Quantization and the Dirac operator

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Note: most of what I’m describing is an advertisement for results available in Meinrenken, *Clifford algebras and Lie theory* and Huang-Pandzic, *Dirac operators in representation theory* + work in progress, paper to appear later this summer.
Quantum mechanics has two basic structures:

I: States
The state of a physical system is given by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$.

II: Observables
To each observable quantity $Q$ of a physical system corresponds a self-adjoint operator $O_Q$ on $\mathcal{H}$. If

$$O_Q|\psi\rangle = q|\psi\rangle$$

(i.e. $|\psi\rangle$ is an eigenvector of $O_Q$ with eigenvalue $q$) then the observed value of $Q$ in the state $|\psi\rangle$ will be $q$. 
Symmetries give observables

Time translations

There is a distinguished observable corresponding to energy: the Hamiltonian $H$. $-iH$ gives the infinitesimal action of the group $\mathbb{R}$ of time translations. This is the Schrödinger equation (for $\hbar = 1$)

$$\frac{d}{dt}\langle \psi \rangle = -iH\langle \psi \rangle$$

Note: $-iH$ is skew-adjoint, so the action of time translation is unitary.

Other examples

- Spatial translations, group $\mathbb{R}^3$, momentum operators $P_j$
- Rotations, group $SO(3)$, angular momentum operators $L_j$
- Phase transformations, group $U(1)$, charge operator $\hat{Q}$
Symmetry and quantum mechanics

General principle

If a Lie group $G$ acts on a physical system, expect a unitary representation of $G$ on $\mathcal{H}$, i.e. for $g \in G$ have unitary operators $\pi(g)$ such that

$$\pi(g_1)\pi(g_2) = \pi(g_1 g_2)$$

Differentiating, get a unitary representation of the Lie algebra $\mathfrak{g}$ on $\mathcal{H}$, i.e. for $X \in \mathfrak{g}$ have skew-adjoint operators $\pi'(X)$ such that

$$[\pi'(X_1), \pi'(X_2)] = \pi'([X_1, X_2])$$

The $i\pi'(X)$ will give observables (self-adjoint operators). If

$$[\pi'(X), H] = 0$$

these are “symmetries” and there will be “conservation laws”: states that are eigenvectors of $i\pi'(X)$ at one time will remain eigenvectors at all times, with the same eigenvalue.
Example: \(SU(2)\)

\(SU(2)\) and \(\mathfrak{su}(2)\)

The group \(G = SU(2)\) (2 by 2 unitary matrices with determinant 1) has Lie algebra \(\mathfrak{su}(2) = \mathbb{R}^3\), with Lie bracket the cross-product.

Representations of \(SU(2)\) and \(\mathfrak{su}(2)\)

Corresponding to elements of a basis \(X_j \in \mathfrak{su}(2)\) have “spin” observables \(S_j = i\pi'(X_j)\). These act on a state space \(\mathcal{H}\) that is a representation of \(SU(2)\). All such representations will be a sum of copies of irreducible representations. The irreducible representations are labeled by \(n = 0, 1, 2, \ldots\), and called by physicists the representations of “spin \(n/2\)”, of dimension \(n + 1\).
Example: canonical quantization and the Heisenberg Lie algebra

Main way to create quantum systems (Dirac). Start with phase space $\mathbb{R}^{2n}$, coordinates $q_j, p_j$ and Poisson bracket satisfying

$$\{q_j, p_j\} = 1$$

The $1, q_j, p_j$ are a basis of a Lie algebra, the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$, with Poisson bracket the Lie bracket. This is the Lie algebra of a group, the Heisenberg Lie group.

“Canonical quantization” is given by operators $Q_j, P_j$, satisfying the Heisenberg commutation relations (for $\hbar = 1$).

$$[−iQ_j, −iP_j] = −i1$$

Stone von-Neumann theorem

Up to unitary equivalence, there is only one non-trivial representation of the Heisenberg group.
For much much more detail, see my recent book available from Springer or at
The universal enveloping algebra \( U(\mathfrak{g}) \)

Given a Lie algebra representation, we also can consider the action of products of the \( \pi'(X_j) \). Can do this by defining, for any Lie algebra \( \mathfrak{g} \) with basis \( X_j \), the universal enveloping algebra \( U(\mathfrak{g}) \). This is the algebra with basis (here \( j < k \)),

\[
1, X_j^{n_j}, X_j^{n_j} X_k^{n_k}, \ldots
\]

and when one takes products one uses \( X_j X_k - X_k X_j = [X_j, X_k] \).

