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SUPERSYMMETRIC QUANTUM MECHANICS, SPINORS AND THE STANDARD MODEL

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The quantization of the simplest supersymmetric quantum mechanical theory of a free fermion on a riemannian manifold requires the introduction of a complex structure on the tangent space. In 4 dimensions, the subgroup of the group of frame rotations that preserves the complex structure is SU(2) × U(1), and it is argued that this symmetry can be consistently interpreted to be an internal gauge symmetry for the analytically continued theory in Minkowski space. The states of the theory carry the quantum numbers of a generation of leptons in the Weinberg-Salam model. Examination of the geometry of spinors in four dimensions also provides a natural SU(3) symmetry and a very simple construction of a multiplet with the standard model quantum numbers.

1. Introduction

The anticommuting *c*-number quantum mechanics [1–4] with world-line supersymmetry that describes free fermions on a riemannian manifold M (also known as the $N = 1$ supersymmetric sigma model in $0 + 1$ dimensions with target space M) has been used recently to prove the Atiyah-Singer index theorem [5, 6]. In this application, the index of the Dirac operator appears as the partition function of the theory when periodic boundary conditions are imposed on the fermionic variables. Witten [7] has used this theory to show that the existence of a spin structure on M is equivalent to the orientability of the loop space of M. The connection between these two applications of the theory has been discussed by Atiyah [8]. This theory can be extended to provide a path integral formalism for describing fermions coupled to gauge fields and in principle can thus reproduce the predictions of the corresponding fermion field theory.

Since the canonical conjugates of the fermionic variables are the variables themselves, the quantization of the theory requires the introduction of a complex structure which splits the fermionic variables into a set of variables half as big and their canonical conjugates. The fermionic variables carry vector indices and thus transform under the gauge group SO(4) of local frame rotations. A U(2) subgroup

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of SO(4) preserves the complex structure, and we will argue that this can consistently be interpreted to be the internal symmetry group of the Weinberg-Salam model of leptons. Consideration of the geometry of spinors in 4 dimensions provides a natural SU(3) symmetry and thus one has the full gauge group of the standard model. Furthermore, we will see that there is a very simple geometrical construction which gives states transforming under these symmetries with the standard model quantum numbers.

These arguments provide answers to the questions "Why $SU(3) \times SU(2)_L \times U(1)$?" and "Why the standard model quantum numbers?" of a very different nature than those provided by other unification schemes. Such schemes typically involve assuming the existence of either a large compact group of symmetries containing the standard model symmetries as a subgroup (grand unification), or extra space-time dimensions of extremely small size (Kaluza-Klein theories). These models all involve a dramatic increase in the number of elementary degrees of freedom beyond the observed ones of the standard model, and they provide no explanation of why one Lie group rather than another is to be chosen, or why a certain number of compact dimensions should exist. Recently popular superstring models combine aspects of both Kaluza-Klein and grand unified theories. They give numerological explanations for a ten dimensional space-time and for the symmetry group being for instance either SO(32) or $E_8 \times E_8$, but these models now provide no way of predicting how many dimensions will compactify and in what manner, or what the symmetry group of the resulting low energy theory will be.

Since it was the first to be developed, the field theoretical approach to describing fermion dynamics has historically dominated the study of fundamental interactions. The path integral approach, while it has recently become widely used in rigorous work on scalar field theories, has not been fully developed in its application to fermions. One aim of this paper is to encourage further study of fermionic path integrals by pointing out that it is quite possible that a deeper understanding of the subject may lead to a unified theory of fundamental interactions. One is beginning with all the correct symmetries and they arise from the study of the frame bundle which makes not unreasonable the hope that gravity can also be quantized within this context. Furthermore the ideas involved are both relatively simple and at the core of modern geometry and should thus lead to fruitful interaction between mathematicians and physicists.

This paper is organized as follows. It begins with a review of supersymmetric quantum mechanics of free particles on a background manifold M. The symmetries of this theory and how they are affected by quantization is then described. The transformation properties of the states under these symmetries are then calculated and one sees that they behave like a generation of leptons of the standard model. Sects. 2 and 3 of this paper review the geometry of spinors in four dimensions, a subject which is not as well known among physicists as it should be. The relationship between spinors in euclidean and Minkowski space is explored in some detail.

Sect. 4 expresses the states of the first section in terms of the spinor geometry and then shows that states with the quark quantum numbers naturally appear. Sect. 5 makes a few comments about the Higgs field in this situation and the concluding section summarizes the argument of this paper.

