

From Quantum Mechanics to Number Theory via the Oscillator Representation

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June 1, 2023

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1 Introduction

These notes are an expanded version of some of the material covered in a spring 2023 graduate course on Lie groups and representations at Columbia University. The topics involved are central to the relations between representation theory, quantum mechanics and number theory, but are not part of most conventional courses or textbooks. Much of the interest of this material is in the way it ties together different fundamental parts of mathematics with fundamental physics.

Textbooks on representation theory of Lie groups typically focus on classification of all the possible such groups and representations, with focus on the beautiful and somewhat intricate (while still tractable) story for semi-simple Lie groups and their finite-dimensional representations. In quantum mechanics on the other hand, the focus is on a single non-semi-simple group and a single infinite-dimensional representation, together with the group of automorphisms of this structure. There will be little discussion of general classification problems

for any class of groups or representations. Since this is a topic of interest not just to mathematicians, but also to physicists, the style of exposition will be adjusted to try to make this material as accessible as possible to physicists with mathematical interests. For mathematicians, we'll begin with a quick discussion of the basics of the physics.

Note that this is still a preliminary document, very much a work in progress. Comments on how it could be improved are welcome.

2 Quantum mechanics background

What physicists call “canonical quantization” can be understood in terms of the unique non-trivial representation of the Heisenberg group and Lie algebra, which will be described in detail in the next section. In this one, we'll motivate the representation theory with a standard description of how it occurs in every physics textbook dealing with quantum theory.

The space of possible states for a quantum system is a complex vector space \mathcal{H} with Hermitian inner product. For one degree of freedom this space can be taken to be the space of wavefunctions (functions $\psi(q)$ of a position variable q) in $L^2(\mathbf{R})$. This version of the state space is called the Schrödinger representation and acting on it are powers of the self-adjoint operators

$$Q = q, \quad P = -i\hbar \frac{d}{dq}$$

which satisfy the Heisenberg commutation relations

$$[Q, P] = i\hbar 1$$

The dynamics of the system is determined by specification of an operator (defined in terms of the Q, P operators), the Hamiltonian H . This operator generates translations in time, with wavefunctions evolving in time according to the Schrödinger equation

$$i\hbar \frac{d}{dt} \psi = H\psi$$

The connection between this formalism and what ones observes, measures and often interprets in a classical picture of the world is given by two principles:

- Self-adjoint operators like Q and P correspond to observable quantities, with eigenfunctions of such an operator states with a well-defined measurable value of the observable quantity, given by the eigenvalue.
- If one tries to measure the value of an observable quantity when the state is not an eigenfunction, the result will be one of the eigenvalues, with probability given by the norm-squared of the inner product between the (normalized) state and eigenfunction with that eigenvalue (this is called the “Born rule”).

For a single quantum particle moving in one dimension, subject to a potential $V(q)$, the Hamiltonian is

$$H = \frac{1}{2m}P^2 + V(Q) = -\frac{\hbar^2}{2m} \frac{d^2}{dq^2} + V(q)$$

One would like to find the eigenfunctions and eigenvalues of this operator, i.e. find $E, \psi_E(q)$ such that

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dq^2} + V(q)\right)\psi_E(q) = E\psi_E(q)$$

and then expand wavefunctions at an initial time $t = 0$ in terms of the energy eigenfunctions $\psi_E(q)$. The Schrödinger equation implies that these evolve in time as

$$\psi_E(q)e^{-\frac{i}{\hbar}Et}$$

For much more detail about the following basic examples, see any physics textbook on quantum mechanics, or [1].

2.1 The free particle

The case of the free particle is the case $V(q) = 0$. Using Fourier analysis, one finds that the energy eigenvalues and eigenfunctions are parametrized by $p \in \mathbf{R}$ and are given by

$$E_p = \frac{p^2}{2m}, \quad \psi_{E_p}(q) = e^{i\frac{p}{\hbar}q}$$

The spectrum of the Hamiltonian is continuous, all non-negative values in \mathbf{R} .

The eigenfunctions of H are also eigenfunctions of the momentum operator P with eigenvalue p . P commutes with H , so if one prepares a state at time 0 with wavefunction $\psi_{E_p}(q)$ and measures its momentum at any later times, one will always get the value p (the momentum is a conserved quantity). Just as H is the generator of time-translations on states, P is the generator of spatial translations.

The eigenfunctions of the operator Q are delta-functions $\delta(q - q')$, with eigenvalue $q' \in \mathbf{R}$. Unlike the case for momentum P , one has $[Q, H] \neq 0$ and these are not energy eigenfunctions. If one prepares a state at time 0 with wavefunction $\delta(q - q')$, so localized at $q = q'$, it will immediately evolve into a linear combination of states with all possible eigenvalues of Q . Measurement of position at later times t may give all possible different values.

Note that the eigenfunctions of Q and P are not functions in $L^2(\mathbf{R})$ and in addition, the operators Q and P don't preserve $L^2(\mathbf{R})$ (multiplying or differentiating by q can take a function that is square-integrable to one that isn't). To deal with these problems simultaneously, one can define the Schwartz space $\mathcal{S}(\mathbf{R})$ of functions such that the function and its derivatives fall off faster than any power at $\pm\infty$. The dual space $\mathcal{S}'(\mathbf{R})$ of continuous linear functionals on

$\mathcal{S}(\mathbf{R})$ is called the space of tempered distributions, and includes the eigenfunctions of Q and P . One has the sequence of dense inclusions

$$S(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$$

The Fourier transform takes each term in this sequence to itself.

A problem here is that elements of $\mathcal{S}'(\mathbf{R})$ like the eigenfunctions of Q and P are not in $L^2(\mathbf{R})$. They do not have well-defined norms, so will not be vectors in a unitary representation and the Born rule can't be used for them. However, they are linear functionals on $S(\mathbf{R})$ and one can use this to replace their inner products with elements of $S(\mathbf{R})$.

To get a well-defined formalism one has two options:

- Work with states $\psi \in L^2(\mathbf{R})$, taking great care with domains and ranges of operators like P, Q and H that are applied to states. In this case, eigenfunctions of these operators are not in the state space.
- Work with the space $\mathcal{S}'(\mathbf{R})$ and distributional states, but be careful to properly pair these only with physical states in $\mathcal{S}(\mathbf{R})$ (sometimes called “wavepackets”).

2.2 The harmonic oscillator

The quantum harmonic oscillator is the case of a particle moving in a quadratic potential $V(q) = \frac{1}{2}m\omega^2 q^2$

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2$$

The energy eigenvalues and eigenfunctions are given by

$$E_n = \hbar\omega(n + \frac{1}{2}), \quad \psi_n(q) = H_n \left(\sqrt{\frac{m\omega}{\hbar}} q \right) e^{-\frac{m\omega}{2\hbar} q^2}$$

where $n = 0, 1, 2, \dots$ and $H_n(q)$ are Hermite polynomials. In this case the spectrum of the operator H is discrete, energy eigenfunctions are in $L^2(\mathbf{R})$, and arbitrary $t = 0$ wavefunctions in $L^2(\mathbf{R})$ can be written as linear combinations of the $\psi_{E_n}(q)$.

The easiest way to get these results is to work not with Q and P , but with complex linear combinations of these. For simplicity, rescaling so that $\hbar = m = \omega = 1$, one can choose

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{1}{\sqrt{2}}\left(q + \frac{d}{dq}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = \frac{1}{\sqrt{2}}\left(q - \frac{d}{dq}\right)$$

a, a^\dagger are each others adjoints and satisfy the commutation relation

$$[a, a^\dagger] = \mathbf{1}$$

The Hamiltonian is

$$H = \frac{1}{2}(Q^2 + P^2) = \frac{1}{2}(a^\dagger a + a a^\dagger) = a^\dagger a + \frac{1}{2}$$

One can easily see that a^\dagger increases the eigenvalue of H by 1, a reduces it by 1. To have a spectrum bounded below, one needs a non-zero state $\psi_0(q)$ satisfying

$$a\psi_0(q) = \frac{1}{\sqrt{2}}\left(q + \frac{d}{dq}\right)\psi_0(q) = 0$$

This state will have energy $\frac{1}{2}$ and is given by

$$\psi_0(q) = e^{-\frac{1}{2}q^2}$$

The other energy eigenstates will have energy $n + \frac{1}{2}$ for $n = 1, 2, \dots$ and can be found explicitly by applying the operator a^\dagger n -times to $\psi_0(q)$, so evaluating

$$\left(q - \frac{d}{dq}\right)^n e^{-\frac{1}{2}q^2}$$

Note that for the harmonic oscillator, $V(q)$ is not translation invariant, and one has $[P, H] \neq 0$ as well as $[Q, H] \neq 0$ so neither position nor momentum are conserved quantities.

For more general potentials one can have both discrete (with eigenfunctions in $L^2(\mathbf{R})$) and continuous (with eigenfunctions not in $L^2(\mathbf{R})$) components of the spectrum. The physical interpretation will involve both “bound states” which correspond to particles localized in some regions of \mathbf{R} and “scattering states” which correspond to particles with possible positions extending to $+\infty$ or $-\infty$.

3 The Heisenberg group and its representations

Quantum mechanics as we know it was born in 1925 in a series of conceptual breakthroughs which began with Heisenberg’s creation of a theory involving non-commuting quantities, soon reformulated (by Max Born) in terms of position and momentum operators Q and P satisfying the commutation relations

$$[Q, P] = i\hbar\mathbf{1}$$

(now known as the Heisenberg commutation relations). We are for now considering just one degree of freedom. \hbar is a constant that depends on units used to measure position and momentum. We will choose units such that $\hbar = 1$. The mathematician Hermann Weyl soon recognized these relations as those of a unitary representation of a Lie algebra now known as the Heisenberg Lie algebra, and described the corresponding Heisenberg group.

Late in 1925, Schrödinger formulated a seemingly different version of quantum mechanics, in terms of wave-functions satisfying a differential equation. What Schrödinger had found was a construction of a representation of the

Heisenberg Lie algebra on the vector space of functions $\psi(q)$ of a position variable q , with Q the multiplication by q operator and P the differential operator

$$P = -i \frac{d}{dq}$$

We'll begin with the Lie algebra corresponding to the Heisenberg commutation relations, then find the group with this Lie algebra and show that Schrödinger's wave-functions give an irreducible unitary representation of the Lie algebra and group. It turns out that any irreducible unitary representation of the Heisenberg group is essentially equivalent to this one (Stone-von Neumann theorem), but the family of different ways of constructing these representations carries an intricate structure.

