WEIL-BOTT AND INDEX THEORY

Geometric quantization is a general program for producing a quantum theory associated to a given classical system. The quantization of the flag manifold G/T was one of the inspirations for the geometric quantization program[7] and not surprisingly this is the case where it works most simply. In this case geometric quantization essentially coincides with the Borel-Weil-Bott theorem[8,9]. This theorem states that the representation of G with highest weight $\lambda$ can be identified with the space of holomorphic sections of a holomorphic line bundle $L_\lambda$. Recalling that G is a principal T bundle over G/T and that the weight $\lambda$ gives a representation of T on $\mathbf{C}$, $L_\lambda$ is the associated line bundle over G/T given by this representation. Note that the condition of a section being holomorphic picks out a finite dimensional subspace $\Gamma_{hol}(L_\lambda)$ of the infinite dimensional space of sections $\Gamma(L_\lambda)$.

We wish to work here not in the very general context of the theory of geometric quantization, but in the context of index theory which will turn out to be an equivalent point of view[10,11]. What the Borel-Weil-Bott theorem does is construct a map

$$R(T) \to R(G)$$

from the representation ring of T to the representation ring of G. The highest weight $\lambda_V$ gives a representation of T and thus an element of R(T) and the Borel-Weil-Bott theorem gives a construction of $V \in R(G)$.

From the point of view of index theory the natural framework for our discussion is that of equivariant K-theory. The definition of the cohomology classes $H^*(M)$ of a manifold is well known to physicists. When a group G acts on M one can define equivariant cohomology classes $H^*_G(M)$, if the action of G is free these reduce to $H^*(M/G)$. The group K(M) is defined in terms of equivalence classes of vector bundles over M and has similar properties to a cohomology group. When G acts on M one can define $K_G(M)$, the equivariant K-theory of M, in terms of equivariant vector bundles on M (see for instance [12]).

In equivariant K-theory we have

$$K_G(pt.) = R(G)$$

and

$$K_T(pt.) = K_G(G/T) = R(T)$$

Just as for a map

$$\pi : M \to pt.$$

there is a push-forward or integration map $\pi_*$ in cohomology or in equivariant cohomology, there is an integration map

$$\pi_! : K_G(M) \to K_G(pt.) = R(G)$$

in equivariant K-theory. If we take M=G/T, this is precisely the map that appears in the Borel-Weil-Bott version of representation theory. It is best described concretely in terms of the index of the Dirac operator on G/T.

If E is a vector bundle representing the class $\alpha_E$ in K(M), then

$$\pi_!(\alpha_E) = index D_E = \dim \ker D_E - \dim \operatorname{coker} D_E$$

where $D_E$ is the Dirac operator on spinors twisted by E. In the equivariant case $\alpha_E \in K_G(M)$ the kernel and cokernel of the Dirac operator are representation spaces for G and their difference is in R(G). Then

$$\pi_!(\alpha) = [\ker D_\alpha] - [\operatorname{coker} D_\alpha] \in R(G)$$

We want to understand the Borel-Weil-Bott construction of the representation V in these K-theoretic terms. We have seen that V is isomorphic to the space $\Gamma_{hol}(L_{\lambda_V})$ of holomorphic sections of the line bundle $L_{\lambda_V}$. In other words

$$V = H^0(G/T; L_{\lambda_V})$$

One can show that for $q > 0$

$$H^{0,q}(G/T; L_{\lambda_V}) = 0$$

Together these facts imply that for the Dolbeault operator

$$\bar{\partial} + \bar{\partial}^* : \Gamma(L_{\lambda_V} \otimes \Lambda^{0,*}) \to \Gamma(L_{\lambda_V} \otimes \Lambda^{0,*})$$

we have

$$V = index(\bar{\partial} + \bar{\partial}^*) = ker(\bar{\partial} + \bar{\partial}^*) - coker(\bar{\partial} + \bar{\partial}^*)$$

