

# Some Notes on Spinors and Vectors

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## 1 Spinors and vectors

In these notes we'll review the conventional formalism for describing four-dimensional spacetime using vectors and spinors. This formalism works most simply if one allows spacetime coordinates to be complex, so it is complex spacetime that we will consider in this first section. Note that the conventions chosen here are for complex coordinates, so slightly different than the usual ones for real coordinates, which will be discussed in the next section.

### 1.1 Spinors and vectors in complex spacetime

#### 1.1.1 Spinors and vectors

Let  $V = \mathbf{C}^4$  be the space of complex four-dimensional spacetime vectors. It is the tensor product

$$V = S_L \otimes S_R$$

of two different spaces of complex two-dimensional spinors (sometimes known as “Weyl spinors” or “half-spinors”). We'll call one of these right-handed spinors ( $S_R$ ), the other left-handed spinors ( $S_L$ ). Choosing bases in  $S_R$  and  $S_L$ , we can write elements in terms of their coordinates as

$$\mu^{\dot{\alpha}} \in S_L, \quad \lambda^{\alpha} \in S_R$$

Here the indices take two values 1, 2 and come in dotted (for  $S_L$ ) and undotted (for  $S_R$ ) versions. We'll use lower indices for coordinates on the dual spaces

$$\mu_{\dot{\alpha}} \in S_L^*, \quad \lambda_{\alpha} \in S_R^*$$

Spacetime vectors  $x \in V$  can then be written in coordinates as

$$x^{\dot{\alpha}\alpha} \in S_L \otimes S_R$$

The tensor product  $S_L \otimes S_R$  can be identified with the space  $\text{Hom}(S_R^*, S_L)$  of linear maps from  $S_R^*$  to  $S_L$  and the coordinates of  $x$  are the entries of a matrix that acts on vectors in  $S_R^*$  and takes values in  $S_L$

$$x \in \text{Hom}(S_R^*, S_L) : \lambda_\alpha \rightarrow \mu^{\dot{\alpha}} = x^{\dot{\alpha}\alpha} \lambda_\alpha$$

To relate these matrix entry coordinates to the usual ones, one must choose an identification of a basis of  $V$  (which we'll label  $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ ) with matrices. A possible choice would be:

$$\mathbf{e}_0 = \mathbf{1}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

With this choice of basis, vectors  $x$  in  $V$  get identified with matrices by

$$(x^0, x^1, x^2, x^3) \leftrightarrow \begin{pmatrix} x^0 + x^3 & x^1 + x^2 \\ x^1 - x^2 & x^0 - x^3 \end{pmatrix}$$

Note that

$$\det x = (x^0)^2 - (x^1)^2 + (x^2)^2 - (x^3)^2 \quad (1)$$

gives a non-degenerate quadratic form on  $V$ . The complex orthogonal group  $SO(4, \mathbf{C})$  of linear transformations preserving this quadratic form is double covered by the group of pairs  $g_L, g_R$  of pairs of elements of  $SL(2, \mathbf{C})$ . This is the group

$$\text{Spin}(4, \mathbf{C}) = SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$$

The first copy of  $SL(2, \mathbf{C})$ , labeled  $SL(2, \mathbf{C})_L$ , acts as the fundamental representation on  $S_L$ , while the second acts as the fundamental representation on  $S_R$ .  $\text{Spin}(4, \mathbf{C})$  acts on  $V = \text{Hom}(S_R^*, S_L)$  by the action on matrices

$$x \rightarrow g_L x g_R^{-1}$$

There is an  $SL(2, \mathbf{C})$ -invariant antisymmetric bilinear form on  $\mathbf{C}^2$ , and this can be used to provide both an identification of  $S_L, S_R$  with their duals and the quadratic form on  $V$  of equation 1.

The usual formalism that physicists use for dealing with spinors tries to be symmetric between  $S_R$  and  $S_L$ , putting them together into the direct sum  $S_L \oplus S_R^*$  (these are called "Dirac spinors"). Then one defines the complex Clifford algebra as four by four matrices acting as linear maps from this space to itself, with the block diagonal form

$$\begin{pmatrix} S_L \\ S_R^* \end{pmatrix} \rightarrow \begin{pmatrix} \text{Hom}(S_L, S_L) & \text{Hom}(S_R^*, S_L) \\ \text{Hom}(S_L, S_R^*) & \text{Hom}(S_R^*, S_R^*) \end{pmatrix} \begin{pmatrix} S_L \\ S_R^* \end{pmatrix} \quad (2)$$