3.1 The Heisenberg Lie algebra and Lie group

The Heisenberg Lie algebra \mathfrak{h}_3 will be the three-dimensional Lie algebra with a basis X, Y, Z and Lie bracket relations

$$[X, Z] = [Y, Z] = 0, \quad [X, Y] = Z$$

This Lie algebra can be identified with the Lie algebra of three by three strictly upper-triangular matrices by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A unitary representation (which we'll call π') will be given by three skew-adjoint operator $\pi'(X), \pi'(Y), \pi'(Z)$ satisfying

$$[\pi'(X), \pi'(Y)] = \pi'(Z), \quad [\pi'(X), \pi'(Z)] = 0, \quad [\pi'(Y), \pi'(Z)] = 0$$

These become the Heisenberg commutation relations if we identify

$$\pi'(X) = -iQ, \quad \pi'(Y) = -iP, \quad \pi'(Z) = -i\mathbf{1}$$

Note that factors of i are appearing here just because physicists like to work with self-adjoint operators (since their eigenvalues are real), but for unitary representations the Lie algebra representation operators are skew-adjoint.

In terms of matrices, exponentiating elements of \mathfrak{h}_3 as in

$$\exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

gives the elements of the Heisenberg group H_3 . This is the group of upper triangular matrices with 1s on the diagonal. Using x, y, z as ("exponential") coordinates on the group, H_3 is the space \mathbf{R}^3 with multiplication law

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$$

For computations with the Heisenberg group it is often convenient to use the Baker-Campbell-Hausdorff formula, which simplifies greatly in this case since all Lie brackets except $[X, Y] = Z$ vanish. As a result, for $A, B \in \mathfrak{h}_3$ one has

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

This group is a central extension

$$0 \rightarrow (\mathbf{R}, +) \rightarrow H \rightarrow (\mathbf{R}^2, +) \rightarrow 0$$

of the additive group of \mathbf{R}^2 by the additive group of \mathbf{R} .

A slightly different version of the Heisenberg group (which we'll call $H_{3,red}$) that is sometimes used takes a quotient by \mathbf{Z} and replaces the central \mathbf{R} with a central $U(1)$, so is a central extension

$$0 \rightarrow U(1) \rightarrow H_{3,red} \rightarrow (\mathbf{R}^2, +) \rightarrow 0$$

Elements are labeled by (x, y, u) where x and y are in \mathbf{R} and $u \in U(1)$, and the group law is

$$(x, y, u)(x', y', u') = (x + x', y + y', uu' e^{i\frac{1}{2}(xy' - x'y)})$$

3.2 The Schrödinger representation

The Schrödinger representation π_S will be a representation on a vector space \mathcal{H} of complex valued functions $\psi(q)$ on \mathbf{R} , with derivative the Lie algebra representation

$$\pi'_S(X) = -iQ = -iq, \quad \pi'_S(Y) = -iP = -\frac{d}{dq}, \quad \pi'_S(Z) = -i1$$

Exponentiating these operators gives unitary operators that generate π_S

$$\pi_S(x) = e^{-ixq}, \quad \pi_S(y) = e^{-y\frac{d}{dq}}, \quad \pi_S(z) = e^{-iz}\mathbf{1} \quad (1)$$

Note that $\pi_S(y)$ acts on the representation space by translation

$$\pi_S(y)\psi(q) = \psi(q - y)$$

Definition (Schrödinger representation). *The Schrödinger representation of the Heisenberg group H is given by*

$$\pi_S(x, y, z)\psi(q) = e^{-iz} e^{i\frac{1}{2}xy} e^{-ixq} \psi(q - y) \quad (2)$$

for $(x, y, z) \in H$.

One can easily check that this is a representation, since it satisfies the homomorphism property

$$\pi_S(x, y, z)\pi_S(x', y', z') = \pi_S(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$$

Taking as representation space $\mathcal{H} = L^2(\mathbf{R})$, for the Lie algebra representation π'_S there will be domain (functions on which operators not defined) and range (operators take something in $L^2(\mathbf{R})$ to something not in $L^2(\mathbf{R})$) problems. As an alternative, one can take $\mathcal{H} = \mathcal{S}(\mathbf{R})$ so that the representation operators are well-defined (but then the dual space is something different, the tempered distributions $\mathcal{S}'(\mathbf{R})$). For the group representation, the operators π_S are well defined on $\mathcal{H} = L^2(\mathbf{R})$. Giving up on a well-defined inner-product and unitarity, one can take $\mathcal{H} = \mathcal{S}'(\mathbf{R})$ and have both a Lie algebra and Lie group representation.

This multiplicity of closely related versions of the representation is a general phenomenon for infinite-dimensional representations of non-compact Lie groups, where one has inequivalent representations on a sequence of dense inclusions of representation spaces, here

$$\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$$

3.3 The Stone-von Neumann theorem

The remarkable fact about representations of the Heisenberg group is that there is essentially only one representation (once one has specified the constant by which Z acts, but non-zero choices are related by a rescaling). More specifically, any irreducible representation of H_3 will be unitarily equivalent to the Schrödinger representation. One has the following theorem

Theorem (Stone-von Neumann). *For any irreducible unitary representation π of H_3 (with action of the center $\pi(0, 0, z) = e^{-iz}$) on a Hilbert space \mathcal{H} , there is a unitary operator $U : \mathcal{H} \rightarrow L^2(\mathbf{R})$ such that*

$$U\pi U^{-1} = \pi_S$$

We will not give a proof here, since the analysis is somewhat involved, but what follows should make clear some problems that any proof needs to overcome and motivate the strategy for an actual proof.

Recall that one can define the adjoint pair of operators

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{1}{\sqrt{2}}\left(q + \frac{d}{dq}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = \frac{1}{\sqrt{2}}\left(q - \frac{d}{dq}\right)$$

and for the harmonic oscillator Hamiltonian the lowest energy eigenspace is the one-dimensional space of solutions in $L^2(\mathbf{R})$ of

$$a\psi_0(q) = 0$$

These are all proportional to

$$\psi_0 = e^{-\frac{1}{2}q^2}$$

The rest of the state space can be generated by repeatedly applying the operator a^\dagger to ψ_0 .

Exercise. *Use this basis to prove that the Schrödinger representation is irreducible.*

A possible approach to the Stone-von Neumann theorem would be to look at the operators

$$b = UaU^{-1}, \quad b^\dagger = Ua^\dagger U^{-1}$$

show that b has a one-dimensional kernel, and that the rest of the representation is given by repeated applications of b^\dagger . Unfortunately, this can't work, since the domain/range problems mean there is no guarantee that vectors in the range of b^\dagger will be in its domain, so generating the representation by repeatedly applying b^\dagger won't work. It turns out that the Stone-von Neumann theorem is not true for general Lie algebra representations of \mathfrak{h}_3 , only works for Lie algebra representations that integrate to give a group representation.

To get a proof that does work, one needs to work not with Q, P or a, a^\dagger , but with their exponentiated versions. For details, see [2], chapter 14.

An important example of an irreducible representation unitarily equivalent to the Schrödinger representation is given by using the Fourier transform \mathcal{F}

$$\psi(q) \rightarrow \tilde{\psi}(p) = (\mathcal{F}\psi)(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ipq} \psi(q) dq$$

This is a unitary transformation on $L^2(\mathbf{R})$, with inverse $\tilde{\mathcal{F}}$ given by Fourier inversion

$$\tilde{\psi}(p) \rightarrow (\tilde{\mathcal{F}}\tilde{\psi})(q) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ipq} \tilde{\psi}(p) dp$$

The Stone-von Neumann theorem applies, with $U = \tilde{\mathcal{F}}$, $U^{-1} = \mathcal{F}$.

Note that we will generically refer to the essentially unique representation of the Heisenberg using \mathcal{H} for the representation space and π for the homomorphism from the group to operators on \mathcal{H} , with π' for the Lie algebra representation. When we want to specify a specific construction, the π may acquire a subscript (e.g. π_S for the Schrödinger construction) and \mathcal{H} may get further specified (e.g. $L^2(\mathbf{R})$). Terminology in this subject can be a bit confusing, since instead of the usual multiple representations to keep track of, here there is only one, but with multiple quite different constructions.

3.4 The Bargmann-Fock representation

The Stone-von Neumann theorem also applies to very different constructions of representations on other versions of Hilbert space. In particular, it is clear from looking at the harmonic oscillator calculations that energy eigenstates can be identified with monomials in a complex variable, with a and a^\dagger decreasing and increasing the degree. To find a construction of the Heisenberg group irreducible representation on $\mathbf{C}[w]$, one needs a Hilbert space structure, which one can define as follows:

Definition (Fock Space). *Fock space \mathcal{H}_F is the space of entire functions on \mathbf{C} , with finite norm in the inner product*

$$\langle f(w), g(w) \rangle = \frac{1}{\pi} \int_{\mathbf{C}} \overline{f(w)} g(w) e^{-|w|^2}$$

An orthonormal basis of \mathcal{H}_F is given by appropriately normalized monomials. Since

$$\begin{aligned}\langle w^m, w^n \rangle &= \frac{1}{\pi} \int_{\mathbf{C}} \bar{w}^m w^n e^{-|w|^2} \\ &= \frac{1}{\pi} \int_0^\infty \left(\int_0^{2\pi} e^{i\theta(n-m)} d\theta \right) r^{n+m} e^{-r^2} r dr \\ &= n! \delta_{n,m}\end{aligned}$$

we see that the functions $\frac{w^n}{\sqrt{n!}}$ are orthonormal.

To get a representation of the (complexified) Heisenberg Lie algebra on this space, define

$$a = \frac{d}{dw}, \quad a^\dagger = w$$

Exercise. Show that these operators are each other's adjoints with respect to the inner product on Fock space.

On the real Heisenberg Lie algebra, this representation exponentiates to a representation of the Heisenberg group. By the Stone-von Neumann theorem it is unitarily equivalent to the Schrödinger representation on $L^2(\mathbf{R})$.

To explicitly write the Bargmann-Fock representation of the Heisenberg Lie algebra, one can complexify and work with operators that depend on complex linear combinations of the real basis X, Y, Z . If one does this first in the Schrödinger representation one has

$$\pi'_S(iX) = Q, \quad \pi'_S(iY) = P, \quad \pi'_S(iZ) = \mathbf{1}$$

and so

$$\pi'_S\left(\frac{1}{\sqrt{2}}(iX + i(iY))\right) = a = \frac{1}{\sqrt{2}}\left(q + \frac{d}{dq}\right)$$

(with a similar formula for a^\dagger). To get Bargmann-Fock one wants a π'_{BF} that takes the same linear combinations to $\frac{d}{dw}$ and w , acting on \mathcal{H}_F . Thus

$$\pi'_{BF}\left(\frac{1}{\sqrt{2}}(iX + i(iY))\right) = a = \frac{d}{dw}, \quad \pi'_{BF}\left(\frac{1}{\sqrt{2}}(iX - i(iY))\right) = a^\dagger = w, \quad \pi'_{BF}(iZ) = \mathbf{1}$$

We won't work this out here, but these operators can be exponentiated to get operators for a Heisenberg Lie group representation. By Stone-von Neumann, there will be a unitary operators

$$U : \mathcal{H}_F \rightarrow L^2(\mathbf{R}), \quad U^{-1} : L^2(\mathbf{R}) \rightarrow \mathcal{H}_F$$

These operators are quite non-trivial and interesting in analysis, giving unitary isomorphisms between two very different kinds of function spaces. The explicit form for U^{-1} is often called the Bargmann transform and is given by

$$(U^{-1}\psi)(w) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}w^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}q^2} e^{\sqrt{2}wq} \psi(q) dq$$

The relation between the Schrödinger and Bargmann-Fock operators will be given by

$$U \frac{d}{dw} U^{-1} = \frac{1}{\sqrt{2}} \left(q + \frac{d}{dq} \right), \quad U w U^{-1} = \frac{1}{\sqrt{2}} \left(q - \frac{d}{dq} \right)$$

For more on the Bargmann-Fock representation and the Bargmann transform a good source is Chapter 1, Section 6 of [3].