\( U(\mathfrak{g}) \) has a subalgebra \( Z(\mathfrak{g}) \) (the center) of elements that commute with all \( X_j \). These will act as scalars on irreducible representations.

Example: the Casimir operator of \( \mathfrak{su}(2) \)

\[
X_1^2 + X_2^2 + X_3^2 \in Z(\mathfrak{su}(2))
\]

and the Casimir operator

\[
(\pi'(X_1))^2 + (\pi'(X_2))^2 + (\pi'(X_3))^2
\]

acts as \( n(n + 1) \) on the spin \( \frac{n}{2} \) representation.
Quantizing $g^*$

For any Lie algebra, the dual space $g^*$ can be thought of as a generalized classical phase space, with Poisson bracket given by the Lie algebra bracket. The Poisson bracket is determined by its values on linear functions on the phase space, but a linear function on $g^*$ is just an element $X \in g$, and one has

$$\{X, Y\} = [X, Y]$$

Quantization of this classical system should give an algebra of operators, the obvious choice is $U(g)$.

Philosophy of quantization

Quantum systems should correspond to Lie algebras, with classical phase space $g^*$, and algebra of operators $U(g)$.

Problem: how does one construct representations of $g$ (and thus $\mathcal{H}$)?
The "orbit philosophy"

Constructing quantum systems corresponding to a Lie algebra $\mathfrak{g}$ is thus equivalent to the (well-known and hard) problem of constructing (unitary) representations of $\mathfrak{g}$. One approach to this problem is

**Orbit philosophy of Kirillov-Kostant-Souriau**

For any $\mu \in \mathfrak{g}^*$, consider the orbit $\mathcal{O}_\mu \subset \mathfrak{g}^*$ under the co-adjoint action of $G$ on $\mathfrak{g}^*$. This turns out to be a symplectic manifold. Take this as a classical phase space and "quantize" it, with the state space of the quantum system giving an irreducible representation. Co-adjoint orbits should thus parametrize irreducible representations, and provide the material for their construction.

To do this, in some cases one can consult physicists to find out how to construct a quantum system given a classical system. In general, "geometric quantization" is supposed to provide such a quantization.
Examples

The Heisenberg Lie algebra

Non-trivial co-adjoint orbits are copies of $\mathbb{R}^{2n} \subset \mathfrak{h}_{2n+1}$, parametrized by a constant $c \neq 0$. Physicists have many unitarily equivalent ways of producing from this a representation of $\mathfrak{h}_{2n+1}$, i.e. a state space $\mathcal{H}$ and operators $Q_j, P_j, C$. Here $C = \pi'(1)$ is central and must act as a scalar. Taking this to be $-i\hbar$ gives the Heisenberg commutation relations

$$[-iQ_j, -iP_j] = -i\hbar 1$$

$\mathfrak{su}(2)$

The non-trivial co-adjoint orbits are spheres in $\mathbb{R}^3$ of arbitrary radius. Only if the radius satisfies an integrality condition does one get an irreducible representation (one of the spin $\frac{n}{2}$ representations). Geometric quantization: construct a holomorphic line bundle over the sphere, realize the representation as holomorphic sections (Borel-Weil construction).
Fermionic quantization

The version of quantization considered so far is “bosonic”, starting with classical observables polynomial functions on $\mathfrak{g}^*$, or equivalently the symmetric algebra $S^*(\mathfrak{g})$, with this commutative algebra quantized as $U(\mathfrak{g})$.

“Fermionic” quantization starts by replacing the symmetric tensor algebra $S^*(\mathfrak{g})$ by $\Lambda^*(\mathfrak{g})$, the anti-symmetric tensor algebra. One can think of this as “polynomials in anti-commuting generators”.