More explicitly, the complex Clifford algebra is the matrix algebra  $M(4, \mathbf{C})$  of four by four complex matrices, generated over  $\mathbf{C}$  by the matrices (Dirac  $\gamma$ -matrices)

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \quad (3)$$

## 1.2 Real forms of complex spacetime: split and Euclidean signature

The simple relation between vectors and spinors in the case of complex spacetime becomes significantly more complicated when one considers “real forms” of spacetime. These are real vector spaces  $V_{\mathbf{R}}$  that become isomorphic to  $V$  after complexification, i.e.

$$V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = V$$

While over  $\mathbf{C}$  all non-degenerate quadratic forms are isomorphic to the one of equation 1, for real vector spaces there are inequivalent non-degenerate quadratic forms, which give inner products of different signatures. Such real forms are determined by a choice of an involution  $\sigma$  on  $V$  that is anti-holomorphic, i.e.

$$\sigma : V \rightarrow V, \quad \sigma^2 = \mathbf{1}, \quad \sigma(cv) = \bar{c}\sigma(v)$$

Given such a  $\sigma$ , one gets a real form of  $V$  as the subset of  $V$  on which  $\sigma$  acts trivially.

### 1.2.1 Split signature spacetime

The simplest case of such a  $\sigma$  is the complex conjugation operator on  $V$ , which is obviously an anti-holomorphic involution. The corresponding real form is the subspace of real two by two matrices. On this subspace the non-degenerate quadratic form of equation 1 gives an indefinite inner product of signature  $(2, 2)$  and we’ll call the real form  $V_{2,2}$ .

For this real form, the spinor spaces  $S_L, S_R$  are real two-dimensional spaces, acted on by  $SL(2, \mathbf{R})_L \times SL(2, \mathbf{R})_R$ . This is the spin group  $Spin(2, 2)$  double covering the orthogonal group  $SO(2, 2)$ . Vectors are again tensor products of spinors, but the tensor product is over the real numbers

$$V_{2,2} = S_L \otimes_{\mathbf{R}} S_R$$

The Clifford algebra is  $M(4, \mathbf{R})$ , with generators real Dirac  $\gamma$ -matrices.

This simplest real form of  $V$  does not seem to appear in fundamental physics. We’ll now turn to the two much more subtle real forms that are of significance for fundamental physics.

### 1.2.2 Euclidean spacetime

The case of the real form with positive-definite inner product we’ll call the Euclidean case, denoting the four-dimensional real spacetime  $V_E$ . To understand the behavior of spinors and vectors in this case, it is a good idea to use the quaternions  $\mathbf{H}$ . The spinor spaces  $S_L, S_R$  can each be identified with  $\mathbf{H}$ , and the spin group will be

$$Spin(4) = Sp(1)_L \times Sp(1)_R$$

the product of two copies of the group  $Sp(1)$  of unit quaternions. The spinor space  $S_L$  will be identified with  $\mathbf{H}$ , with the spinor representation the action of  $g_L \in Sp(1)_L$  by left multiplication. The spinor space  $S_R$  will also be  $\mathbf{H}$ , but the representation will be the right action of  $g_R \in Sp(1)_R$  by multiplication by  $g_R^{-1}$ . The spacetime vectors will be the tensor product

$$V_E = S_L \otimes_{\mathbf{H}} S_R$$

This can be identified with  $\mathbf{H}$ , and as a representation of  $Spin(4)$ , pairs  $g_L, g_R$  of unit quaternions act by

$$v \rightarrow g_L v g_R^{-1}$$

In order to exploit holomorphicity, one usually wants to work instead with an identification of  $\mathbf{H}$  with  $\mathbf{C}^2$ , so  $S_R$  and  $S_L$  can be treated in the same way as for complex spacetime. When one does this, the origin in the quaternions implies that multiplication by  $\mathbf{j}$  provides  $S_R$  and  $S_L$  with an antilinear map with square  $-1$  (unlike the  $+1$  of a real structure). We'll write this map as

$$\lambda^\alpha \rightarrow \widehat{\lambda}^\alpha, \quad \mu^{\dot{\alpha}} \rightarrow \widehat{\mu}^{\dot{\alpha}}$$

without now writing it out more explicitly (since it depends on the identification  $\mathbf{C}^2 = \mathbf{H}$ ).