3.5 The Weyl algebra

A closely related algebra to the Heisenberg Lie algebra is the Weyl algebra, which can be defined as the non-commutative algebra of polynomial coefficient differential operators for a complex variable w . The generators of the algebra are

- Multiplication by w .
- Differentiation by w : $\frac{d}{dw}$

These satisfy the same commutation relations as a, a^\dagger

$$\left[\frac{d}{dw}, w \right] = 1$$

since

$$\frac{d}{dw}(wf) - w \frac{df}{dw} = f$$

Recall that one can think of representations of a Lie algebra \mathfrak{g} as modules for the associative algebra $U(\mathfrak{g})$ (the universal enveloping algebra of \mathfrak{g}). It is convenient here also to complexify, and for any Lie algebra we'll use the notation $U(\mathfrak{g})$ to refer to $U(\mathfrak{g}) \otimes \mathbf{C} = U(\mathfrak{g} \otimes \mathbf{C})$. For the Heisenberg Lie algebra \mathfrak{h}_3 , $U(\mathfrak{h}_3)$ is given by all complex linear combinations of products of basis elements X, Y, Z , modulo the relations

$$[X, Z] = [Y, Z] = 0, \quad [X, Y] = Z$$

The center of $U(\mathfrak{h}_3)$ (denoted here $\mathcal{Z}(\mathfrak{h}_3)$) is the commutative algebra $\mathbf{C}[Z]$ of polynomials in Z . In any irreducible representation π' of a Lie algebra \mathfrak{g} , by Schur's lemma elements of the center $\mathcal{Z}(\mathfrak{g})$ act by scalars. This gives a homomorphism

$$\chi_{\pi'} : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbf{C}$$

called the infinitesimal character of the representation. In the case of $\mathfrak{g} = \mathfrak{h}_3$, since $\mathcal{Z}(\mathfrak{h}_3)$ is an algebra of the polynomial functions in one variable, the infinitesimal character is evaluation of the polynomial at some $c \in \mathbf{C}$. This c is the scalar given by the action of $\pi'(Z)$ on the representation space. The Schrödinger representation as we have defined it is an irreducible representation with $c = -i$.

For general Lie algebra representations of the complexified Lie algebra $\mathfrak{h}_3 \otimes \mathbf{C}$, for each $c \neq 0$ we have the irreducible representation unitarily equivalent to the

Schrödinger representation (rescaled from $c = -i$. These will be unitary for c imaginary.

Z acts by a scalar we'll call $c_{\pi'}$. Polynomials in Z also act by a scalar, the evaluation of the polynomial at $c_{\pi'}$. The Schrödinger representation as we have defined it is an irreducible representation with $c_{\pi'_S} = -i$. Restricting attention to Lie algebra representations for which $\pi'(Z) = c\mathbf{1}$ for a chosen $c \in \mathbf{C}$, these will be modules for the quotient algebra

$$U(\mathfrak{h}_3)/(Z - c)$$

By rescaling X and Y , for $c \neq 0$, we get the Weyl algebra, and so an irreducible Heisenberg algebra representation will be a module for the Weyl algebra. Among these modules is the standard one on polynomials on w , which corresponds to the one we have studying, which is integrable to a unitary Heisenberg group representation. But there are many different modules for the Weyl algebra, with the study of these modules the beginning of the subject of D-modules in algebraic geometry (see for instance [4]).

3.6 The Heisenberg group and symplectic geometry

The three-dimensional Heisenberg group that we have been studying has a simple generalization that behaves in much the same way. For any n , define the $2n+1$ dimensional Heisenberg Lie algebra \mathfrak{h}_{2n+1} to be the Lie algebra with basis X_j, Y_j, Z ($j = 1, 2, \dots, n$) and all Lie brackets zero except

$$[X_j, Y_k] = \delta_{jk}Z$$

One can easily get a corresponding Heisenberg Lie group H_{2n+1} generalizing the $n = 1$ case by exponentiating.

Instead of working with a basis like this, one can define this Lie group in a more coordinate-invariant way, starting with any symplectic form on $M = \mathbf{R}^{2n}$ (which will have a physical interpretation as a phase space), where

Definition (Symplectic form). *A symplectic form Ω on a vector space M is a non-degenerate anti-symmetric bilinear form*

$$(v_1, v_2) \in M \times M \rightarrow \Omega(v_1, v_2) \in \mathbf{R}$$

on M .

This is the same definition as that of an inner product on a vector space V , with “symmetric” replaced by “antisymmetric.” For any even-dimensional real vector space M with a symplectic form Ω , one can define a Lie algebra structure on $M \oplus \mathbf{R}$ by taking the Lie bracket to be

$$[(v, z), (v', z')] = (0, \Omega(v, v'))$$

where (v, z) are elements of $M \oplus \mathbf{R}$. One gets a corresponding Lie group by taking as group law on $M \oplus \mathbf{R}$

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\Omega(v, v'))$$

In the inner product case, by Gram-Schmidt orthonormalization one can always find an orthonormal basis of V , with any other basis related to this one by an element of $GL(V)$. The subgroup of $GL(V)$ preserving the inner product and thus taking orthonormal bases to orthonormal bases is the orthogonal group $O(V)$. In the symplectic case, M has to be even-dimensional (to have a non-degenerate Ω).

Exercise. Show that one can always find a “symplectic basis”: X_j and Y_j for $j = 1, 2, \dots, n$ satisfying

$$\Omega(X_j, X_k) = \Omega(Y_j, Y_k) = 0, \quad \Omega(X_j, Y_k) = \delta_{jk}$$

and that in this basis one recovers the earlier definition of the Heisenberg Lie algebra and Lie group of dimension $2n + 1$.

The subgroup of $GL(M)$ preserving Ω and taking symplectic bases to symplectic bases is by definition the symplectic group $Sp(M)$. Choosing a basis, this group will be a matrix group that can be denoted $Sp(2n, \mathbf{R})$. Note that this is different than the group often written as $Sp(n)$, the group of n by n quaternionic matrices preserving the standard hermitian form on \mathbf{H}^n . The groups $Sp(n)$ and $Sp(2n, \mathbf{R})$ are different real forms of the group $Sp(2n, \mathbf{C})$ of linear transformations preserving a non-degenerate anti-symmetric bilinear form on \mathbf{C}^{2n} .

4 The symplectic group and the oscillator representation

The irreducible representation of the Heisenberg group we have been studying provides a projective representation of the symplectic group. This has various names, of which we’ll choose Roger Howe’s “oscillator representation.” For more details, a good source is [3].

4.1 The symplectic group and automorphisms of the Heisenberg Lie group

Since the definition of the Heisenberg Lie algebra and Lie group only depend on the antisymmetric bilinear form Ω on $M = \mathbf{R}^{2n}$, the group $Sp(2n, \mathbf{R})$ of linear maps preserving Ω acts on this Lie algebra and group as automorphisms. Using $(v, z) \in V \oplus \mathbf{R}$ as coordinates on H_{2n+1} , the action of $g \in Sp(2n, \mathbf{R})$ on the Heisenberg group is

$$\Phi_g(v, z) = (gv, z)$$

Using this automorphism, one can construct the semi-direct product

$$H_{2n+1} \rtimes Sp(2n, \mathbf{R})$$

which is sometimes called the “Jacobi group.”

We also can use these automorphisms to act on the set of representations of H_{2n+1} , taking

$$\pi \rightarrow \pi_g$$

where

$$\pi_g(v, z) = \pi(\Phi_g(v, z))$$

If π is irreducible, π_g will also be irreducible, and by the Stone-von Neumann theorem there will be unitary operators U_g such that

$$\pi_g = U_g \pi_S U_g^{-1}$$

By Schur's lemma, these operators will be unique up to a phase factor. They will then provide a representation of $Sp(2n, \mathbf{R})$ up to a phase factor (a projective representation)

$$U_{g_1} U_{g_2} = e^{i\theta(g_1, g_2)} U_{g_1 g_2}$$

By changing the U_g by a phase factor

$$U_g \rightarrow V(g) = e^{i\phi(g)} U(g)$$

one can try and remove the projective factor from the multiplication law. It turns out though that this can only be done up to sign, a problem much like that which occurs in the case of the spin representation of the rotation group. As in the case of the rotation group, one can get a true representation by going to a double cover of $Sp(2n, \mathbf{R})$, which we'll denote $Mp(2n, \mathbf{R})$ and call the "metaplectic group." Two differences from the rotation group case are:

- In the rotation group case $\pi_1(SO(n)) = \mathbf{Z}_2$ and the double cover $Spin(n)$ is the universal cover. In the symplectic case $\pi_1(Sp(2n, \mathbf{R})) = \mathbf{Z}$ and the metaplectic double cover is just one of many possible covering groups.
- $Spin(n)$ can be identified with a group of finite-dimensional matrices. This is not true for $Mp(2n, \mathbf{R})$, a group which has no finite-dimensional faithful representations. It provides a very unusual example of where thinking of Lie theory just in terms of matrix groups is inadequate.

We will refer to the representation of $Mp(2n, \mathbf{R})$ as the "oscillator representation (it goes by many other names, including Weil representation, Segal-Shale-Weil representation, etc.). The representation will be on the same space \mathcal{H} as the Schrödinger representation, extending the action of the Heisenberg Lie group, so we will often denote it by the same symbol π_S and also call the representation of the Heisenberg group by the same name. We will also describe this representation as being "essentially unique", meaning that all versions of it are the same up to unitary transformations, possible rescaling, and differences in the definition of \mathcal{H} related by dense inclusions.

4.2 The Poisson bracket and the Lie algebras \mathfrak{h}_{2n+1} and $\mathfrak{sp}(2n, \mathbf{R})$

As is usual in the subject of Lie groups and their representations, it is much easier to work with a Lie algebra and its representation than with the corresponding Lie group and its representation. This is especially true in the case we are now considering: the semi-direct product of the Heisenberg and metaplectic groups and the oscillator representation. One aspect of the problem is that the metaplectic group is not a matrix group and thus hard to describe explicitly. On the other hand, the Lie algebra of the metaplectic group is the same as the Lie algebra of the symplectic group and can be described by matrices.

