Quantization and the Dirac operator

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Outline

Note: most of what I'm describing is an advertisement for results available in Meinrenken, *Clifford algebras and Lie theory* and Huang-Pandzic, *Dirac operators in representation theory* + work in progress, paper to appear later this summer.

Quantum mechanics

Quantum mechanics has two basic structures:

I: States

The state of a physical system is given by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$.

II: Observables

To each observable quantity Q of a physical system corresponds a self-adjoint operator O_Q on $\mathcal{H}$. If

$$O_Q|\psi\rangle = q|\psi\rangle$$

(i.e. $|\psi\rangle$ is an eigenvector of O_Q with eigenvalue q) then the observed value of Q in the state $|\psi\rangle$ will be q.

Symmetries give observables

Time translations

There is a distinguished observable corresponding to energy: the Hamiltonian H. $-iH$ gives the infinitesimal action of the group $\mathbf{R}$ of time translations. This is the Schrödinger equation (for $\hbar = 1$)

$$\frac{d}{dt}|\psi\rangle = -iH|\psi\rangle$$

Note: $-iH$ is skew-adjoint, so the action of time translation is unitary.

Other examples

- Spatial translations, group $\mathbf{R}^3$, momentum operators P_j
- Rotations, group $SO(3)$, angular momentum operators L_j
- Phase transformations, group $U(1)$, charge operator $\widehat{Q}$

Symmetry and quantum mechanics

General principle

If a Lie group G acts on a physical system, expect a unitary representation of G on $\mathcal{H}$, i.e. for $g \in G$ have unitary operators $\pi(g)$ such that

$$\pi(g_1)\pi(g_2) = \pi(g_1 g_2)$$

Differentiating, get a unitary representation of the Lie algebra $\mathfrak{g}$ on $\mathcal{H}$, i.e. for $X \in \mathfrak{g}$ have skew-adjoint operators $\pi'(X)$ such that

$$[\pi'(X_1), \pi'(X_2)] = \pi'([X_1, X_2])$$

The $i\pi'(X)$ will give observables (self-adjoint operators). If

$$[\pi'(X), H] = 0$$

these are "symmetries" and there will be "conservation laws": states that are eigenvectors of $i\pi'(X)$ at one time will remain eigenvectors at all times, with the same eigenvalue.

Example: $SU(2)$

$SU(2)$ and $\mathfrak{su}(2)$

The group $G = SU(2)$ (2 by 2 unitary matrices with determinant 1) has Lie algebra $\mathfrak{su}(2) = \mathbf{R}^3$, with Lie bracket the cross-product.

Representations of $SU(2)$ and $\mathfrak{su}(2)$

Corresponding to elements of a basis $X_j \in \mathfrak{su}(2)$ have "spin" observables $S_j = i\pi'(X_j)$. These act on a state space $\mathcal{H}$ that is a representation of $SU(2)$. All such representations will be a sum of copies of irreducible representations. The irreducible representations are labeled by $n = 0, 1, 2, \ldots$, and called by physicists the representations of "spin $n/2$", of dimension $n + 1$.

Example: canonical quantization and the Heisenberg Lie algebra

Main way to create quantum systems (Dirac). Start with phase space $\mathbf{R}^{2n}$, coordinates q_j, p_j and Poisson bracket satisfying

$$\{q_j, p_j\} = 1$$

The $1, q_j, p_j$ are a basis of a Lie algebra, the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$, with Poisson bracket the Lie bracket. This is the Lie algebra of a group, the Heisenberg Lie group.
"Canonical quantization" is given by operators Q_j, P_j, satisfying the Heisenberg commutation relations (for $\hbar = 1$).

$$[-iQ_j, -iP_j] = -i\mathbf{1}$$

Stone von-Neumann theorem

Up to unitary equivalence, there is only one non-trivial representation of the Heisenberg group

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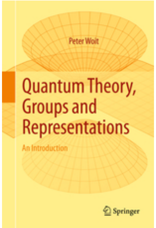
For much much more detail, see my recent book



available from Springer or at
http://www.math.columbia.edu/~woit/QM/qmbook.pdf

The universal enveloping algebra $U(\mathfrak{g})$

Given a Lie algebra representation, we also can consider the action of products of the $\pi'(X_j)$. Can do this by defining, for any Lie algebra $\mathfrak{g}$ with basis X_j, the universal enveloping algebra $U(\mathfrak{g})$. This is the algebra with basis (here $j < k$),

$$1, X_j^{n_j}, X_j^{n_j} X_k^{n_k}, \cdots$$

and when one takes products one uses $X_j X_k - X_k X_j = [X_j, X_k]1$.
$U(\mathfrak{g})$ has a subalgebra $Z(\mathfrak{g})$ (the center) of elements that commute with all X_j. These will act as scalars on irreducible representations.