2. Supersymmetric quantum mechanics

By supersymmetric quantum mechanics, we will mean the theory with lagrangian

$$L = \int \mathrm{d}r \, \tfrac{1}{2}\dot{x}^{\mu}\dot{x}_{\mu} + \tfrac{1}{2} i \Psi^{a} D_{\tau} \Psi_{a} ,$$

where

$$D_{\tau}\Psi^{a} = \frac{\mathrm{d}}{\mathrm{d}\tau}\Psi^{a} + \dot{x}^{\mu}\omega^{a}_{b,\mu}\Psi^{b} .$$

This theory will be defined with respect to a riemannian metric, assuming that analytic continuation to a Minkowski signature space takes place at the end of a calculation. The Ψ^{a} are anticommuting c-number functions of τ which transform under the group SO(4) of local frame rotations. Greek indices refer to local coordinates, latin indices to coordinates with respect to local frames (vierbeins). $\omega^{a}_{b,\mu}$ is usually taken to be the Levi-Civita connection determined by the metric, but we will leave it as an arbitrary SO(4) connection. The discussion of the quantization of this model normally begins (and ends) with the remark that the canonical anticommutation relations for the Ψ^{a}'s are

$$\{\Psi^{a}, \Psi^{b}\} = \delta^{ab}$$

and thus the Ψ^{a}'s satisfy the same relations as generators of the Clifford algebra. The Clifford algebra has a representation on spinors, among others, and it is assumed that the states of the theory will thus be spinors.

Within this sort of formalism one can also introduce anti-commuting c-numbers coupled to arbitrary gauge fields, and then use the rules of path integration for anticommuting variables to derive path integral expressions for the propagator and the closed loop amplitude in a background gauge field. This should give results equivalent to a proper-time formulation of the fermion field theory in the loop expansion. Functional integration over the gauge fields then gives the full interacting theory of fermions and gauge fields. Some of the details of this formalism have been worked out by Rajeev [9] for the case of U(1) gauge theory.

Understanding the nature of the states of the theory requires dealing with a problem not encountered in the ordinary quantum mechanics. The lagrangian is first order in the time derivative and as a result the canonical momentum to Ψ^{a} is proportional to Ψ^a itself. Thus the space of the Ψ^a's should be thought of as a phase space and the configuration space on which the states will be functions should be only half as big. This is normally accomplished by writing

$$\eta^1 = \Psi^1 + i\Psi^2,$$

$$\eta^2 = \Psi^3 + i\Psi^4,$$

and then demanding that the states be functions only of the $\bar{\eta}$'s and not of the η's. Since the $\bar{\eta}$'s are anticommuting variables a basis for the space of functions on them will be

$$f_0(x),$$

$$f_1(x)\bar{\eta}^1, \qquad f_2(x)\bar{\eta}^2,$$

$$f_{12}(x)\bar{\eta}^1\bar{\eta}^2.$$

What we have done is chosen a complex structure on the tangent space to the manifold at each point, this is what gives us a way of writing the four real variables Ψ^a as two complex variables. A $U(2) = SU(2) \times U(1)/Z_2$ subgroup of SO(4) leaves this complex structure invariant and the η's transform as the fundamental representation of this U(2). Identifying the SU(2) with weak isospin and the U(1) with weak hypercharge, the states transform exactly as a generation of Weinberg-Salam leptons: a singlet (ν_R) which has no interactions, a hypercharge $Y = -1$ isodoublet (ν_L, e_L) and a hypercharge $Y = -2$ isosinglet (e_R).

What has happened to the spin degree of freedom? It turns out that it is hidden in the choice of complex structure. Understanding this and the other issues involved in making sense of the above explanation for the Weinberg-Salam multiplet structure will require developing the geometry of spinors in four dimensions in some detail, and this will be the subject of the next two sections.

3. Projective geometry

While physicists have become very familiar with riemannian geometry, they are less familiar with a simpler sort of geometry that is historically prior and in which the metric is not the central concept. This is projective geometry, a subject that dominated 19th century work on geometry and provided the context that Riemann's work grew out of. Furthermore, it was simple examples provided by projective geometry that led to the formulation of the general concept of a fiber bundle, a concept which is central to 20th century geometry and topology. Dirac has explained [10] how it was considerations of projective geometry that led him to the discovery of spinors and the Dirac equation. The connection between spinors and projective geometry was investigated by the mathematician Veblen [11] and later formed the basis for Penrose's work on twistors (see refs. [12] and [13] for reviews and references). While physicists have recently made much use of the geometry of gauge fields, the concepts of projective geometry that underly the geometry of spinors are less well known. This section aims to provide a basic introduction to the subject.