It is however often much more convenient to work with a realization of the Lie algebra not as matrices, but as low degree polynomial functions. The Lie bracket is then the Poisson bracket on functions, which we'll now describe. We take $P = \mathbf{R}^{2n}$, with coordinates q_j, p_k for $j, k = 1, 2, \dots, n$. Then

Definition (Poisson bracket). *The Poisson bracket of two functions f_1, f_2 on P is the function*

$$\{f, g\} = \sum_{j=1}^n \left(\frac{\partial f_1}{\partial q_j} \frac{\partial f_2}{\partial p_j} - \frac{\partial f_2}{\partial q_j} \frac{\partial f_1}{\partial p_j} \right)$$

In the Hamiltonian form of classical mechanics, P will be the phase space and functions on P observables. There will be a distinguished function, the Hamiltonian h , which determines the dynamics, with time dependence of observables given by Hamilton's equation:

$$\frac{df}{dt} = \{f, h\}$$

The Poisson bracket can easily be seen to satisfy the following properties:

- Anti-symmetry:

$$\{f_1, f_2\} = -\{f_2, f_1\}$$

- Jacobi identity:

$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$$

- Leibniz rule (derivation property)

$$\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2 \{f_1, f_3\}$$

The first two properties imply that the Poisson bracket provides a Lie algebra structure on the space of functions on P . This is an infinite-dimensional Lie algebra. The corresponding infinite dimensional group is the subgroup of all diffeomorphisms of \mathbf{R}^{2n} that preserve the standard symplectic form ("symplectomorphisms").

The Leibniz rule implies that, at least for polynomial functions, the Poisson bracket is determined by what it does on linear functions, where

$$\{q_j, q_k\} = \{p_j, p_k\} = 0, \quad \{q_j, p_k\} = \delta_{jk}$$

These are just the Lie bracket relations for the $2n + 1$ -dimensional Heisenberg Lie algebra \mathfrak{h}_{2n+1} . Thinking of the q_j, p_k as basis elements of P^* , we have a Lie algebra structure on

$$\mathbf{R} \oplus P^*$$

the polynomial functions of degree less than or equal to one on P . As we have seen earlier, a more basis-independent point of view is that we have a symplectic form Ω on P^* , with q_j, p_k the basis of P^* that puts Ω in standard form.

From now on we will refer to P^* as M . The Poisson bracket on linear functions on P gives a symplectic form Ω on $M = P^*$, and the Lie algebras \mathfrak{h}_{2n+1} and $\mathfrak{sp}(2n, \mathbf{R})$ are defined in terms of M , not P . See appendix A for a general discussion of quantization and general symplectic manifolds.

The space of degree two monomials on P has a basis elements $q_j p_k$ for all j, k and $q_j q_k, p_j p_k$ for $j \leq k$. The Poisson bracket of two of these is a linear combination of degree two monomials, so these provide a real Lie algebra of dimension $2n^2 + n$.

Exercise. *Show that this is isomorphic to the Lie algebra $\mathfrak{sp}(2n, \mathbf{R})$*

Here we will work out explicitly what happens for $n = 1$. The Heisenberg Lie algebra \mathfrak{h}_3 has a basis $q, p, 1$ with only non-zero Lie bracket

$$\{q, p\} = 1$$

The symplectic Lie algebra $\mathfrak{sp}(2, \mathbf{R})$ has basis q^2, p^2, qp with non-zero Lie brackets

$$\left\{ \frac{q^2}{2}, \frac{p^2}{2} \right\} = qp, \quad \{qp, p^2\} = 2p^2, \quad \{qp, q^2\} = -2q^2$$

This is isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbf{R})$ of 2 by 2 traceless real matrices, with bracket the commutator, where a conventional basis is

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The isomorphism is explicitly given by

$$\frac{q^2}{2} \leftrightarrow E, \quad -\frac{p^2}{2} \leftrightarrow F, \quad -qp \leftrightarrow G$$

or by

$$-aqp + \frac{bq^2}{2} - \frac{cp^2}{2} \leftrightarrow \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

The semi-direct product of H_3 and $SL(2, \mathbf{R})$ puts the above two Lie algebras together, with the action of $SL(2, \mathbf{R})$ on H_3 by automorphisms reflected in the non-zero Lie brackets

$$\{qp, q\} = -q, \quad \{qp, p\} = p$$

$$\left\{\frac{p^2}{2}, q\right\} = -p, \quad \left\{\frac{q^2}{2}, p\right\} = q$$

From these relations one can see that

$$-qp \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

generates a group \mathbf{R} acting on the q direction in the qp plane by e^t , on the p direction by e^{-t} . The element

$$\frac{1}{2}(q^2 + p^2) \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generates an $SO(2)$ subgroup of rotations in the qp plane.

4.3 The Schrödinger model for the oscillator representation

We have seen that the Schrödinger representation is given as a representation of \mathfrak{h}_3 by the operators

$$\pi'_S(q) = -iQ = -iq, \quad \pi'_S(p) = -iP = -\frac{d}{dq}, \quad \pi'_S(1) = -i\mathbf{1}$$

Dirac's original definition of "quantization" asked for an extension of this representation from linear functions to all functions on phase space, i.e. a choice of operators that would take any polynomial in q and p to an operator, with Poisson bracket of functions going to commutator of operators, so a Lie algebra homomorphism. But going from functions of q and p to operators built out of Q and P , one runs into "operator-ordering" ambiguities since Q and P do not commute. It turns out that one can get a Lie algebra homomorphism for polynomials up to degree two, but this is impossible in higher degree (Groenewold-van Hove theorem).

What works in degree two is to extend the Schrödinger representation to a representation of $\mathfrak{sl}(2, \mathbf{R})$ (and of the semi-direct product with \mathfrak{h}_3) by taking

$$\pi'_S(q^2) = -iQ^2 = -iq^2, \quad \pi'_S(p^2) = -iP^2 = i\frac{d^2}{dq^2}$$

and making the choice

$$\pi'_S(qp) = -i\frac{1}{2}(QP + PQ) = -i\frac{1}{2}(2QP - i\mathbf{1}) = -q\frac{d}{dq} - \frac{1}{2}\mathbf{1}$$

(which gives a skew-adjoint operator).

These operators will satisfy the commutation relations given by the Lie bracket of $\mathfrak{sl}(2, \mathbf{R})$, so give a representation, which is the oscillator representation. (it has many other names, including the "Weil representation"). The

representation will be on the same space as the Schrödinger representation, extending the action of the Heisenberg Lie algebra, so we will often denote it by the same symbol π_S .

One would like to exponentiate the Lie algebra representation operators to get a representation of the Lie group $SL(2, \mathbf{R})$. In the case of $\pi'_S(qp)$ the operator exponentiates to an operator on functions which rescales in the q variable. It is though not so easy to exponentiate the second order differential operator

$$-iP^2 = i \frac{d^2}{dq^2}$$

If one takes a Fourier transform to turn derivatives in q into multiplication operators, the problem just moves to the operator $-iQ^2$ which changes from a multiplication operator to a second-order differential operator.

The problem is best thought of as having to do with exponentiating the Lie algebra element

$$\frac{1}{2}(q^2 + p^2)$$

which generates the $SO(2) \subset SL(2, \mathbf{R})$ subgroup of rotations in the qp plane. So, for the oscillator representation, we need to explicitly construct the operator

$$e^{\theta \pi'_S(\frac{1}{2}(q^2 + p^2))}$$

where

$$\pi'_S\left(\frac{1}{2}(q^2 + p^2)\right) = -i\frac{1}{2}(Q^2 + P^2) = -i\frac{1}{2}\left(q^2 - \frac{d^2}{dq^2}\right)$$

Changing notation from θ to t , this is just the standard physics problem of solving the Schrödinger equation for the Hamiltonian $H = \frac{1}{2}(Q^2 + P^2)$ and so constructing the unitary operator

$$U(t) = e^{-it\frac{1}{2}(Q^2 + P^2)} \quad (3)$$

With some effort (see for instance exercises 4 and 5 of chapter III of [5]), one can derive a formula for the kernel $K_t(q, q')$ (known in physics as the “propagator”) where

$$(U(t)\psi)(q) = \int_{\mathbf{R}} K_t(q, q')\psi(q')dq'$$

One finds

$$K_t(q, q') = \frac{1}{\sqrt{2\pi \sin t}} \exp\left(-\frac{1}{2}\begin{pmatrix} q & q' \end{pmatrix} \begin{pmatrix} \frac{\cos t}{\sin t} & -\frac{1}{\sin t} \\ -\frac{1}{\sin t} & \frac{\cos t}{\sin t} \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix}\right) \quad (4)$$

This expression requires interpretation as a distribution defined as a boundary value of a holomorphic function, replacing t by $t - i\epsilon$ and taking the limit as positive ϵ vanishes.

One can show that

$$\lim_{\epsilon \rightarrow 0^+} U\left(\frac{\pi}{2} - i\epsilon\right) = e^{i\frac{\pi}{4}} \mathcal{F}$$

This is the oscillator representation operator for an element of the symplectic group corresponding to a $\frac{\pi}{2}$ rotation in the q, p plane, interchanging the role of q and p . As expected from the Stone-von Neumann theorem, one gets the Fourier transform, up to a phase factor. The calculation of the propagator fixes the phase factor. In some sense, rotations by arbitrary values of t will give “fractional Fourier transforms.”

Rotation by π in the q, p plane is given by

$$i\mathcal{F}^2$$

The \mathcal{F}^2 is as expected since \mathcal{F}^2 acts on functions by

$$\psi(q) \rightarrow \mathcal{F}^2\psi(q) = \psi(-q)$$

corresponding to a rotation by π taking q to $-q$. Rotation by 2π is given by $-\mathcal{F}^4 = -\mathbf{1}$ rather than the $\mathbf{1}$ expected if $U(t)$ is to be a true (rather than up to ± 1) representation of $SO(2) \subset SL(2, \mathbf{R})$. This is a precise analog of what happens when we take the spinor Lie algebra representation of $SO(3)$ and exponentiate: we find that rotating around an axis by 2π gives a factor of -1 . The representation is only a projective (up to sign) representation of $SO(3)$. To get a true representation, one needs the double cover $Spin(3) = SU(2)$. Here again we have a representation up to sign and need a double cover of $Sp(2, \mathbf{R})$. This will be the metaplectic group $Mp(2, \mathbf{R})$, which is not a matrix group.

4.4 The Bargmann-Fock model for the oscillator representation

The best way to calculate the phase factors in the exponentiated version of the oscillator representation is not to use the Schrödinger version of the representation and the complicated formula 4 for the propagator, but to instead use the Bargmann-Fock version. Here the representation is on the space of polynomials $\mathbf{C}[w]$ (with the Bargmann-Fock inner product) and the operators

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{d}{dw}, \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = w$$

provide a representation of the complexified Heisenberg Lie algebra (which is the standard one on the real Lie algebra).