Since G/T is a Kähler manifold, this Dolbeault operator operating on $\Gamma(L_{\lambda_V} \otimes \Lambda^{0,*})$ is identical to the Dirac operator acting on $\Gamma(L_{\lambda_V} \otimes S \otimes (\Lambda^{0,n})^{1/2})$ where $S$ is the spinor bundle and $(\Lambda^{0,n})^{1/2}$ is a square root of the canonical bundle. On G/T, $(\Lambda^{0,n})^{1/2} = L_\delta$ (where $\delta$ is half the sum of the positive roots) and the Dirac operator acts as

$$D_{(\lambda_V+\delta)} : \Gamma(S \otimes L_{(\lambda_V+\delta)}) \to \Gamma(S \otimes L_{\lambda_V+\delta})$$

Finally we see that we have the isomorphism

$$V = index(D_{\lambda_V+\delta})$$

What we have done is shown that the Borel-Weil-Bott construction of a representation of G from a highest weight representation of T is just the integration map

$$\pi_! : K_G(G/T) = R(T) \to K_G(pt.) = R(G)$$

for the map

$$\pi : G/T \to pt.$$

An important example of how this works out is that of V=**1**, the trivial representation. Here the Atiyah-Singer index theorem tells us

$$\begin{aligned} dimV = Tr_1(1) &= index(D_\delta) \\ &= \hat{A} \wedge ch(L_\delta)[G/T] \\ &= \tau[G/T] = 1 \end{aligned}$$

where $\tau$ is the Todd class. Note that here even the trivial representation involves a non-trivial calculation, and this will be reflected in our path integral calculations by a non-trivial path integral that corresponds to this representation.

We have seen that for a phase space M=G/T quantization is equivalent to integration in K-theory. It turns out that this is also true in other less trivial contexts. While the geometric quantization program has tried with partial success to provide a geometric description of quantization for general symplectic manifolds, thinking of quantization as integration in K-theory gives a conceptually simpler picture when it is applicable.

As another example of this general principle consider the quantization of the harmonic oscillator with phase space M=**C**. This is different than the M=G/T case since **C** is not compact. There is a well-known description of the Hilbert space of the harmonic oscillator in terms of holomorphic sections of the trivial bundle over **C**. The group $\mathbf{C}^*$ of non-zero complex numbers acts on this line bundle and the infinite dimensional Hilbert space decomposes as a sum of one-dimensional representations of

$\mathbf{C}^*$. Thus the Hilbert space of the harmonic oscillator can be thought of as a sum of finite-dimensional spaces of zero modes of a Dirac operator. Note that the $\frac{1}{2}$ that occurs as the ground state energy of the harmonic oscilator has the same origin as the term $\delta$, half the sum of the positive roots, in the G/T case.

The harmonic oscillator thus corresponds to the quantization of $\mathbf{C}^*$, the complexification of U(1). One can also consider the quantization of GL(n,**C**), the complexification of U(n). Here the Hilbert space is $L^2(U(n))$ which could be thought of as the space of holomorphic sections of a certain trivial line bundle over GL(n,**C**). By the Peter-Weyl theorem this Hilbert space contains all the irreducible unitary representations of U(n). In our earlier discussion we restricted attention to one irreducible representation V by looking at not all of $L^2(G)$ but at $\Gamma(L_{\lambda_V})$, which is a subspace of this space satisfying a certain equivariance property. The passage from the complexification of G to G/T is an example of Marsden-Weinstein[13] reduction in symplectic geometry.

We will see later that the principle of quantization as integration in K-theory also seems to apply in the very non-trivial cases of Wess-Zumino-Witten models in conformal field theory and in Witten's Chern-Simons gauge theory. Undoubtedly there are other examples where this principle is valid, the full range of its validity has not yet been investigated.

## INDEX THEORY AND SUPERSYMMETRIC QUANTUM MECHANICS

The Atiyah-Singer index theorem tells us that the index of the Dirac operator on M coupled to a vector bundle E is given by $\pi_!(\alpha_E)$. This push-forward map was defined by Atiyah and Singer in terms of embedding M in $S^{2N}$ for some large N and using the Thom isomorphism for the normal bundle of M to construct an element in $K(S^{2N})$ from the class $\alpha_E \in K(M)$. Bott periodicity implies that $K(S^{2N}) = \mathbf{Z}$ and this integer will be the index.