Example: the Casimir operator of $\mathfrak{su}(2)$

$X_1^2 + X_2^2 + X_3^2 \in Z(\mathfrak{su}(2))$ and the Casimir operator

$$(\pi'(X_1))^2 + (\pi'(X_2))^2 + (\pi'(X_3))^2$$

acts as $n(n+1)$ on the spin $\frac{n}{2}$ representation.

Quantizing $\mathfrak{g}^*$

For any Lie algebra, the dual space $\mathfrak{g}^*$ can be thought of as a generalized classical phase space, with Poisson bracket given by the Lie algebra bracket. The Poisson bracket is determined by its values on linear functions on the phase space, but a linear function on $\mathfrak{g}^*$ is just an element $X \in \mathfrak{g}$, and one has

$$\{X, Y\} = [X, Y]$$

Quantization of this classical system should give an algebra of operators, the obvious choice is $U(\mathfrak{g})$.

Philosophy of quantization

Quantum systems should correspond to Lie algebras, with classical phase space $\mathfrak{g}^*$, and algebra of operators $U(\mathfrak{g})$.

Problem: how does one construct representations of $\mathfrak{g}$ (and thus $\mathcal{H}$)?

The "orbit philosophy"

Constructing quantum systems corresponding to a Lie algebra $\mathfrak{g}$ is thus equivalent to the (well-known and hard) problem of constructing (unitary) representations of $\mathfrak{g}$. One approach to this problem is

Orbit philosophy of Kirillov-Kostant-Souriau

For any $\mu \in \mathfrak{g}^*$, consider the orbit $\mathcal{O}_\mu \subset \mathfrak{g}^*$ under the co-adjoint action of G on $\mathfrak{g}^*$. This turns out to be a symplectic manifold. Take this as a classical phase space and "quantize" it, with the state space of the quantum system giving an irreducible representation. Co-adjoint orbits should thus parametrize irreducible representations, and provide the material for their construction.

To do this, in some cases one can consult physicists to find out how to construct a quantum system given a classical system. In general, "geometric quantization" is supposed to provide such a quantization.

Examples

The Heisenberg Lie algebra

Non-trivial co-adjoint orbits are copies of $\mathbf{R}^{2n} \subset \mathfrak{h}_{2n+1}$, parametrized by a constant $c \neq 0$. Physicists have many unitarily equivalent ways of producing from this a representation of $\mathfrak{h}_{2n+1}$, i.e. a state space $\mathcal{H}$ and operators Q_j, P_j, C. Here $C = \pi'(1)$ is central and must act as a scalar. Taking this to be $-i\hbar$ gives the Heisenberg commutation relations

$$[-iQ_j, -iP_j] = -i\hbar\mathbf{1}$$

$\mathfrak{su}(2)$

The non-trivial co-adjoint orbits are spheres in $\mathbf{R}^3$ of arbitrary radius. Only if the radius satisfies an integrality condition does one get an irreducible representation (one of the spin $\frac{n}{2}$ representations). Geometric quantization: construct a holomorphic line bundle over the sphere, realize the representation as holomorphic sections (Borel-Weil construction).

Fermionic quantization

The version of quantization considered so far is "bosonic", starting with classical observables polynomial functions on $\mathfrak{g}^*$, or equivalently the symmetric algebra $S^*(\mathfrak{g})$, with this commutative algebra quantized as $U(\mathfrak{g})$.

"Fermionic" quantization starts by replacing the symmetric tensor algebra $S^*(\mathfrak{g})$ by $\Lambda^*(\mathfrak{g})$, the anti-symmetric tensor algebra. One can think of this as "polynomials in anti-commuting generators".

Quantization then replaces commutators by anticommutators, and the algebra $U(\mathfrak{g})$ by the Clifford algebra $\mathrm{Cliff}(\mathfrak{g})$. This requires the choice of a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$. We will later need this to be invariant and non-degenerate (Lie algebras that have this are called "quadratic").