As in the Schrödinger case, one can extend this representation to the oscillator representation of $\mathfrak{sp}(2n, \mathbf{R})$ by taking quadratic combinations of the Heisenberg Lie algebra operators. In particular, using

$$\frac{1}{2}(Q^2 + P^2) = \frac{1}{2}(a^\dagger a + a a^\dagger) = a^\dagger a + \frac{1}{2}$$

one has (writing elements of $\mathfrak{sl}(2, \mathbf{R})$ both as quadratic polynomials and as matrices)

$$\pi'_{BF}\left(\frac{1}{2}(q^2 + p^2)\right) = \pi'_{BF}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = -i(a^\dagger a + \frac{1}{2}) = -i\left(w\frac{d}{dw} + \frac{1}{2}\right)$$

This operator can easily be exponentiated:

$$e^{\theta \pi'_{BF}(\frac{1}{2}(q^2+p^2))}$$

acts on $\mathbf{C}[w]$ by multiplying the monomial w^n by $e^{-i\theta(n+\frac{1}{2})}$. This gives the minus sign previously discussed for $\theta = 2\pi$.

In this representation the other two basis elements of $\mathfrak{sl}(2, \mathbf{R})$ are

$$\pi'_{BF}(-qp) = \pi'_{BF}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = -\frac{1}{2}((a^\dagger)^2 - a^2)$$

$$\pi'_{BF}(q^2 - p^2) = \pi'_{BF}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = -\frac{i}{2}((a^\dagger)^2 + a^2)$$

Note that these operators do not change the parity of monomials they act on, and you can get from any monomial of a given parity to any other of the same parity by applying these operators repeatedly. So, the oscillator representations we have constructed here is the sum of two irreducibles (all polynomials of even degree, and all polynomials of odd degree).

5 Choice of polarization

5.1 Real polarizations and the Schrödinger representation

From the discussion above, M can be written as

$$M = L \oplus L^*$$

where L is an n -dimensional vector space with basis X_j and L^* is the dual vector space with basis elements Y_j dual to the X_j (i.e. $Y_j(X_k) = \delta_{jk}$). Note that for any vectors $x, x' \in L \subset M$ one has $\Omega(x, x') = 0$. A subspace with this property is called “isotropic”. The maximal dimension of a subspace of M on which Ω is zero is n , and such isotropic subspaces are called “Lagrangian”. L^* is also Lagrangian.

Since the definition of the Heisenberg Lie algebra and Lie group depend only on the symplectic form S , and by Stone-von Neumann there is only one irreducible representation, one might expect that the definition of this irreducible representation should depend just on S . It turns out though that all constructions of this representation depend upon a choice of additional structure. We have seen that the construction of the Schrödinger representation depends on a choice of n position coordinates q_j , corresponding to the basis elements X_j of the Lie algebra, which span a Lagrangian subspace of \mathbf{R}^{2n} . The Fourier transform takes this construction to a different one, depending on n momentum coordinates p_j , corresponding to the basis elements Y_j of the Lie algebra, which span a complementary Lagrangian subspace of \mathbf{R}^{2n} .

More generally, one can construct a version of the Schrödinger representation for any choice of Lagrangian subspace $\ell \subset \mathbf{R}^{2n}$. By the Stone-von Neumann

theorem, for each ℓ there will be an operator U_ℓ giving a unitary equivalence with the construction for the standard Schrödinger choice of $\ell = L$ spanned by the X_j . For $\ell = L^*$ spanned by the Y_j , U_ℓ will be the Fourier transform, but for more general ℓ its construction is rather non-trivial. A choice of a Lagrangian ℓ and thus a decomposition $M = \ell \oplus \ell^*$ is called a “real polarization” of M .

Exercise. *Show that the choices of Lagrangian subspace ℓ are parametrized by the space $U(n)/O(n)$.*

For the case $n = 1$, $U(1)/O(1) = \mathbf{R}P^1$, which is a circle, so real polarizations ℓ are parametrized by an angle θ . The operators U_ℓ are the operators $U(\theta)$ of equation 3, going once around $\mathbf{R}P^1$ as θ goes from 0 to π .

5.2 Complex polarizations

The Bargmann-Fock construction involves a different sort of polarization, called a “complex polarization.” Here one complexifies M and asks for Lagrangian subspaces W and \overline{W} such that

$$M \otimes \mathbf{C} = W \oplus \overline{W}$$

where W and \overline{W} are interchanged by the conjugation map on \mathbf{C} .

Such a decomposition is equivalent to the choice of a compatible complex structure on M , where

Definition (Complex structure). *A complex structure on a real vector space M is a (real)-linear map*

$$J : M \rightarrow M$$

satisfying $J^2 = -\mathbf{1}$.

and

Definition (Compatible complex structure). *A complex structure on M is compatible with a symplectic form Ω on M when*

$$\Omega(Jv_1, Jv_2) = \Omega(v_1, v_2)$$

Such J only exist if the dimension of M is even and one can think of them as ways of making M a complex vector space (so identifying $\mathbf{R}^{2n} = \mathbf{C}^n$), with multiplication by i given by J . J has no eigenvectors in M , but it does have complex eigenvalues $\pm i$, giving a decomposition

$$V \otimes \mathbf{C} = M_J^+ \oplus M_J^-$$

into $\pm i$ eigenspaces for J . This will be a polarization of M when J is compatible with Ω since then M_J^+ and M_J^- are Lagrangian subspaces. To see this, note that for $w_1, w_2 \in V_J^+$

$$S(w_1, w_2) = S(Jw_1, Jw_2) = S(iw_1, iw_2) = -S(w_1, w_2)$$

so must be zero.

Given both a symplectic form Ω and a compatible complex structure J on M , M becomes not just a complex vector space, but a complex vector space with Hermitian inner product, defined by

$$\langle v_1, v_2 \rangle_J = \Omega(v_1, Jv_2) + i\Omega(v_1, v_2)$$

One can easily check that this is Hermitian, but it is not necessarily positive. To get a positive Hermitian structure one needs to impose an additional condition on J , that, for non-zero $v \in M$ one has

$$\Omega(v, Jv) > 0$$

The possible choices of general complex structure J are parametrized by

$$GL(2n, \mathbf{R})/GL(n, \mathbf{C})$$

The compatibility condition implies that $J \in Sp(2n, \mathbf{R})$.

Exercise. Show that the space of possible positive complex structures compatible with Ω is $Sp(2n, \mathbf{R})/U(n)$. This is called the Siegel upper half space.

5.2.1 The $n = 1$ case

For the case $n = 1$, the geometry of the space $SL(2, \mathbf{R})/U(1)$ is best understood in terms of the geometry of $\mathbf{C}P^1$, the space of complex lines in \mathbf{C}^2 . This is also the best way to understand the holomorphic line bundles on $SL(2, \mathbf{R})/U(1)$ and how representations of $SL(2, \mathbf{C})$ and its subgroups can be constructed geometrically (see Appendix A).

$SL(2, \mathbf{C})$ acts linearly on \mathbf{C}^2 and transitively on the the space $\mathbf{C}P^1$. The space $\mathbf{C}P^1$ is a complex manifold, the Riemann version of the sphere S^2 , and the action of $SL(2, \mathbf{C})$ is holomorphic and thus an action by conformal transformations. One can choose the coordinate of the line in \mathbf{C}^2 generated by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

to be $z = z_1/z_2$. This gives a good coordinate system away from one point, that of the line generated by $z_1 = 1, z_2 = 0$.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbf{C})$$

acts on this coordinate by the fractional linear transformation

$$z \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot z = \frac{\alpha z + \beta}{\gamma z + \delta}$$

The subgroup $SL(2, \mathbf{R})$ of real matrices acts in the coordinate z preserving the sign of $\text{Im}z$ and so does not act transitively. There are three orbits of the

action: the upper and lower open half planes, and the real line. On \mathbf{CP}^1 , the three orbits are two open hemispheres and the equator separating them. The correspondence of the three orbits in the z coordinate with the three orbits on \mathbf{CP}^1 is that the point where z is not a good coordinate is on the equator orbit, and approached as one goes off to infinity in any direction in the z -plane.

Picking the point $z = i$ in the upper half plane, the subgroup of elements of $SL(2, \mathbf{R})$ of elements stabilizing the point is the an $SO(2) = U(1)$ subgroup given by elements of the form

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

We can identify the upper half plane (which we'll denote \mathcal{H}) with $SL(2, \mathbf{R})/U(1)$.

The Cayley transform

$$z \rightarrow z' = \frac{z - i}{z + i}$$

takes the upper half plane to the unit disk. Conjugating an element of $SL(2, \mathbf{R})$ by this transformation gives a matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

where α, β are complex numbers satisfying $|\alpha|^2 - |\beta|^2 = 1$. Such matrices give the subgroup $SU(1, 1)$ of $SL(2, \mathbf{C})$ preserving a $(1, 1)$ signature Hermitian form. It is isomorphic to $SL(2, \mathbf{R})$ by the conjugation map. At each point in the open unit disk, $SU(1, 1)$ acts with stabilizer a $U(1)$ subgroup. The Cayley transform takes $z = i$ to $z' = 0$, which is stabilized by elements of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

The subgroup of such elements acts by rotation of the unit disk about its center.

In this $n = 1$ case, changing complex polarization corresponds to changing the linear combinations of Q and P that define annihilation and creation operators. One gets an analog of the Bargmann-Fock construction for any $\tau \in \mathbf{C}$ with positive imaginary part by changing

$$a = \frac{1}{\sqrt{2}}(Q - iP) \rightarrow a_\tau = \frac{1}{\sqrt{2}} \frac{|\tau|}{\sqrt{\text{Im } \tau}} (Q - \frac{1}{\tau}P)$$

$$a^\dagger = \frac{1}{\sqrt{2}}(Q + iP) \rightarrow a_\tau^\dagger = \frac{1}{\sqrt{2}} \frac{|\tau|}{\sqrt{\text{Im } \tau}} (Q - \frac{1}{\tau}P)$$

a_τ and a_τ^\dagger are adjoint operators satisfying the commutation relation

$$[a_\tau, a_\tau^\dagger] = 1$$

and the representation is constructed by starting with a distinguished vector annihilated by a_τ and generating the rest of the representation by applying powers of a_τ^\dagger .

The unitary transformation to the Schrödinger representation will then take the distinguished vector to a solution of

$$a_\tau \psi(q) = \frac{1}{\sqrt{2}} \frac{|\tau|}{\sqrt{\text{Im } \tau}} (Q - \frac{1}{\tau} P) \psi(q) = \frac{1}{\sqrt{2}} \frac{|\tau|}{\sqrt{\text{Im } \tau}} (q + \frac{i}{\tau} \frac{d}{dq}) \psi(q) = 0$$

Solutions will be proportional to

$$\psi(q) = e^{\frac{i}{2} \tau q^2}$$

and normalizable for $\text{Im } \tau > 0$.

To visualize the entire space of possible choices of polarization that give constructions of the oscillator representation for $n = 1$, one should think of the unit disk, with interior points corresponding to complex polarizations and the Bargmann-Fock construction for different τ given above. As one approaches the boundary, the distinguished vectors annihilated by a_τ become non-normalizable and leave the space $L^2(\mathbf{R})$ (they will still be distributions in $\mathcal{S}'(\mathbf{R})$).