There are three different types of projective spaces we will consider, corresponding to the three types of number fields, the real numbers **R**, the complex numbers **C** and the quaternions **H**. The projective spaces $\mathbf{HP}^n$ will be defined explicitly, the other two cases $\mathbf{RP}^n$ and $\mathbf{CP}^n$ are defined in exactly the same way although the order of multiplication becomes irrelevant.

A point in $\mathbf{HP}^n$ is an equivalence class of sets of $n+1$ quaternions $[q_0, q_1, \dots q_n]$ where two such sets are equivalent when they are equal up to multiplication by a constant quaternion q:

$$[q_0, q_1, \dots q_n] \sim [q_0', q_1', \dots q_n'] \qquad \text{iff} \quad q_i = qq_i' \quad \text{for} \quad i = 0, 1, \dots n\,.$$

$\mathbf{HP}^n$ can be thought of as the space of "quaternionic" lines in the space $\mathbf{H}^{n+1}$ of $n+1$ quaternions. Away from the point $q_0 = 0$, one can take as coordinates on $\mathbf{HP}^n$ the set of quaternions q_i/q_0 for $i = 1 \dots n$.

Associated to each projective space there is a so-called "tautological" bundle T. The fiber above each point in $\mathbf{HP}^n$ is the quaternionic line **H** that defines that point. More formally, the tautological bundle is the subbundle of the trivial bundle $\mathbf{HP}^n \times \mathbf{H}^{n+1}$ given by $\mathrm{T} = \{(l, p) \in \mathbf{HP}^n \times \mathbf{H}^{n+1} | p \in l\}$. One can also construct T*, the hyperplane bundle or dual bundle to T, by taking as fiber at the point q the space dual to the fiber of T above q. Finally one can construct the quotient bundle $\mathrm{T}^{\perp}$ by taking as fiber the $\mathbf{H}^n$ orthogonal to the line l in $\mathbf{H}^{n+1}$ that is the fiber of T at q. This involves a choice of metric on $\mathbf{H}^{n+1}$, a natural choice is given by identifying $\mathbf{H}^{n+1}$ with $\mathbf{R}^{4(n+1)}$ and using the euclidean metric. This is called the quotient bundle because, if one wants to evade the use of the metric on $\mathbf{H}^{n+1}$, one can define the fibers as being the quotient space $\mathbf{H}^{n+1}/l$.

The tangent space to $\mathbf{HP}^n$ at q is the set of quaternionic linear maps from the fiber of T to the fiber of $\mathrm{T}^{\perp}$ at q. An infinitesimal change in q moves one to a slightly different fiber of T. The corresponding linear map will then be projection of this new fiber onto the fiber of $\mathrm{T}^{\perp}$ at q.

Corresponding to the vector bundles T and $\mathrm{T}^{\perp}$ are the bundles of orthonormal frames in each fiber. For the bundle T, since the fiber is **H**, the unit frames are just the unit length quaternions, which make up the group $\mathrm{Sp}(1) = \mathrm{SU}(2)$. The fibers of $\mathrm{T}^{\perp}$ are $\mathbf{H}^n$, the orthonormal frames in $\mathbf{H}^n$ make up the group $\mathrm{Sp}(n)$. The unit length vectors in $\mathbf{H}^{n+1}$ make up a $4n+3$ dimensional sphere and can also be seen to be the set of unit vectors in the fibers of the tautological bundle. Taking the coordinates of such a vector in $\mathbf{H}^{n+1}$ to be the homogeneous coordinates of a point in $\mathbf{HP}^n$ defines the Hopf fibration $S^{4n+3} \to \mathbf{HP}^n$. The same construction in the real and complex cases gives the fibrations $S^n \to \mathbf{RP}^n$ and $S^{2n+1} \to \mathbf{CP}^n$ respectively.

The space $\mathbf{HP}^n$ has an alternative description as a coset space. The group $\mathrm{Sp}(n+1)$ acts transitively on the lines in $\mathbf{H}^{n+1}$ and thus on $\mathbf{HP}^n$. A point q in $\mathbf{HP}^n$ is left invariant under the subgroup $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ of $\mathrm{Sp}(1)$ transformations that leave the fiber of T invariant, and $\mathrm{Sp}(n)$ transformations that leave the fiber of $\mathrm{T}^{\perp}$ invariant. Thus

$$\mathbf{HP}^n = \mathrm{Sp}(n+1)/(\mathrm{Sp}(1) \times \mathrm{Sp}(n)),$$

so $\mathrm{Sp}(n+1)$ is an $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ bundle over $\mathbf{HP}^n$. The notion of a projective space can be further generalized to that of a grassmannian. The grassmannian $\mathrm{G}_{i,n+i}(\mathbf{H})$ is the set of i-planes $\mathbf{H}^i$ through the origin of $\mathbf{H}^{n+i}$. $\mathrm{G}_{1,n+1}(\mathbf{H})$ is another name for $\mathbf{HP}^n$, $\mathrm{G}_{2,n+2}(\mathbf{H})$ can be thought of as the set of lines in $\mathbf{HP}^{n+1}$. These spaces again have tautological bundles above them, a similar description of their tangent bundles and a coset space description

$$\mathrm{G}_{i,n+i}(\mathbf{H}) = \mathrm{Sp}(n+i)/(\mathrm{Sp}(i) \times \mathrm{Sp}(n)).$$