On vectors  $V = S_L \otimes S_R$ , the  $\widehat{\phantom{x}}$  operation squares to  $+1$  and becomes a real structure  $\sigma_E$ , with fixed points ( $\widehat{x} = x$ ) the Euclidean subspace  $V_E \subset V$ . Elements  $x \in V_E$  can be identified with matrices of the form

$$\begin{pmatrix} ix^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & ix^0 - x^3 \end{pmatrix}$$

for  $x_0, x_1, x_2, x_3$  real. On both  $S_L$  and  $S_R$  the subgroup of the corresponding  $SL(2, \mathbf{C})$  is an  $Sp(1) = SU(2)$ , and the subgroup of  $SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$  preserving  $V_E$  is

$$Spin(4) = SU(2)_L \times SU(2)_R$$

### 1.3 Minkowski spacetime

For  $V$  identified with complex two by two matrices, the Minkowski real form corresponds to the real form  $\sigma_M$  that takes a matrix to its Hermitian adjoint:

$$\sigma_M x = x^\dagger \quad \text{or} \quad \sigma_M x^{\dot{\alpha}\alpha} = \overline{x^{\alpha\dot{\alpha}}}$$

Minkowski spacetime  $V_M$  is the four real dimensional vector space of two by two self-adjoint matrices. These will be of the form

$$x = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

with an indefinite non-degenerate quadratic form given by

$$-\det x = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

This gives the usual signature (3, 1) metric for physical spacetime used in special relativity.

The subgroup of  $Spin(4, \mathbf{C}) = SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$  that commutes with  $\sigma_M$  and so preserves the subspace  $V_M \subset V$  is the group of pairs of elements of  $SL(2, \mathbf{C})$  of the form

$$(g, \bar{g}) \tag{4}$$

which act on  $V_M$  by

$$x \rightarrow gxg^\dagger$$

taking self-adjoint matrices to self-adjoint matrices. This subgroup is isomorphic to  $SL(2, \mathbf{C})$ , which is the Lorentz group of special relativity. It is a two-fold cover of the group  $SO^+(3, 1)$ , the subgroup of  $SO(3, 1)$  preserving time orientation.

Within the complex group  $SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$  one can analytically continue between the Lorentz group  $Spin^+(3, 1) = SL(2, \mathbf{C})$  (considered as a real group, embedded as in 4), and the real groups  $Spin(4) = Sp(1)_L \times Sp(1)_R$  or  $SL(2, \mathbf{R})_L \times SL(2, \mathbf{R})_R$ .

In the Minkowski spacetime case, there is no real or quaternionic structure on the spinors. Instead, there is an Hermitian form  $\Phi$  on  $T = S_L \oplus S_R^*$ , which is usually defined using the generator  $\gamma_0$  of the Clifford algebra by

$$\Phi(\psi, \psi') = \psi^\dagger \gamma_0 \psi'$$

The ‘‘Dirac adjoint’’ is then defined as

$$\bar{\psi} \equiv \psi^\dagger \gamma_0$$

The Hermitian form  $\Phi$  provides not a real form (an antiholomorphic map from  $T$  to itself) but instead an antiholomorphic map from  $T$  to its dual  $T^*$ .

With respect to the decomposition  $T = S_L \oplus S_R^*$  and a choice of basis on  $S_L, S_R^*$ , the usual choice of  $\gamma_0$  has the block diagonal form

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

Note that diagonalizing  $\gamma_0$ , it has signature (2, 2), so  $\Phi$  is a Hermitian form of signature (2, 2). The subgroup of the complex conformal group  $SL(4, \mathbf{C})$  that preserves  $\Phi$  is the group  $SU(2, 2)$ , which is the conformal group of Minkowski spacetime.  $SU(2, 2)$  is a real form of the complex group  $SL(4, \mathbf{C})$ , and also the spin group  $Spin(4, 2)$ , double cover of the conformal group  $SO(4, 2)$ .

In this case there is only one  $SL(2, C)$ , embedded in  $SL(2, C)_L \times SL(2, C)_R$  as in 4. Instead of two spinor representations  $S_L, S_R$ , we now have a single one (which we’ll call  $S$ ), together with its conjugate representation  $\bar{S}$ , embedded as  $S = S_L$  and  $\bar{S} = S_R$ . The vector representation is the fixed points in

$$S \otimes \bar{S} \text{ or } \text{Hom}(\bar{S}^*, S)$$

of the real structure

$$\sigma_M(s \otimes t) = \bar{t} \otimes \bar{s}$$

Dirac spinors are the  $SL(2, \mathbf{C})$  representation  $S \oplus \bar{S}^*$