For more details and to see how this picture generalizes to $n \geq 1$, see Graeme Segal's notes on *Symplectic manifolds and quantization* [6].

6 Representations and holomorphic line bundles

While the oscillator representation is essentially unique, any construction of the representation requires specification of an additional structure. For complex polarizations, this additional structure is a complex subspace

$$W \subset M \otimes \mathbf{C} \subset \mathfrak{h}_{2n+1} \otimes \mathbf{C}$$

There are corresponding operators on the oscillator representation, the annihilation operators, and a distinguished vector

$$|0\rangle_\tau \subset \mathcal{H}$$

annihilated by these operators. Here the notation reflects that in the physical interpretation in which annihilation and creation operators annihilate and create quanta, this is the state with zero quanta. The subscript τ in general labels points in the Siegel upper half space. In the $n = 1$ case, τ is a complex number with positive imaginary part, and in the Schrödinger representation one has explicitly

$$|0\rangle_\tau = e^{\frac{i}{2} \tau q^2}$$

More precisely, what the choice of τ picks out is the one-dimensional complex line in \mathcal{H} generated by $|0\rangle_\tau \subset \mathcal{H}$. The space of these complex lines gives a complex line bundle \mathcal{L} over $Sp(2n, \mathbf{R})/U(n)$. This is not quite an $Sp(2n, \mathbf{R})$ equivariant line bundle (i.e. with an action of $Sp(2n, \mathbf{R})$ on \mathcal{L} that projects to the action by left multiplication on the base $Sp(2n, \mathbf{R})$), since the only the double cover $Mp(2n, \mathbf{R})$ of $Sp(2n, \mathbf{R})$ acts on \mathcal{H} . \mathcal{L} is a $Mp(2n, \mathbf{R})$ equivariant

bundle over the Siegel upper half space, described as $Mp(2n, \mathbf{R})/\widetilde{U}(n)$ where $\widetilde{U}(n)$ is a double-cover of $U(n)$.

In the case $n = 1$, recall that $SL(2, \mathbf{C})$ acts on $\mathbf{C}P^1$, with a subgroup $SU(2)$ acting transitively, identifying $SU(2)/U(1) = \mathbf{C}P^1$. Two ways to form $SU(2)$ equivariant line bundles over $\mathbf{C}P^1$ are

- Consider the product

$$SU(2) \times \mathbf{C}$$

and quotient by the action

$$(g, w) \rightarrow (gh_\theta, e^{ik\theta}w)$$

of $U(1)$, where

$$h_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

This will give a line bundle we'll call L^k , with sections

$$\Gamma(L^k) = \{\phi : SU(2) \rightarrow \mathbf{C}, \phi(gh_\theta) = e^{ik\theta}\phi(g)\}$$

- Since a point in $\mathbf{C}P^1$ is a complex line, one tautologically gets a line bundle (the fiber above a point is the point).

One can show that the tautological line bundle is isomorphic with L^{-1} , which is the dual line bundle to $L = L^1$. The bundle of holomorphic one-forms $\mathbf{C}P^1$ is the line bundle L^2 , so in some sense L is the bundle of 1/2-forms (the spinors in this two-dimensional geometry).

The L^k for all $k \in \mathbf{Z}$ can be thought of as tensor powers of L or L^{-1} . It turns out that these are holomorphic line bundles and one can consider their holomorphic sections

$$\Gamma_{hol}(L^k)$$

(in the algebraic geometer's notation, this is $H^0(\mathbf{C}P^1, \mathcal{O}(k))$). The action of $SU(2)$ takes holomorphic sections to holomorphic sections and one finds (this is a simple example of the Borel-Weil theorem) that

$$\Gamma_{hol}(L^k) = \begin{cases} V^k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

where V^k is the irreducible representation of $SU(2)$ of dimension $k + 1$ (in physicist's language, the spin $\frac{k}{2}$ representation).

For the subgroup $SU(1, 1) \subset SL(2, \mathbf{C})$, the story is quite different, since the action of $SU(1, 1)$ on $\mathbf{C}P^1$ is not transitive. Instead there are three orbits: two hemispheres and the equator between them. On a hemisphere D , one can use the same definition of the line bundle L^k as a quotient given above (replacing $SU(2)$ by $U(1)$) and get an irreducible representation of $SU(1, 1)$ on $\Gamma_{hol}(L^k)$ but this space of sections is now infinite dimensional. This representation will

be the discrete series representation D_k^+ . Say something about what happens for positive and negative k , what the weights are. Problem is unitarity.

The even irreducible component of the oscillator representation can be realized as holomorphic sections of the line bundle \mathcal{L} , and one can show that $\mathcal{L} \otimes \mathcal{L} = L$. \mathcal{L} is a square root of L , and a fourth-root of the holomorphic one-forms.

On the subset $D \subset \mathbf{C}P^1$ the line bundle L is the trivial bundle $D \times \mathbf{C}$, so one can choose coordinates on D and work with the first description of L described above, in which sections are holomorphic functions on D . L is an equivariant bundle under the action of $SL(2, \mathbf{R})$ and one wants to choose coordinates that transform simply under $SL(2, \mathbf{R})$.

We'll denote the upper-half-plane by \mathfrak{H} . The point $z = i \in \mathfrak{H}$ is stabilized by the subgroup $SO(2) \subset SL(2, \mathbf{R})$ of elements of the form

$$k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and one can identify $SL(2, \mathbf{R})/SO(2) = \mathfrak{H}$. The line bundle L^k is $\mathfrak{H} \times \mathbf{C}$, with coordinates (z, w) and an action of $SL(2, \mathbf{R})$ by

$$(z, w) \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, w) = \left(\frac{az + b}{cz + d}, (cz + d)^k w \right)$$

Holomorphic sections in $\Gamma_{hol}(L^k)$ are given by holomorphic functions $f : \mathfrak{H} \rightarrow \mathbf{C}$ with $SL(2, \mathbf{R})$ acting by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) = (cz + d)^k f\left(\frac{az + b}{cz + d}\right)$$

This needs to be fixed, need to get inverses straight

The discrete series of $SL(2, \mathbf{R})$ as holomorphic sections of L^k . Discussion of the inner product, including the version that generalizes to twistors, see chapter 10 of [7].

Instead of working with sections as holomorphic functions f on the base, one can work with equivariant functions ϕ_f on the bundle. Show how to go back and forth.

Discuss the metaplectic double cover, and the oscillator representation.

More about the analogy with the conformal group in 4d?

7 Spinor-oscillator analogy

The oscillator representation of a symplectic group that we have been discussing is closely analogous to the spinor representation of the orthogonal group. Here we'll make this analogy very explicit. This parallelism is well-known in physics, where the "canonical formalism" in quantum mechanics comes in both a "bosonic" version, with canonical commutation relations, and a "fermionic" version, with canonical anti-commutation relations. Much of this material is worked out in great detail in [1].

7.1 Classical theory, Lie groups and Lie algebras

Q : Symmetric non-degenerate bilinear form on $V = \mathbf{R}^d$	Ω : Antisymmetric non-degenerate bilinear form on $M = \mathbf{R}^{2n}$
Lie group $SO(d)$ preserving Q , with Lie algebra $\mathfrak{so}(d)$.	Lie group $Sp(2n, \mathbf{R})$ preserving Ω , with Lie algebra $\mathfrak{sp}(2n)$
$\pi_1(SO(d)) = \mathbf{Z}_2$.	$\pi_1(Sp(2n), \mathbf{R}) = \mathbf{Z}$.
$Spin(d)$, double cover of $SO(d)$.	$Mp(2n, \mathbf{R})$, double cover of $Sp(2n, \mathbf{R})$.
$\Lambda^*(V)$: anti-symmetric algebra on V . Polynomials in “anti-commuting variables ξ_j , $j = 1, 2, \dots, d$. For physicists these are “fermionic” variables.	$S^*(M)$: symmetric algebra on M . Polynomial functions on M^* . Generated by a basis q_j, p_k , $j, k = 1, 2, \dots, n$ of M . For physicists these are “bosonic” variables.
Poisson bracket $\{\cdot, \cdot\}_+$. Lie bracket for Lie super-algebra of “anti-commuting functions” on V^* , determined by Q .	Poisson bracket $\{\cdot, \cdot\}$. Lie bracket for Lie algebra of functions on M^* , determined by Ω .

$$\{v_1, v_2\}_+ = Q(v_1, v_2)$$

Lie superalgebra of anticommuting polynomials on V^* of degree 0, 1, 2. Semi-direct product of a Lie superalgebra (degree 0 and 1) and the orthogonal Lie algebra $\mathfrak{so}(d, \mathbf{R})$ (degree 2).	Lie algebra of polynomials on M^* of degree 0, 1, 2. Semi-direct product of the Heisenberg Lie algebra \mathfrak{h}_{2n+1} (degree 0 and 1) and the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbf{R})$ (degree 2).
Pseudo-classical mechanics.	Classical mechanics.

7.2 Quantum theory and representations

Spin representation S (unitary) on a complex vector space of dimension 2^n for $d = 2n$ even.	Oscillator representation (unitary) on \mathcal{H} , an infinite-dimensional Hilbert space.
Clifford algebra $\text{Cliff}(d, \mathbf{C})$. For $d = 2n$ even this is the algebra $\text{End}(S)$, isomorphic to the matrix algebra $M(2^n, \mathbf{C})$.	Weyl algebra $U(\mathfrak{h}_{2n+1})/(Z - 1)$. This algebra is infinite-dimensional over \mathbf{C} .
The group $SO(d)$ acts by automorphisms on $\text{Cliff}(d, \mathbf{C})$.	The group $Sp(2n, \mathbf{R})$ acts by automorphism on the Weyl algebra.
For $d = 2n$ even, $\text{Cliff}(2n, \mathbf{C})$ has a unique irreducible module, the spin module S . This is the spin representation as a Lie algebra representation of $\mathfrak{so}(2n)$. Integrating to the group, one gets a	Stone-von Neumann theorem: the Weyl algebra has an essentially unique irreducible module \mathcal{H} that integrates to a representation of the Heisenberg group on \mathcal{H} . Integrating to the group, one

projective (up to \pm) representation of $SO(2n)$, a true representation of the double cover $Spin(2n)$.

For $d = 2n$ even, the spin representation has two irreducible components, the half-spinors S^+, S^- , each of dimension 2^{n-1} .

Generators γ_j of the Clifford algebra. On the spinor module S , identifying the Clifford algebra with a matrix algebra, these are the physicist's Dirac γ -matrices.

In even dimension, the Lie algebra representation operators for the spin representation are given by quadratic combinations of γ -matrices.

Spin 1/2 degree of freedom in d dimensions.

gets a projective (up to \pm) representation of $Sp(2n, \mathbf{R})$, a true representation of the double cover $Mp(2n, \mathbf{R})$.

The oscillator representation has two irreducible components (an "even" and an "odd" component).

Generators Q_j, P_k of the Weyl algebra.