For the complex projective spaces $\mathbf{CP}^n$, the symplectic groups are replaced by the unitary groups. As a coset space

$$\mathbf{CP}^n = \mathrm{U}(n+1)/(\mathrm{U}(1) \times \mathrm{U}(n)) = \mathrm{SU}(n+1)/\mathrm{U}(n),$$

so $\mathrm{SU}(n+1)$ is a $\mathrm{U}(n)$ bundle over $\mathbf{CP}^n$. The real projective spaces $\mathbf{RP}^n$ are given by (since $\mathrm{O}(1) = \mathrm{Z}_2$)

$$\mathbf{RP}^n = \mathrm{O}(n+1)/(\mathrm{Z}_2 \times \mathrm{O}(n)) = \mathrm{SO}(n+1)/(\mathrm{Z}_2 \times \mathrm{SO}(n))$$

and geometrically can be realized as one half of an n-sphere, cut along the equator, with opposite points on the equator identified.

The spaces $\mathbf{RP}^1$, $\mathbf{CP}^1$, and $\mathbf{HP}^1$, are especially simple since they can be identified with spheres by stereographic projection. $\mathbf{RP}^1 = \mathrm{S}^1$, $\mathbf{CP}^1 = \mathrm{S}^2$, and $\mathbf{HP}^1 = \mathrm{S}^4$. The bundle of orthonormal frames in the tautological bundles T above these spaces are well known: the Möbius strip for $\mathbf{RP}^1$, the Dirac monopole bundle for $\mathbf{CP}^1$ (the Hopf fibration $\mathrm{S}^3 \to \mathrm{S}^2$) and the BPST instanton bundle for $\mathbf{HP}^1$ (the Hopf fibration $\mathrm{S}^7 \to \mathrm{S}^4$). Note that these three bundles fit together, the fiber of one of them is the total space of the preceding one.

4. Spinors and twistors

The geometry of the spin bundle in 4 dimensions is simplest if one deals with the conformal compactification of flat space, which will be S^4 in the euclidean case and

$S^3 \times S^1$ in the Minkowski case. This compactification is physically irrelevant since these spaces are locally flat and one can make their size as big as one likes. We have no intention of saying anything about the global structure of space-time since our concern is with the properties of particles and their interactions and only in a very peculiar theory would these depend upon the structure of space-time infinity.

Twistor theory [12–14] relies upon the observation that the conformal compactification of Minkowski space is a certain real slice of the 4 complex dimensional space $G_{2,4}(\mathbf{C})$, the grassmannian of complex 2-planes in $\mathbf{C}^4$. The tautological bundle above this space will be called $\mathbf{S}^+$, it is the bundle of Weyl spinors. This formalism is inherently parity asymmetric, the tautological bundle is the bundle of spinors of one chirality, we will choose to call these the right-handed spinors, but this is a matter of convention. A point in Minkowski space corresponds to a $\mathbf{C}^2$ in $\mathbf{C}^4$, this $\mathbf{C}^2$ is the space of right-handed Weyl spinors at that point and the $\mathbf{C}^4$ is twistor space. The conformal compactification of euclidean space is S^4 and this is another real slice of $G_{2,4}(\mathbf{C})$, one that intersects Minkowski space on a constant time S^3. This identification of $G_{2,4}(\mathbf{C})$ with the complexification of space-time has two advantages: it provides a clear way of understanding the analytic continuation between Minkowski and euclidean space that is an essential part of quantum field theory, and it provides a simple geometrical construction of the spinor bundle.

It is perhaps easiest to study $G_{2,4}(\mathbf{C})$ as the space of complex projective lines ($\mathbf{CP}^1$'s) on $\mathbf{CP}^3$. There is a natural metric on this space with the property that for any two points x_1 and x_2 the distance between them satisfies $|x_1 - x_2|^2 = 0$ if and only if the corresponding lines in $\mathbf{CP}^3$ intersect. On the Minkowski slice this corresponds to the Minkowski metric, on the S^4 slice it gives the standard positive definite metric. This latter property is a reflection of the fact that $\mathbf{CP}^3$ has a fibration by a family of non-intersecting $\mathbf{CP}^1$'s parametrized by S^4.