Clifford algebras and spinors

Clifford algebra

For any vector space V with symmetric bilinear form $(\cdot,\cdot)$, Cliff(V) is the algebra generated by $v \in V$, with relations

$$v_1 v_2 + v_2 v_1 = 2(v_1, v_2)$$

As a vector space Cliff$(V) = \Lambda^*(V)$ (multiplication is different)

The structure of irreducible modules is much simpler for Cliff$(\mathfrak{g})$ than for $U(\mathfrak{g})$. For V complex and $dim\ V = 2n$

Spinors

$$\mathrm{Cliff}(V) = \mathrm{End}(S) = S \otimes S^*$$

where $S = \mathbf{C}^{2^n}$.

Quantum Weil algebra

One can combine the "bosonic" and "fermionic" quantizations into what physicists would call a "supersymmetric" quantization

The quantum Weil algebra

Given a Lie algebra $\mathfrak{g}$ with invariant non-degenerate bilinear form $(\cdot, \cdot)$, the quantum Weil algebra is the algebra

$$\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{g})$$

with (super)commutation relations

$$[X \otimes 1, Y \otimes 1]_{\mathcal{W}} = [X, Y] \otimes 1, [X \otimes 1, 1 \otimes Y]_{\mathcal{W}} = 0, [1 \otimes X, 1 \otimes Y]_{\mathcal{W}} = 2(X, Y)$$

Note that this is a $\mathbf{Z}_2$ graded algebra, with generators of $U(\mathfrak{g})$ even, generators of $\mathrm{Cliff}(\mathfrak{g})$ odd.

The Dirac operator

For quadratic Lie algebras one can identify $\mathfrak{g} = \mathfrak{g}^*$, and one has a quadratic Casimir element of $U(\mathfrak{g})$ given by

$$\Omega = \sum e^j e_j$$

where e_j is an orthonormal basis of $\mathfrak{g}$, e^j the dual basis. A wonderful discovery of Dirac in 1928, was that the introduction of a Clifford algebra and spinors allowed the construction of a square root of the Casimir operator (which in his case was a Laplacian). Here one defines:

Kostant Dirac operator

Ω has (up to a constant), a square root given by

$$\not{D}_{\mathfrak{g}} = \sum (e^j \otimes e_j) + 1 \otimes q(\phi)$$

where $\phi \in \Lambda^3(\mathfrak{g})$ and q is the quantization map.

The differential super Lie algebra $\widehat{\mathfrak{g}}$

For another point of view on $\mathcal{W}(\mathfrak{g})$, one can define a "super Lie algebra"

$$\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \epsilon\mathfrak{g} \oplus \mathbf{R}c$$

Where $\epsilon^2 = 0$ and the super Lie bracket relations are

$$[X, Y]_{\widehat{\mathfrak{g}}} = [X, Y], \quad [X, \epsilon Y]_{\widehat{\mathfrak{g}}} = \epsilon[X, Y], \quad [\epsilon X, \epsilon Y]_{\widehat{\mathfrak{g}}} = (X, Y)c$$

The operator $\frac{\partial}{\partial \epsilon}$ provides a differential d on $\widehat{\mathfrak{g}}$ satisfying $d^2 = 0$.
One then has

$$\mathcal{W}(\mathfrak{g}) = U(\widehat{\mathfrak{g}})/(c-1)$$

with replacing $\mathfrak{g}$ by $\widehat{\mathfrak{g}}$ (with its differential) an alternate motivation for the replacement of $U(\mathfrak{g})$ by $\mathcal{W}(\mathfrak{g})$.

Dirac cohomology

Since the square of the Dirac operator $\not{D}_{\mathfrak{g}}$ is a Casimir operator (up to a constant) and thus is central, the operator

$$d(\cdot) = [\not{D}_{\mathfrak{g}}, \cdot]_{\mathcal{W}}$$

satisfies $d^2 = 0$.
One can define

Dirac cohomology

The Dirac cohomology $H_{\not{D}}(\mathfrak{g})$ of a quadratic Lie algebra $\mathfrak{g}$ is the cohomology of d on $\mathcal{W}(\mathfrak{g})$. The Dirac cohomology of a representation V of $\mathfrak{g}$ is given by

$$H_{\not{D}}(\mathfrak{g}, V) = \ker \not{D}_{\mathfrak{g}|V\otimes S}$$

$H_{\not{D}}(\mathfrak{g})$ acts on $H_{\not{D}}(\mathfrak{g}, V)$.