Quantization then replaces commutators by anticommutators, and the algebra $U(\mathfrak{g})$ by the Clifford algebra $\text{Cliff}(\mathfrak{g})$. This requires the choice of a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$. We will later need this to be invariant and non-degenerate (Lie algebras that have this are called “quadratic”).
Clifford algebras and spinors

Clifford algebra

For any vector space \( V \) with symmetric bilinear form \((\cdot, \cdot)\), \( \text{Cliff}(V) \) is the algebra generated by \( v \in V \), with relations

\[
v_1 v_2 + v_2 v_1 = 2(v_1, v_2)
\]

As a vector space \( \text{Cliff}(V) = \Lambda^*(V) \) (multiplication is different)

The structure of irreducible modules is much simpler for \( \text{Cliff}(g) \) than for \( U(g) \). For \( V \) complex and \( \text{dim } V = 2n \)

Spinors

\[
\text{Cliff}(V) = \text{End}(S) = S \otimes S^*
\]

where \( S = \mathbb{C}^{2^n} \).
Quantum Weil algebra

One can combine the “bosonic” and “fermionic” quantizations into what physicists would call a “supersymmetric” quantization.

The quantum Weil algebra

Given a Lie algebra \( \mathfrak{g} \) with invariant non-degenerate bilinear form \((\cdot, \cdot)\), the quantum Weil algebra is the algebra

\[
\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{g})
\]

with (super)commutation relations

\[
[X \otimes 1, Y \otimes 1]_\mathcal{W} = [X, Y] \otimes 1, [X \otimes 1, 1 \otimes Y]_\mathcal{W} = 0, [1 \otimes X, 1 \otimes Y]_\mathcal{W} = 2(X, Y)
\]

Note that this is a \( \mathbb{Z}_2 \) graded algebra, with generators of \( U(\mathfrak{g}) \) even, generators of \( \text{Cliff}(\mathfrak{g}) \) odd.
The Dirac operator

For quadratic Lie algebras one can identify $\mathfrak{g} = \mathfrak{g}^*$, and one has a quadratic Casimir element of $U(\mathfrak{g})$ given by

$$\Omega = \sum e^j e_j$$

where $e_j$ is an orthonormal basis of $\mathfrak{g}$, $e^j$ the dual basis. A wonderful discovery of Dirac in 1928, was that the introduction of a Clifford algebra and spinors allowed the construction of a square root of the Casimir operator (which in his case was a Laplacian). Here one defines:

Kostant Dirac operator

$\Omega$ has (up to a constant), a square root given by

$$\mathcal{D}_\mathfrak{g} = \sum (e^j \otimes e_j) + 1 \otimes q(\phi)$$

where $\phi \in \Lambda^3(\mathfrak{g})$ and $q$ is the quantization map.
The differential super Lie algebra $\hat{\mathfrak{g}}$

For another point of view on $\mathcal{W}(\mathfrak{g})$, one can define a “super Lie algebra”

$$\hat{\mathfrak{g}} = \mathfrak{g} \oplus \epsilon \mathfrak{g} \oplus \mathbb{R} c$$

Where $\epsilon^2 = 0$ and the super Lie bracket relations are

$$[X, Y]_{\hat{\mathfrak{g}}} = [X, Y], \quad [X, \epsilon Y]_{\hat{\mathfrak{g}}} = \epsilon [X, Y], \quad [\epsilon X, \epsilon Y]_{\hat{\mathfrak{g}}} = (X, Y)c$$

The operator $\frac{\partial}{\partial \epsilon}$ provides a differential $d$ on $\hat{\mathfrak{g}}$ satisfying $d^2 = 0$. One then has

$$\mathcal{W}(\mathfrak{g}) = U(\hat{\mathfrak{g}})/(c - 1)$$

with replacing $\mathfrak{g}$ by $\hat{\mathfrak{g}}$ (with its differential) an alternate motivation for the replacement of $U(\mathfrak{g})$ by $\mathcal{W}(\mathfrak{g})$. 
Dirac cohomology

Since the square of the Dirac operator $\mathcal{D}_g$ is a Casimir operator (up to a constant) and thus is central, the operator

$$d(\cdot) = [\mathcal{D}_g, \cdot]_\mathcal{W}$$

satisfies $d^2 = 0$.