Remarkably, it turns out that there is an alternate formulation of this topological index map in terms of a certain supersymmetrical quantum mechanical system. The essential idea is that at least formally one can identify $K(M)$ with $K_{S^1}(LM)$ the equivariant K-theory of the free loop space LM of parametrized loops in M with respect to the circle action given by rotation of the loop. The calculation of the index is reformulated in terms of an integration map in the $S^1$ equivariant cohomology of the loop space. This abstract concept when carried out using differential forms on the loop space is equivalent to the calculation of the partition function in a simple supersymmetrical quantum mechanical system [14].

Following Atiyah's exposition of an idea originally due to Witten[15], the index of the Dirac operator can be expressed as an integral over the loop space $\Omega M$ of the equivariantly closed form

$$\mu = e^{\frac{1}{2}(d-i_X)\alpha}$$

Here X is the vector field on LM that generates rotations around the loops, $\alpha$ is the 1-form dual to this vector field (we are assuming that M has a metric and using the induced metric on LM), and $i_X$ is contraction with the vector field $X$.

For a finite 2n-dimensional manifold $\tilde{M}$ with an $S^1$ action generated by the vector field $X$ there is a localization formula[16,17] which expresses the integral of an equivariantly closed form in terms of an integral over the submanifold F left fixed by the

$S^1$ action. For a form $\mu$ on $\tilde{M}$ such that $(d - i_X)\mu = 0$,

$$\int_{\tilde{M}} \mu = \int_F \frac{\mu|_F}{\chi(X,N)}$$

In this formula $\chi(X,N)$ is an equivariant Euler characteristic defined as

$$\chi(X,N) = det(J_X - \frac{\Omega}{2\pi i})$$

where N is the normal bundle to F in $\tilde{M}$, $J_X \in \Gamma(EndN)$ is the infinitesimal action of X in N and $\Omega \in \Lambda^2 T^* \tilde{M}_0 \otimes EndN$ is the curvature of an $S^1$ invariant linear connection on N.

For the special case of $\tilde{M}$ a symplectic manifold and

$$\mu = e^{-H} \frac{\sigma^n}{n!}$$

this formula is due to Duistermaat and Heckman[18]. In this context it states that the stationary phase approximation for $\int_{\tilde{M}} \mu$ is exact.

If we assume that a suitably regularized version of this formula applies to our infinite dimensional case of $\tilde{M} = \Omega M$, Atiyah has shown that one finds

$$\text{index } D = \int_{\Omega M} \mu = \hat{A}(M)$$

giving the standard result for the index of the Dirac operator. In this case the fixed point set is the manifold M itself, and Fourier expansion of vectors in the tangent space to a point loop gives the normal bundle as the infinite direct sum

$$N = (T \otimes \mathbf{C})_1 \oplus (T \otimes \mathbf{C})_2 \oplus \ldots$$

where $(T \otimes \mathbf{C})_p$ is the complexified tangent bundle of M with $S^1$ acting with rotation number p.

The integrand that occurs here is identical with the integrand in the path integral form of the simple supersymmetric quantum mechanics system where the Lagrangian is written

$$L = \int d\tau \frac{1}{2}(|\frac{dx}{d\tau}|^2 - \psi^i D_\tau \psi^i)$$

In SSQM the partition function is evaluated as

$$Z = \int [dx][d\psi] e^{-L}$$

which can be understood as a way of rewriting the integral expressed through differential forms on loop space in terms of the Riemannian volume form on loop space and a fermionic integration. In the physics literature[14] this integral is evaluated in the stationary phase approximation, which we have seen is exact in this situation.

We will actually need the generalization of this to the case of the Dirac operator coupled to a line bundle L. If $A$ is a connection 1-form for this line bundle then one can formally define a 1-form on LM by

$$\alpha_1 = 2i \oint d\tau A(x(\tau))$$

(the way we have written the connection assumes a choice of section of L, but the final result will be independent of this choice).