Relative Dirac cohomology

It turns out the Dirac cohomology itself is trivial, with $H_{\not{D}}(\mathfrak{g}) = \mathbf{R}$ and $H_{\not{D}}(\mathfrak{g}, V)$ zero. What is interesting is, for $\mathfrak{r} \subset g$ ($\mathfrak{r}$ a quadratic Lie subalgebra), a relative version of the Dirac cohomology. The super Lie algebra $\widehat{\mathfrak{r}}$ acts by the adjoint representation on $\mathcal{W}(\mathfrak{g})$, with

$$\mathcal{W}(\mathfrak{g})^{\widehat{\mathfrak{r}}} = (U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{s}))^{\mathfrak{r}}$$

where $\mathfrak{s}$ is the orthogonal complement of $\mathfrak{r}$ in $\mathfrak{g}$.
Since $\not{D}_{\mathfrak{g}/\mathfrak{r}} = \not{D}_{\mathfrak{g}} - \not{D}_{\mathfrak{r}}$ preserves $(U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{s}))^{\mathfrak{r}}$ and has square an element of the center, one can define

Relative Dirac cohomology

$H_{\not{D}}(\mathfrak{g}, \mathfrak{r})$ is the cohomology of $d = [\not{D}_{\mathfrak{g}/\mathfrak{r}}, \cdot]_{\mathcal{W}}$ on $(U(\mathfrak{g}) \otimes \mathrm{Cliff}(\mathfrak{s}))^{\mathfrak{r}}$. It acts on

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{r}, V) = \ker \not{D}_{\mathfrak{g}|V \otimes S_{\mathfrak{s}}}$$

Examples

The relative Dirac cohomology $H_{\not D}(\mathfrak{g}, \mathfrak{r}, V)$ with its action by $H_{\not D}(\mathfrak{g}, \mathfrak{r})$ provides an interesting invariant of V. Some examples:

- $\mathfrak{g}$ complex semi-simple, $\mathfrak{r} = \mathfrak{h}$ the Cartan subalgebra and $\mathfrak{g} = \mathfrak{n} \oplus \overline{\mathfrak{n}} \oplus \mathfrak{h}$. Here $H_{\not D}(\mathfrak{g}, \mathfrak{h}, V)$ is the Lie algebra cohomology $H^*(\mathfrak{n}, V)$ (up to ρ-twist).
- For many other cases of $\mathfrak{g}$ reductive, see Huang-Pandzic, *Dirac operators in representation theory*
- (Work in progress): $\mathfrak{g}$ the Heisenberg Lie algebra. Not a quadratic Lie algebra, but an extension (the oscillator Lie algebra) is.
- (Work in progress): $\mathfrak{g}$ the Poincaré Lie algebra. Not a quadratic Lie algebra, but get the actual Dirac operator used in physics.

Constructing representations using Dirac operators

Some known relations between constructions of representation and Dirac operators:

- Compact Lie groups: Borel-Weil-Bott construction of irreducible representations as sheaf cohomology groups of holomorphic line bundles on G/T can be reinterpreted as kernels of a Dirac operator.
- Real semi-simple Lie groups: can construct discrete series representations using Dirac operators.
- Freed-Hopkins-Teleman: provide a construction using families of Dirac operators that associates an orbit to an irreducible representation (for compact Lie groups).

Conclusions

Modern philosophy of quantization

Quantum systems should correspond to Lie algebras $\mathfrak{g}$, with classical phase space $\mathfrak{g}^*$ and algebra of operators $U(\mathfrak{g})$. Ad hoc "geometric quantization" techniques associate irreducible representations to orbits in $\mathfrak{g}^*$.

Post-modern philosophy of quantization

Quantum systems should correspond to Lie algebras, with classical phase space a derived geometry based on $\widehat{\mathfrak{g}}$ and algebra of operators $\mathcal{W}(\mathfrak{g})$ with differential given by the Dirac operator $\not{D}_{\mathfrak{g}}$.

May give a Dirac operator-based geometric quantization.
Applications in physics:

- New construction of elementary particle states using representations of the Poincaré Lie algebra.
- New version of BRST method for dealing with gauge symmetries.