The fibration of $\mathbf{CP}^3$ by $\mathbf{CP}^1$'s is given by choosing an arbitrary identification of the twistor space $\mathbf{C}^4$ with $\mathbf{H}^2$, and associating to each complex line through the origin in $\mathbf{C}^4$ the quaternionic line through the origin in $\mathbf{H}^2$ that it generates. As an example, consider the identification of $\mathbf{C}^4$ with $\mathbf{H}^2$ given by

$$(z_1, z_2, z_3, z_4) \leftrightarrow (z_1 + z_2 j, z_3 + z_4 j).$$

Taking these to be homogeneous coordinates on $\mathbf{CP}^3$ and $\mathbf{HP}^1$ respectively this provides a map

$$\pi: \mathbf{CP}^3 \to \mathbf{HP}^1$$

which is given in terms of homogeneous coordinates by associating to any non-zero point in $\mathbf{C}^4$ the corresponding point in $\mathbf{H}^2$ and then taking the quaternionic line through zero generated by this point. This map is a fibration with fiber $\mathbf{CP}^1$ since a point in $\mathbf{HP}^1$ is a quaternionic line, thus a copy of $\mathbf{C}^2$ and the set of complex lines lying in this $\mathbf{C}^2$ is a copy of $\mathbf{CP}^1$.

Furthermore, each point in the fiber $\mathbf{CP}^1$ above a point in $\mathbf{HP}^1 = S^4$ provides an identification of the 4-dimensional tangent space to S^4 with $\mathbf{C}^2$. This occurs because for each point x in $\mathbf{CP}^3$ there is a distinguished complex line L_x passing through it given by the fibration. In terms of the map π

$$L_x = \pi^{-1}\pi(x).$$

The tangent space $T_x\mathbf{CP}^3$ can be used to construct the quotient space $T_x\mathbf{CP}^3/L_x$ which naturally has a complex structure and furthermore is naturally identified by the map π with the tangent space to S^4 at the point $\pi(x)$. Although we restrict our attention to the case of S^4 we should point out that this sort of fiber bundle exists in general and is of great importance in the study of 4-dimensional riemannian geometry. Atiyah, Hitchin and Singer [15] have studied solutions to the Yang-Mills equations over arbitrary 4-dimensional riemannian manifolds by looking at the generalized twistor space of all complex structures on the tangent spaces for each point. This gives a bundle over the 4-manifold, which for self-dual 4-manifolds is actually a complex manifold (the analogue of $\mathbf{CP}^3$ in our case).

The tangent space to $G_{2,4}(\mathbf{C})$ is given by complex linear maps from $\mathbf{S}^+$ to $\mathbf{S}^-$, equivalently by elements in the tensor product $\mathbf{S}^{+*} \otimes \mathbf{S}^-$, where $\mathbf{S}^+$ is the tautological bundle above $G_{2,4}(\mathbf{C})$, (right-handed spin bundle), $\mathbf{S}^-$ is the quotient subbundle (left-handed spin bundle) and $\mathbf{S}^{+*}$ is the dual bundle to $\mathbf{S}^+$. Once we have chosen coordinates on the fibers of $\mathbf{S}^+$ and $\mathbf{S}^-$, a point in the tangent space will be given by an arbitrary 2×2 complex matrix M:

$$M = i\begin{pmatrix} z_0 + z_3 & z_1 + iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}.$$

The metric function on the tangent space of $G_{2,4}(\mathbf{C})$ is given by the determinant of this matrix:

$$\det M = -z_0^2 + z_1^2 + z_2^2 + z_3^2.$$

It is invariant under the group $SO(4,\mathbf{C}) = SL(2,\mathbf{C})_L \times SL(2,\mathbf{C})_R/Z_2$ which acts on M by

$$M \to AMB,$$

where A and B are $SL(2,\mathbf{C})$ matrices (the Z_2 arises because $(A, B) = (-1, -1)$ has the same effect as $(A, B) = (1,1)$). The Minkowski slice (z_0, z_1, z_2, z_3 real) of the tangent space is left invariant by the diagonal $SL(2,\mathbf{C})$ subgroup ($B = A^\dagger$), this is the Lorentz group. The euclidean slice (z_0 imaginary, z_1, z_2, z_3 real) is left invariant by the compact subgroup $SU(2)_L \times SU(2)_R/Z_2$ ($A, B \in SU(2)$), this is the euclidean rotation group SO(4). It is double covered by the group $\text{Spin}(4) = SU(2)_L \times SU(2)_R$. The diagonal SU(2) ($A = B^{-1}$) gives the spatial rotations and acts on both the euclidean and Minkowski slices. The boosts in the Lorentz SL(2, C) take points in euclidean space out of the euclidean slice, and the other "anti-diagonal" SU(2) in Spin(4) ($A = B$) similarly does not take Minkowski space-time points to Minkowski space points.