The coupling to L simply has the effect of adding $\alpha_1$ to the 1-form $\alpha$ and the integrand that gives the index in this case will be

$$\mu = e^{\frac{1}{2}(d - i_X)(\alpha + \alpha_1)}$$

Of the two new terms in the exponent, one just gives the holonomy around the loop, the other involves the curvature and is the standard term familiar in QED that couples the spin to the magnetic field. Formally applying the localization formula gives the standard cohomological form of the index theorem. At the fixed point set only the curvature term survives and the equivariant Euler characteristic of the normal bundle is unchanged from the untwisted case.

Given the above, the main point that we would like to make is quite simple. Instead of the standard coherent state path integral and its attendant problems, the appropriate path integral quantization of G/T is given by the SSQM path integral for the index of the Dirac operator twisted by a certain line bundle. Notice that this path integral contains the holonomy term that appears in the coherent state path integral, but it also contains a term

$$e^{-\frac{1}{2}\int |\dot{x}|^2}$$

as well as fermionic variables. The effect of the fermionic variables is to provide for a cancellation of the contribution to the partition function of all states except those corresponding to the zero modes of the Dirac operator. Thus this path integral resolves the problem with the coherent state path integral that we noted before.

In order to avoid problems about exactly how to normalize these integrals (equivalently, how to define the equivariant Euler characteristic of the normal bundle to the point loops in LM), we can compute ratios of path integrals, taking the ratio of the path integral for the representation we want to study with that for the trivial representation. Thus we get the formula for the dimension of a representation

$$dimV = tr(1) = \frac{\int_{\mathcal{O}(\lambda_V+\delta)} e^{\omega}}{\int_{\mathcal{O}(\delta)} e^{\omega'}}$$

Here $O(\lambda_V + \delta)$ and $\mathcal{O}(\delta)$ are the flag manifolds determined by the highest weight $\lambda_V$ and the weight $\delta$, and $\omega$ and $\omega'$ are the standard symplectic 2-forms on the two orbits. This orbital integral formula for the dimension of a representation is well known

## CHARACTER FORMULAS

We have so far just been discussing the formula for the dimension of the representation, which is an integer index. The equivariant index theorem gives the character of the representation, and our integration formula in this case is a trace formula (see [19] for further discussion of the relation between index theory and the Kirillov formula). The modification that needs to be made is simply that of changing the vector field X on L(G/T) by adding a constant vector field proportional to the vector field on G/T that corresponds to the action of the group element g whose trace we wish to calculate. One gets the formula

$$tr(e^X) = \frac{\int_{\mathcal{O}(\lambda_V+\delta)} e^{if(X)+\omega}}{\int_{\mathcal{O}(\delta)} e^{if(X)+\omega'}}$$

Here $f(X)$ is the moment map corresponding to the action of $e^X$ on G/T. Further applying the fixed-point localization formula to the denominator gives

$$tr(e^X) = det^{-\frac{1}{2}}(\frac{e^{ad\frac{X}{2}} - e^{-ad\frac{X}{2}}}{adX}) \int_{\mathcal{O}(\lambda_V+\delta)} e^{if(X)+\frac{\omega}{2\pi}}$$

which is the Kirillov character formula. Yet another application of the fixed point formula to the numerator gives a version of the Weyl character formula.

## LOOP GROUPS

The representation theory for positive energy unitary representations of the loop group LG can be developed in much the same way as the Borel-Weil-Bott representation theory for G[20]. Here LG/T is an infinite dimensional Kähler manifold that plays the role of G/T in the finite-dimensional case. Actually there is a fibration $LG/T \to LG/G$ of Kähler manifolds with fiber G/T. Restricting attention to a fiber gives back the finite-dimensional picture.

The corresponding quantum theory here is the Wess-Zumino-Witten[21] model of conformal field theory which has received much attention from physicists in recent years. This theory is the simplest example of a field theoretical model that can be most simply understood in terms of the equivariant K-theory framework we have developed.