One can define

Dirac cohomology

The Dirac cohomology $H_D(g)$ of a quadratic Lie algebra $g$ is the cohomology of $d$ on $\mathcal{W}(g)$. The Dirac cohomology of a representation $V$ of $g$ is given by

$$H_D(g, V) = \ker \mathcal{D}_g|_{V \otimes s}$$

$H_D(g)$ acts on $H_D(g, V)$. 
Relative Dirac cohomology

It turns out the Dirac cohomology itself is trivial, with \( H_{\mathcal{D}}(\mathfrak{g}) = \mathbb{R} \) and \( H_{\mathcal{D}}(\mathfrak{g}, V) \) zero. What is interesting is, for \( \mathfrak{r} \subset \mathfrak{g} \) (\( \mathfrak{r} \) a quadratic Lie subalgebra), a relative version of the Dirac cohomology. The super Lie algebra \( \hat{\mathfrak{r}} \) acts by the adjoint representation on \( \mathcal{W}(\mathfrak{g}) \), with

\[
\mathcal{W}(\mathfrak{g})^{\hat{\mathfrak{r}}} = (U(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s}))^{\mathfrak{r}}
\]

where \( \mathfrak{s} \) is the orthogonal complement of \( \mathfrak{r} \) in \( \mathfrak{g} \).

Since \( \mathcal{D}_{\mathfrak{g}/\mathfrak{r}} = \mathcal{D}_{\mathfrak{g}} - \mathcal{D}_{\mathfrak{r}} \) preserves \( (U(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s}))^{\mathfrak{r}} \) and has square an element of the center, one can define

Relative Dirac cohomology

\( H_{\mathcal{D}}(\mathfrak{g}, \mathfrak{r}) \) is the cohomology of \( d = [\mathcal{D}_{\mathfrak{g}/\mathfrak{r}}, \cdot]_{\mathcal{W}} \) on \( (U(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s}))^{\mathfrak{r}} \). It acts on

\[
H_{\mathcal{D}}(\mathfrak{g}, \mathfrak{r}, V) = \ker \mathcal{D}_{\mathfrak{g}|V \otimes S_{\mathfrak{s}}}
\]
Examples

The relative Dirac cohomology $H_{\mathcal{D}}(\mathfrak{g}, \mathfrak{r}, V)$ with its action by $H_{\mathcal{D}}(\mathfrak{g}, \mathfrak{r})$ provides an interesting invariant of $V$. Some examples:

- $\mathfrak{g}$ complex semi-simple, $\mathfrak{r} = \mathfrak{h}$ the Cartan subalgebra and $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{\bar{n}} \oplus \mathfrak{h}$. Here $H_{\mathcal{D}}(\mathfrak{g}, \mathfrak{h}, V)$ is the Lie algebra cohomology $H^*(\mathfrak{n}, V)$ (up to $\rho$-twist).

- For many other cases of $\mathfrak{g}$ reductive, see Huang-Pandzic, *Dirac operators in representation theory*

- (Work in progress): $\mathfrak{g}$ the Heisenberg Lie algebra. Not a quadratic Lie algebra, but an extension (the oscillator Lie algebra) is.

- (Work in progress): $\mathfrak{g}$ the Poincaré Lie algebra. Not a quadratic Lie algebra, but get the actual Dirac operator used in physics.
Some known relations between constructions of representation and Dirac operators:

- **Compact Lie groups**: Borel-Weil-Bott construction of irreducible representations as sheaf cohomology groups of holomorphic line bundles on $G/T$ can be reinterpreted as kernels of a Dirac operator.
- **Real semi-simple Lie groups**: can construct discrete series representations using Dirac operators.
- **Freed-Hopkins-Teleman**: provide a construction using families of Dirac operators that associates an orbit to an irreducible representation (for compact Lie groups).
Conclusions

Modern philosophy of quantization
Quantum systems should correspond to Lie algebras \( \mathfrak{g} \), with classical phase space \( \mathfrak{g}^* \) and algebra of operators \( U(\mathfrak{g}) \). Ad hoc “geometric quantization” techniques associate irreducible representations to orbits in \( \mathfrak{g}^* \).

Post-modern philosophy of quantization
Quantum systems should correspond to Lie algebras, with classical phase space a derived geometry based on \( \hat{\mathfrak{g}} \) and algebra of operators \( \mathcal{W}(\mathfrak{g}) \) with differential given by the Dirac operator \( \mathcal{D}_{\mathfrak{g}} \).

May give a Dirac operator-based geometric quantization.
Applications in physics:
- New construction of elementary particle states using representations of the Poincaré Lie algebra.
- New version of BRST method for dealing with gauge symmetries.