Quantum field theories, even those for scalars, are not well defined in Minkowski space and the same problems will plague any equivalent path integral form of the theory. Perturbatively this shows up in ambiguities in the definition of the free propagator. Non-perturbatively, functional integrals are not convergent in Minkowski space even for a regularized theory. These problems disappear when one formulates the theory in euclidean space, where functional integrals are convergent in the regularized theory and propagators are unambiguously defined. Once the theory has been defined in euclidean space, the Minkowski space Green functions are defined to be the analytic continuation of the euclidean space Green functions.

For spinor field theories the difficulties in understanding the relationship between the theory written in Minkowski and in euclidean space are well-known. While technical solutions to these problems have been found allowing the performance of calculations in euclidean space, our point of view is that these difficulties are a symptom that one is misinterpreting the situation. We believe that some of the euclidean space rotational symmetries should be thought of as internal symmetries of the Minkowski space theory. This analytic continuation deserves a much more careful analysis and the natural context for this is in the twistor formalism. While we have not fully carried this out, we will make a few comments on the problem.

The right-handed spin bundle $\mathbf{S}^+$ over $G_{2,4}(\mathbf{C})$ is a holomorphic vector bundle (its transition functions can be chosen to be holomorphic functions). Since right-handed spinor fields are sections of this bundle, the notion of analytic continuation of spinor fields between the Minkowski and euclidean slices of $G_{2,4}(\mathbf{C})$ will make sense provided we take these spinor fields to be holomorphic sections. An arbitrary bundle over $G_{2,4}(\mathbf{C})$ will not necessarily be holomorphic and in such a case there would be no gauge invariant way of making sense of the notion of analytic continuation of the fields from one slice to another. This is the case because only if the transition functions can be chosen to be holomorphic can the gauge transformations also be chosen to be holomorphic.

Under this notion of analytic continuation of the spinor fields, while the value of the field at a Minkowski point is determined in terms of its value at a euclidean point, the transformations of the field under the Minkowski boosts and under the anti-diagonal SU(2) in SO(4) can be performed completely independently. The analytic continuation does not relate this SU(2) to the boosts of the Lorentz group, so it behaves like an internal symmetry not a space-time symmetry. The prejudice that the analytic continuation of the fields should relate the euclidean SO(4) rotation group and the Minkowski SL(2, C) Lorentz group is based upon the fact that these groups share the same complexification, SO(4, C), and thus an analytic continuation in the group parameters from one group to the other can be established. In flat space, using global symmetry transformations, one can use the existence of this complexification to relate Minkowski and euclidean space Green functions. However this is not possible in a curved space and if one considers SO(4) and SL(2, **C**) purely in their role as local frame transformations there is no reason to believe that this analytic continuation in SO(4, **C**) has anything to do with the analytic continuation in the space-time variables.

Gauging the group of space-like rotations would be a mistake because one would then have a gauge field that coupled to particles through their spin, something which is not observed. We are only gauging a U(1) subgroup of this group, the subgroup of transformations which is indistinguishable from an overall U(1) phase transformation on the spinor field at a point.

5. The states of SSQM and the standard model

In the first section of this paper we argued that the quantization of SSQM required the introduction of a complex structure on the tangent space at each point x. The last section showed that this is equivalent to picking a point in the fiber above x in the fibration

$$\pi: \mathbf{CP}^3 \to \mathbf{HP}^1 .$$

In order to avoid breaking the rotational invariance of the theory, we should integrate over all choices of complex structure. This suggests that we should be defining the theory by integrating over paths in $\mathbf{CP}^3$ rather than paths in S^4. The action will remain the same, it is independent of the complex structure so it only depends on the projection of the path onto S^4. The introduction of the complex structure is necessary only to define the space of states of the theory.

A basis for the states is given by the complex exterior algebra and we have seen how this basis transforms under the U(2) subgroup of SO(4) that preserves the complex structure. We begin by identifying this U(2) in terms of the spinor geometry of the last section. Since vectors in the tangent space at a point of S^4 correspond to maps from the fiber of $\mathbf{S}^+$ at the point to the fiber of $\mathbf{S}^-$, the group SO(4) of transformations preserving the metric on the tangent space becomes the group $SU(2)_L \times SU(2)_R/Z_2$ of independent SU(2) transformations preserving the hermitian metrics on the fibers of $\mathbf{S}^+$ and $\mathbf{S}^-$. The complex structures above a point correspond to the complex lines in the corresponding fiber of the right-handed spin bundle $\mathbf{S}^+$. A U(1) subgroup of the group $SU(2)_R$ leaves this complex line invariant, just multiplying vectors in it by a phase. The group $SU(2)_L$ also clearly leaves this line invariant, so we have the full $SU(2)_L \times U(1)$ subgroup of $SU(2)_L \times SU(2)_R$ that leaves the complex structure invariant.