## WITTEN'S CHERN-SIMONS THEORY

Last summer, in a striking paper[2], Witten defined new invariants for links in 3-manifolds using a quantum gauge field theory with action given by the Chern-Simons functional. Witten writes his invariants in terms of a functional integral as

$$Z(C, R, k, N) = \int [dA] W_R(C) e^{2\pi i CS[A]}$$

where C is a link in a 3-manifold $M^3$, k and N are integers, A is a connection on a the trivial principal SU(N) bundle over $M^3$, $[dA]$ is the standard physicist's notion of a formal measure on the space of such connections and $W_R(C)$ is the trace of the holonomy around the curve C with respect to the connection A in the SU(N) representation R. CS[A] the Chern-Simons functional of the connection A.

The functionals $W_R(C)$ are well-known to physicists as Wilson loops, they are exactly the sort of objects that appeared as the partition functions for the simple quantum system of the first part of this paper. Thus we have seen that there is a supersymmetric quantum mechanics path integral expression for these quantities.

Restricting attention to the case $C = \emptyset$, we get invariants of the 3-manifold itself and the functional integral over gauge fields Witten uses is very much analogous to the coherent state path integral we discussed at the beginning of this talk. It has similar problems and Witten performs most of his calculations using the associated geometric quantization of the theory rather than the functional integral.

The analogy we wish to point out is most clearly seen if we specialize to the case of a three manifold of the form $M^3 = \Sigma \times S^1$, where $\Sigma$ is a Riemann surface. The space of connections $\mathcal{A}$ on a principal G bundle over $\Sigma$ is an infinite dimensional symplectic manifold, given a complex structure on $\Sigma$ it is Kähler[22]. The group $\mathcal{G}$ of gauge transformations acts on $\mathcal{A}$ and symplectic reduction with respect to this symmetry gives as reduced phase space the moduli space $\mathcal{M}$ of flat connections on $\Sigma$. This moduli space is again a Kähler manifold and there is a holomorphic line bundle L over it whose first Chern class is the Kähler 2-form.

A loop in $\mathcal{M}$ corresponds to a connection on $M^3$ and the Chern-Simons functional of this connection is the holonomy around the loop in the line bundle L. Witten's path integral for this theory is just an integral over loops in $\mathcal{M}$ weighted by the holonomy, exactly as in the G/T coherent state path integral but with G/T replaced by $\mathcal{M}$. Witten points out that his invariant in this case is the dimension of the space of holomorphic sections of the line bundle $L^k$.

Thus Witten's invariant can be thought of in this case as an integration in K-theory, in particular

$$Z(k,N) = \pi_!(L^k)$$

where

$$\pi : \mathcal{M} \rightarrow pt.$$

and it should have an expression as a supersymmetric quantum field theory.

The functional integral that Witten writes down suffers from the same problems as the coherent state path integral for G/T. It is a trace over all sections of $L^k$, not just the holomorphic ones. In a recent preprint[23], Ramadas, Singer and Weitsman deal with this problem by inserting a term

$$e^{-T\int |\frac{dA}{dt}|^2}$$

into the functional integral and taking the limit $T \rightarrow \infty$. It would seem to be preferable to reformulate Witten's functional integral with fermionic variables that would cancel all but the holomorphic sections from the partition function.

Further evidence for the desirability of an index-theory reformulation of Witten's functional integral lies in the problems associated with framings. Witten finds that to get a well defined semi-classical approximation he must add a term involving $\eta$ invariants to his Chern-Simons action. The effect of this term is to change $k \rightarrow k+N$ in the results of his calculations. An index theory reformulation of Witten's invariant leads to this in a natural way since N plays the same role in this case as $\delta$ (half the sum of the positive roots) plays in the G/T case. The analogy can most clearly be seen by working through the connection between Witten's invariants and the Wess-Zumino-Witten quantum theory version of loop group representation theory. In the loop group case one finds that one should think of N as half the first Chern class of the tangent bundle of LG/G, just as $\delta$ is half the first Chern class of the tangent bundle of G/T.

## CONCLUSIONS

In this talk we have tried to explain a new conceptual approach to the problem of quantization through its application to a simple quantum system. This approach seems also the best way to understand the various topological quantum theories that have excited recent interest. These topological quantum theories contain a great deal of structure of physical as well as of mathematical interest. A more detailed exposition of these ideas and their applications is in preparation.

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