The Lie algebra representation operators for the oscillator representation are given by quadratic combinations of the Q_j, P_k operators.

Harmonic oscillator with n degrees of freedom.

7.3 Real and complex polarizations

When Q has signature (n, n) , choosing a real polarization $V = L \oplus L^*$ ($Q = 0$ on L and on L^*), one can realize the spinor module as anticommuting functions on L . This will be an irreducible representation of the real form $SO(n, n)$, non-unitary.

For $d = 2n$ even, an orthogonal complex structure on V is a linear map J satisfying $J^2 = -\mathbf{1}$ and preserving the bilinear form Q . This picks out a $U(n) \subset SO(2n)$ and the space of such complex structures is the compact space $SO(2n)/U(n)$.

For $d = 2n$ even, such a J gives a complex polarization $V \otimes \mathbf{C} = W_J^+ \oplus W_J^-$ ($\pm i$ eigenspaces of J).

For $d = 2n$ even, taking complex linear combinations of the γ_j in W_J^+ one

Choosing a real polarization $M = L \oplus L^*$ one can realize (the Schrödinger representation) the Q_j, P_j operators respectively as multiplication and differentiation operators on $L^2(L)$. This representation will be unitary, both as a representation of the Heisenberg group and the metaplectic group.

A symplectic complex structure on M is a linear map J satisfying $J^2 = -\mathbf{1}$ and preserving the bilinear form Ω . This picks out a $U(n) \subset Sp(2n, \mathbf{R})$. Such J satisfying the positivity conditions $S(\cdot, J\cdot)$ positive are parametrized by the non-compact space $Sp(2n, \mathbf{R})/U(n)$.

Such a J gives a complex polarization $M \otimes \mathbf{C} = W_J^+ \oplus W_J^-$ ($\pm i$ eigenspaces of J).

Taking complex linear combinations of the Q_j, P_j in W_J^+ one can form ad-

can form adjoint operators a_j, a_j^\dagger on the spinor module, satisfying the canonical anti-commutation relations

$$[a_j, a_k^\dagger]_+ = \delta_{jk} \mathbf{1}$$

For each J there is a unique (up to scalar) vector in S annihilated by all the a operators. These are fibers of the line bundle $\Lambda^n(W_J^+)^{-\frac{1}{2}}$.

Applying a^\dagger operators

$$S = \Lambda^*(W_J^+) \otimes (\Lambda^n(W_J^+))^{-\frac{1}{2}}$$

Half-spinors are holomorphic sections of the line bundle $\Lambda^n(W_J^+)^{-\frac{1}{2}}$ over $SO(2n)/U(n)$

$$S^+ = \Gamma_{hol}(\Lambda^n(W_J^+)^{-\frac{1}{2}})$$

joint operators a_j, a_j^\dagger on the oscillator representation, satisfying the canonical commutation relations

$$[a_j, a_k^\dagger] = \delta_{jk} \mathbf{1}$$

For each J there is a unique (up to scalar) vector (vacuum vector) in \mathcal{H} annihilated by all the a operators. These are fibers of the line bundle $\Lambda^n(W_J^+)^{\frac{1}{2}}$.

Applying a^\dagger operators

$$\mathcal{H} = S^*(W_J^+) \otimes (\Lambda^n(W_J^+))^{\frac{1}{2}}$$

The even component of the oscillator representation is holomorphic sections of the line bundle $\Lambda^n(W_J^+)^{\frac{1}{2}}$ over $Sp(2n, \mathbf{R})/U(n)$.

$$\mathcal{H}_{even} = \Gamma_{hol}(\Lambda^n(W_J^+)^{\frac{1}{2}})$$

7.4 Supersymmetry

One can put together copies of the spinor and oscillator representations, for instance tensoring \mathcal{H} for $n = 3$ (describing a particle degree of freedom moving in 3 spatial dimensions) with S , the spinor module for the $d = 3$ Clifford module (describing a spin1/2 degree of freedom). The tensor product of the Weyl algebra and Clifford algebra acts, and one now has new quadratic operators that are tensor products of a Clifford algebra generator (γ_j) and a Schrödinger operator P_j . An example would be the Dirac operator (in this 3-dimensional case known as the Pauli-Schrödinger operator), which is

$$\not{D} = \sum_{j=1}^3 P_j \otimes \gamma_j$$

acting on

$$L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$$

For much more about this see the article [8] as well as [9] which gives other examples occurring in physics

8 Howe duality and the Theta correspondence

We'll now consider an important application of the oscillator representation. It provides a correspondence between irreducible representations of certain pairs ("Howe duals) of commuting subgroups of the symplectic group $Sp(2n, \mathbf{R})$. Here we'll give some background and then describe certain specific examples of the correspondence. This will be for real Lie groups, but the whole story has a number-theoretic analog which we will turn to in the next section.

8.1 Multiplicity spaces and commuting group actions

We'll begin with some generalities that are part of standard representation theory for finite-dimensional representations. Given the irreducible representations V_i of a group G , for an arbitrary representation V one has an isomorphism

$$\bigoplus_{\text{irreps } i} \text{Hom}_G(V_i, V) \otimes V_i \rightarrow V$$

which shows how the representation decomposes into irreducibles. Here the spaces $\text{Hom}_G(V_i, V)$ are "multiplicity spaces", with dimension given by the multiplicity of the irreducible representation V_i in V . If there is a group $H \subset G$ acting on V , commuting with the G action, then the $\text{Hom}_G(V_i, V)$ will provide representations of the group H . Two examples are:

- The left and right actions of G on itself gives commuting actions of two copies of G , G_L and G_R on functions on G

$$\pi_L(g_L)f(g) = f(g_L^{-1}g), \quad \pi_R(g_R)f(g) = f(gg_R)$$

The Peter-Weyl theorem says that L^2 functions on G decompose under these commuting actions as the completed sum

$$L^2(G) = \widehat{\bigoplus_{\text{irreps } i} V_i^* \otimes V_i}$$

where G_L acts on the irreducible representations V_i , and G_R on their duals V_i^* . In this case

$$\text{Hom}_G(V_i, L^2(G)) = V_i^*$$

is the multiplicity space for the decomposition into irreducibles of the action of G_L on $L^2(G)$ and it provides the representation V_i^* of G_R . Similarly

$$\text{Hom}_G(V_i^*, L^2(G)) = V_i$$

is the multiplicity space for the decomposition into irreducibles of the action of G_R on $L^2(G)$ and it provides the representation V_i of G_L .

- If $V = \mathbf{C}^N$ is the defining representation of $GL(V)$, one can form the tensor product

$$T^k(V) = V \otimes V \cdots \otimes V$$

of k copies of V . This is again a representation of $GL(V)$, but it also is a representation of a commuting group, the symmetric group S_k acting by permuting the factors of $T_k(V)$. For W an irreducible representation of $GL(V)$ one has multiplicity spaces

$$\text{Hom}_{GL(V)}(W, T^k(V))$$

which are representations of S_k . For U an irreducible representation of S_k one has multiplicity spaces

$$\text{Hom}_{S_k}(U, T^k(V))$$

which are representations of $GL(V)$.

These two examples have the property that they provide a duality map between irreducible representations of two groups. In the first case this is rather trivial, relating an irreducible representation V_i of a group G to its dual irreducible representation. In the second case the duality (“Schur-Weyl duality”) is much less trivial. It allows one to parametrize and construct irreducible representations of $GL(V)$ from irreducible representation of S_k and vice-versa, giving a map which takes irreducible representations of one group to either an irreducible representation of the other group, or zero.

8.2 The theta correspondence

Taking $G = Mp(2n, \mathbf{R})$ and its oscillator representation π , one can choose subgroups G_1 and G_2 (“Howe duals”) such that G_1 is all of the elements of G commuting with G_2 and vice-versa. One then finds (“the theta correspondence”) that for π_1 an irreducible representation of G_1

$$\Theta(\pi_1) = \text{Hom}_{G_1}(\pi_1, \pi|_{G_1})$$

is an irreducible representation of G_2 or zero, and for π_2 an irreducible representation of G_2

$$\Theta(\pi_2) = \text{Hom}_{G_2}(\pi_2, \pi|_{G_2})$$

is an irreducible representation of G_1 or zero. Some examples follow. is

8.2.1 A trivial example

Taking $G_1 = O(1) = \mathbf{Z}_2$, which has two irreducible representations, trivial and non-trivial, one finds that $G_2 = Mp(2n, \mathbf{R})$ is a Howe dual, $\Theta(\text{trivial})$ is the irreducible component π^+ of the oscillator representation on even functions, while $\Theta(\text{non-trivial})$ is π^- , the other irreducible component (on odd functions).

8.2.2 Unitary subgroup examples

One can get examples of Howe dual pairs by taking $G_1 = U(n) \subset Sp(2n, \mathbf{R})$ and G_2 the central $U(1)$ subgroup of G_2 . In this case the theta correspondence relates finite-dimensional irreducible representations, taking the irreducible $U(1)$ representation $\pi(k)$ of weight k (for $k \geq 0$) to the irreducible $U(n)$ representation on symmetric tensor products $S^k(\mathbf{C}^n)$ (these are the energy eigenspaces of the quantum harmonic oscillator with n degrees of freedom).

For something much more non-trivial, one can instead take $G_1 = U(n, n) \subset Sp(2n, \mathbf{R})$, which is a non-compact group with infinite dimensional irreducible representations. Again taking G_2 as the central $U(1)$ subgroup, the theta correspondence takes irreducible $U(1)$ representation of weight k (for $k \geq 0$) to irreducible infinite-dimensional representations of $U(n, n)$. One finds

- For $n = 1$, $\Theta(\pi(k))$ is the weight k holomorphic discrete series representation of $U(1, 1)$.
- For $n = 2$, $\Theta(\pi(k))$ is the representation of the conformal group $SU(2, 2) = Spin(4, 2)$ on solutions of the massless helicity $\frac{k}{2}$ wave-equation.

Citations here should include Howe's article, others?

8.2.3 Quadratic forms

If one is given a symplectic form Ω on a vector space $U = \mathbf{R}^m$ and a quadratic form Q on a vector space $V = \mathbf{R}^n$, one gets a symplectic form $\Omega \otimes Q$ on the tensor product $U \otimes V = \mathbf{R}^{mn}$. In particular, for $m = 2$, with $U = L \oplus L^*$ and the standard symplectic form for this case, one gets a Howe dual pair $G_1 = SL(2, \mathbf{R}), G_2 = O(Q)$ of subgroups of the group $Mp(2n, \mathbf{R})$. Note that this is the standard sort of structure one has in a conventional physical system: a configuration space $V = \mathbf{R}^n$ with a positive definite inner product preserved by an orthogonal group $O(n)$. The inner product identifies V and V^* , giving a phase space $V \oplus V^*$ with the standard symplectic form. Quantization gives the oscillator representation π of the groups H_{2n+1} and $Mp(2n, \mathbf{R})$. For $n = 3$ and some choice of Hamiltonian operator, this is exactly conventional non-relativistic quantum mechanics.