In order to make this more explicit, choose coordinates on the fibers of the spin bundles and fix the complex structure by choosing the line generated by [1, 0] in the right-handed spin bundle. Then the U(1) acts on the right-handed spinors as

$$v_R \to e^{ig'\theta\tau_3/2} v_R$$

(g' will be the weak hypercharge coupling constant) and the $SU(2)_L$ leaves them invariant. In other words, the vector bundle $\mathbf{S}^+$ splits into two line bundles, which transform under the U(1) with hypercharges $Y = +1$ and $Y = -1$, where by definition a state with hypercharge Y will transform under the U(1) as

$$v \to e^{ig'Y\theta/2} v .$$

The $SU(2)_L$ acts on the left-handed spinors as

$$v_L \to e^{ig\theta_k\tau_k/2} v_L$$

(where g will be the weak isospin coupling constant) and the U(1) leaves them invariant.

Just as vectors on S^4 are sections of the bundle $\mathbf{S}^{+*} \otimes \mathbf{S}^-$ over S^4, we should think of elements of the complex exterior algebra as sections of a bundle over $\mathbf{CP}^3$, since we have to specify a complex structure as well as a point on S^4. The appropriate bundle will be:

$$T^* \otimes (\pi^*\mathbf{S}^+ \oplus \pi^*\mathbf{S}^-) \to \mathbf{CP}^3 ,$$

where T^* is the dual of the tautological bundle over $\mathbf{CP}^3$. Here π^* means pullback by the projection mapping π, and T is the bundle whose fiber above a point is just the complex line that determines that point. Thus sections of T will transform under the U(1) with $Y = +1$ and sections of T^* will carry $Y = -1$.

Looking first at the even elements of the basis of the exterior algebra (1 and $\bar\eta^1\bar\eta^2$), they correspond to $T^* \otimes \pi^*\mathbf{S}^+$ since they carry $Y = 0$ and $Y = -2$ and are SU(2) singlets. The odd elements $(\bar\eta^1, \bar\eta^2)$ correspond to $T^* \otimes \pi^*\mathbf{S}^-$ which carry $Y = -1$ and are an SU(2) doublet.

Now that we have identified the basis elements of the complex exterior algebra in terms of the spin bundles twisted by the tautological bundle over $\mathbf{CP}^3$ we find it irresistible to wonder what happens if one twists the spin bundles by the other natural bundle over $\mathbf{CP}^3$, the quotient bundle $T^\perp$. Thinking of $\mathbf{CP}^3$ as the coset space $U(4)/(U(3) \times U(1))$, U(4) is the bundle over $\mathbf{CP}^3$ whose fibers $U(3) \times U(1)$ are the frames in the associated vector bundles $T^\perp$ and T. The fibers of $T^\perp$ transform as the fundamental representation of $SU(3) \subset U(3)$. There are now two U(1) groups to be considered. One corresponds to the $U(1) \subset U(4)$ that acts on $\mathbf{C}^4$ as

$$z_i \to e^{i\theta} z_i .$$

This overall U(1) symmetry presumably corresponds to fermion number conservation in the theory, it is the other U(1) that is interesting, the one that is left if one describes $\mathbf{CP}^3$ as SU(4)/U(3). This U(1) is a subgroup of SU(4) which acts both on T and on $T^{\perp}$. If we identify it with the hypercharge U(1) on T it will act as $e^{ig'\theta/2}$ on the fibers of T. Since it is a subgroup of SU(4) it must act on $T^{\perp}$ as $e^{-ig'\theta/6}$.

This shows that sections of

$$T^{\perp *} \otimes (\pi^* S^+ \oplus \pi^* S^-) \to \mathbf{CP}^3$$

will transform in the same way as a generation of leptons except that they will be SU(3) triplets and will have their hypercharges shifted by $Y \to Y - \frac{4}{3}$. These are exactly the transformation properties of a generation of quarks in the standard model. We have shown that a generation of fermions in the standard model transforms as sections of the bundle

$$(T^* \oplus T^{\perp *}) \otimes (\pi^* S^+ \oplus \pi^* S^-) \to \mathbf{CP}^3.$$

This simplifies since $T^* \oplus T^{\perp *}$ is just the trivial $\mathbf{C}^4$ bundle over $\mathbf{CP}^3$. In a sense the topological twisting of the lepton fields coming from the tautological bundle is undone by the twisting of the quark fields, which is quite reminiscent of the way in which the gauge anomaly of a lepton generation is cancelled by the quarks.

6. The Higgs field

The supersymmetric quantum mechanics that we began by considering describes massless fermions and we have now seen that it has an internal U(2) symmetry. If we wish to understand the connection to the standard model, we have to understand how spontaneous symmetry breaking of U(2) to U(1) can occur in this sort of model. This problem is under investigation, but there is one simple remark that can be made about it. As we shall see, it turns out that there is a very natural geometrical object which transforms as the Higgs field in the standard model.

For simplicity we will use a non-linear sigma model description of the Higgs in which it takes values in $S^3 = SU(2) = Sp(1)$, the unit quaternions. It is well known that the Higgs sector of the standard model by itself is SO(4) invariant, with the Higgs field transforming as a 4-vector. At each point x in $\mathbf{HP}^1$, the tangent space is the space of quaternions. Just as there is a $SO(4)/U(2) = \mathbf{CP}^1$ worth of inequivalent ways of identifying $\mathbf{C}^2$ and $\mathbf{R}^4$, there is a $SO(4)/Sp(1) = S^3/Z_2$ worth of inequivalent ways of identifying $\mathbf{H}$ with $\mathbf{R}^4$. One way of seeing this is to note that the real unit vector $\mathbf{1}$ is distinguished from all other unit vectors in $\mathbf{H}$ as defining the real axis and thus is the only unit vector invariant under conjugation. We will identify this distinguished unit vector with the Higgs degree of freedom. It is left invariant by the diagonal SU(2) in the SO(4) of frame rotations. Since these will be the spacelike rotations, this distinguished vector corresponds to the time direction relative to the given frame. While it is a scalar under spatial rotations, it transforms as an isodoublet under $SU(2)_L$ and thus has the correct transformation properties to behave like a Higgs field and break the weak isospin symmetry.

To make the above considerations more explicit, consider the standard identification of $\mathbf{R}^4$ and $\mathbf{H}$:

$$x^\mu \leftrightarrow q = x^\mu \sigma^\mu ,$$

where $\sigma^0 = i1$ SO(4) acts on this by $q \to aqb$ with $a, b \in SU(2)$. The purely spatial rotations correspond to the SU(2) subgroup $a = b^{-1}$; they leave invariant the time coordinate, which is given by $\mathrm{Tr}(q)/2i$. The subgroup $a = b$ changes the trace and thus the time coordinate. It acts transitively on the space of all possible time axes (equivalently, all identifications of $\mathbf{R}^4$ with $\mathbf{H}$) rotating any one into any other.

7. Conclusion

The goal of this work has been to at least raise the possibility of the existence of a very different sort of unified theory of particle interactions, one based upon considering the simplest supersymmetric non-linear sigma model and the geometry of four dimensional spinors. The argument for this is quite simple. Quantization implies that the states of the model transform under a U(2) with the transformation properties of a generation of leptons, and expressing these states in terms of spinor geometry leads one to a simple way of introducing quarks, which are presumably necessary for the consistency of the theory once it is coupled to gauge fields.

This provides a remarkable set of answers to the questions "Why $SU(3) \times SU(2)_L \times U(1)$?" and "Why this particular set of transformation properties for a generation?". The supersymmetric model considered is of great independent interest, it has provided a proof of the Atiyah-Singer index theorem for the Dirac operator, and work of Witten has shown that it gives a measure on the space of paths on a manifold, a measure different from Wiener measure. The fact that spinor and twistor geometry may be involved in understanding the internal symmetries should help bring the beautiful subject of twistor theory into the mainstream of particle physics. Furthermore, the fact that the internal symmetries may be understood as properties of spinors and the frame bundle promises the possibility of new approaches to unifying gravity with the other interactions.

This work raises many more questions than it answers, one of the most important is understanding how spontaneous symmetry breaking occurs and thus where the fermion mass matrix comes from. While it seems likely that most of the successful field theoretical results of the standard model can be derived in a path integral rather than functional integral context by use of this model, this project has never been fully worked out. This now seems a much more interesting problem than it was before, because a reformulation of the field theory in terms of paths may be necessary in order to make progress on the question of unification. Perhaps quantities that are inherently not calculable in the field theory formalism may be calculable in some sort of path integral formalism.

One important difference between the geometric picture suggested here and the standard one is that the SU(3) and U(1) gauge fields will be defined over the twistor space $\mathbf{CP}^3$ rather than over S^4. This allows one a new freedom in defining the dynamics of the theory which may have interesting consequences.

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