In this case one has a decomposition of the oscillator representation as

$$\mathcal{H} = \mathbf{C}[q_1, \dots, q_n] = \bigoplus_{k \geq 0} E_k \otimes H_k$$

where H_k are the irreducible representations of $O(k)$ on spherical harmonics of degree k . The E_k are the multiplicity spaces

$$E_k = \text{Hom}_{O(k)}(H_k, \mathcal{H}) = \Theta(H_k)$$

and provide different irreducible representations of $Mp(2, \mathbf{R})$ associated to each irreducible representation H_k of $O(n)$ by the theta correspondence.

More explicitly, E_k is the space of functions of the form

$$r^k P_k(r^2)$$

for some polynomial P_k . where

$$r^2 = q_1^2 + \cdots + q_n^2$$

The Lie algebra of $Mp(2, \mathbf{R})$ is the same as the Lie algebra $\mathfrak{sl}(2, \mathbf{R})$, with basis elements H, E, F satisfying

$$[H, E] = 2E, [H, F] = -2F, [E, F] = H$$

The representation on E_k is given by

$$\pi'_k(H) = r \frac{d}{dr} + \frac{n}{2}, \quad \pi'_k(E) = -\frac{i}{2} r^2, \quad \pi'_k(F) = -\frac{i}{2} \Delta$$

where Δ is the Laplacian acting on radial functions times an element of H_k , given by

$$\Delta = \frac{d^2}{dr^2} + \frac{n-1}{r} - \frac{k(k+n-2)}{r^2}$$

For more details, see chapter 10 of [10].

The representation E_k is a lowest-weight representation, with lowest weight vector the function r^k , with weight (under the action of H) $k + \frac{n}{2}$. For n odd the weights are half-integral and one needs the double cover $Mp(2n, \mathbf{R})$, but for n even E_k is a representation also of $Sp(2n, \mathbf{R})$.

Note that this is a discrete series representation?

9 Automorphic forms and θ -functions

For a Lie group G , one can often choose a discrete subgroup Γ , and then study $L^2(\Gamma \backslash G)$ instead of $L^2(G)$ as a representation of G (using the right action of G on itself). Functions on $\Gamma \backslash G$ are known as “automorphic forms” and generalize the case $\Gamma = \mathbf{Z}$, $G = \mathbf{R}$ which is that of Fourier series and trigonometric functions. In this section we’ll discuss the extension of that case to the context of the oscillator representation.

9.1 Automorphic forms for the Heisenberg group

One can take $q \in \mathbf{R}$ and $p \in \mathbf{R}$ coordinates each with a \mathbf{Z} subgroup as in the Fourier series case, and extend this to a discrete Heisenberg group $H_3(\mathbf{Z}) \subset H_3(\mathbf{R})$, giving automorphic forms for the Heisenberg group as functions on $H_3(\mathbf{Z}) \backslash H_3(\mathbf{R})$. Explicitly, we use the same Heisenberg group law as before,

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$$

but now $x, y, z, x', y', z' \in \mathbf{Z}$. In the definition 1 of the Schrödinger representation, one can rescale the action of Q and the center by factors of 2π , preserving the representation property, getting

$$\pi_S(x) = e^{-i2\pi xq}, \quad \pi_S(y) = e^{-y\frac{d}{dq}}, \quad \pi_S(z) = e^{-i2\pi z\mathbf{1}} \quad (5)$$

The constant $c = -i$ used to define the oscillator representation is rescaled to $c = -2\pi i$. Note that integer elements $(0, 0, z)$ of the center act trivially, but one can get half-integer elements due to the $\frac{1}{2}$ in the group law, giving an action by a minus sign. One way to deal with this situation is to interpret the group $H_3(\mathbf{Z})$ as the group \mathbf{Z}^2 , lying in $H_3(\mathbf{R})$ as the subgroup of elements of the form

$$(x, y, (-1)^{xy})$$

for $x, y \in \mathbf{Z}$, and we will do that in what follows.

If one looks at the action of the subgroup $H_3(\mathbf{Z})$ on the Schrödinger representation, one finds that there are no invariant vectors in $\mathcal{H} = L^2(\mathbf{R})$. There is however a unique (up to scale) $H_3(\mathbf{Z})$ -invariant element of the dual space \mathcal{H}' , interpreted as linear functionals on the Schwarz space $S(\mathbf{R}) \subset \mathcal{H}$ (and thus these are tempered distributions). Explicitly, this is the distribution

$$\theta = \sum_{n=-\infty}^{\infty} \delta(q - n)$$

i.e the distribution that takes value

$$\theta(\psi) = \sum_{n=-\infty}^{\infty} \psi(q - n)$$

on Schrödinger representation wave-functions.

$e^{-i2\pi q} = 1$ only when $q \in \mathbf{Z}$, so to be invariant a distribution has to be supported only at $q \in \mathbf{Z}$. To be invariant under integer translations in q it must take the same value at those integer points, so, up to scale θ is the only possibility.

With this normalization for the action of Q , the Schrödinger representation vector that will be annihilated by a_τ will be

$$|0\rangle_\tau = e^{i\pi\tau q^2}$$

Evaluating the distribution θ on this vector gives the classical theta function

$$\theta(\tau) = \theta(|0\rangle_\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2}$$

One can get the more general theta function

$$\theta(w, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} e^{i2\pi n w}$$

in terms of matrix elements of the Heisenberg group in the Schrödinger representation since

$$\theta(\pi_S(x, y, 0) |0\rangle_\tau) = e^{i\pi\tau y^2} e^{i\pi xy} \theta(\tau y + x, \tau)$$

(use equation 2). For more details of this and the generalization to higher dimensions, see [11].

One can put this calculation into the language of sections of holomorphic bundles on complex tori, defining these sections as functions on \mathbf{R}^2 with a given transformation property under the group \mathbf{Z}^2 . In this context one gets a holomorphic line bundle over a torus with complex structure parametrized by τ . The theta function $\theta(w, \tau)$ is the unique holomorphic section of such a bundle. Holomorphic sections of tensor powers of L can be used to give a projective embedding of the complex torus. For more details of this story, see for instance the first few chapters of [12].

Comments on Riemann surfaces, Jacobians, Abelian Chern-Simons.

9.2 Automorphic forms for the metaplectic group

Review of basic facts about modular/automorphic forms for $SL(2, \mathbf{Z}) \subset SL(2, \mathbf{R})$.

Modular forms of weight k as holomorphic sections of the line bundle L^k over $SL(2, \mathbf{Z}) \backslash \mathfrak{H}$

The conventional theory of automorphic forms for the subgroup $SL(2, \mathbf{Z}) \subset SL(2, \mathbf{R})$ is about the decomposition of the function space $L^2(SL(2, \mathbf{Z}) \backslash SL(2, \mathbf{R}))$ into irreducible representations (using the right action of $SL(2, \mathbf{R})$ on itself).

Modular forms of weight k as the multiplicity space of intertwiners between a discrete series representation and $L^2(SL(2, \mathbf{Z}) \backslash SL(2, \mathbf{R}))$.

The distribution θ also invariant for the theta-subgroup of $SL(2, \mathbf{Z})$. Note that invariance under the Fourier transform is Poisson summation.

One reference for details is [13].

10 Automorphic representations

It turns out that the full story of the Heisenberg group, Schrödinger representation, metaplectic group and oscillator representation, as well as Howe duality and the theta correspondence, extends naturally from the field \mathbf{R} to the case of the number field \mathbf{Q} . In this section we'll provide some sketches of background from number theory and work out some specific examples of what happens in this general context.

10.1 The case of finite fields

For a simple example of how everything works for other fields, consider the finite field \mathbf{F}_p . Reference for details is [14]

10.2 The case of \mathbf{Q}_p

10.3 Adeles and global fields

Definition of $\mathbf{A}_{\mathbf{Q}}$

Note that $\widehat{\mathbf{Q}} = \mathbf{Q} \backslash \mathbf{A}_{\mathbf{Q}}$

10.4 Automorphic representations for the Heisenberg group over $\mathbf{A}_{\mathbf{Q}}$ and the Θ -distribution

10.5 Automorphic representations for the metaplectic group

Note that splitting over the global $SL(2, \mathbf{Q})$ is quadratic reciprocity

10.6 An example of the theta-correspondence: Jacquet-Langlands and quaternions

11 Further directions

Witten's reformulation of a 2d QFT for a free fermion as an automorphic representation.

What is relation of all of this to geometric Langlands?

A Quantization

In these notes we have been discussing what physicists call “canonical quantization”, which associates to a linear phase space $P = \mathbf{R}^{2n}$ with the oscillator representation on a state space \mathcal{H} . Replacing P by a general symplectic manifold, one will still have a Poisson bracket and can ask for a more general notion of quantization that gives a state space with operators satisfying commutation relations corresponding to the Poisson bracket relations. The methods used in these notes start with a Poisson bracket defined using a symplectic form not on P but on P^* , crucially using the linear structure on P , so cannot be extended to more general symplectic manifolds. The subject of “geometric quantization” attempts to provide a generalization, but does not work in the generality one would like.

In much greater generality, there is a notion of “quantization” valid for any Lie algebra \mathfrak{g} , extending the one used here for $\mathfrak{g} = \mathfrak{h}_{2n+1}$. If one takes as phase space $P = \mathfrak{g}^*$, then the Lie bracket of \mathfrak{g} provides a Poisson bracket on functions on P . Linear functions on $P = \mathfrak{g}^*$ are just elements $X \in \mathfrak{g}$, for them one can define

$$\{X, Y\} = [X, Y]$$

and then the derivation property gives the Poisson bracket on polynomial functions on P . The algebra $S^*(\mathfrak{g})$ of polynomial functions on \mathfrak{g} can be thought of

as an algebra of classical observables. There is one obvious candidate for the quantization of this algebra: the universal enveloping algebra $U(\mathfrak{g})$.

A quantization though requires more: a state space with an action of $U(\mathfrak{g})$ by linear operators, in other words, a representation of \mathfrak{g} . For a general Lie algebra \mathfrak{g} , there are many representations with the problem of classifying and constructing them a major part of representation theory. The “orbit philosophy” in representation theory asks for a construction of general representations in terms of the geometry of \mathfrak{g}^* , positing an association between irreducible representations and orbits of the co-adjoint action of a Lie group G on $(\text{Lie } G)^*$. It turns out that such orbits are symplectic manifolds with a G action. One can think of these as classical phase spaces and ask for a quantization of the symplectic manifold, with the G action giving a representation of G .

For the case of the Heisenberg group H_{2n+1} , the non-trivial co-adjoint orbits are copies of \mathbf{R}^{2n} , with (up to scaling) the symplectic form Ω . For this case quantization is given by any of the constructions of the oscillator representation we have described. For more general groups though, typically the co-adjoint orbits will not be linear spaces and different methods are required to construct a representation. For a summary of the orbit philosophy and how it mostly (but not always) leads to constructions of irreducible representations, see [15].

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