Unifying Foundations for Physics and Mathematics

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Outline

Note: These slides at
`https://www.math.columbia.edu/~woit/paris-slides.pdf`

General Relativity

The gravitational force is described by the

General Theory of Relativity (1915)

Dynamical variables: geometry of space-time
Classical dynamical equations: Einstein equations for metric

The mathematics used is

(Pseudo)-Riemannian geometry

Riemann (1854): first version of geometry with local curvature
Cartan/Ehresmann (1923-1943): general framework for differential geometry in terms of connections and curvature

The Standard Model

Matter and the other known forces (electromagnetic, weak, strong) are described by:

The Standard Model (1973)

Dynamical variables: connections, spinor fields, scalar field (Higgs)
Classical dynamical equations: Yang-Mills equations, Dirac equation, Wave equation

The mathematics used is

Geometry of connections, curvature and spinors

Same Cartan/Ehresmann (1923-1943) geometry formalism in terms of connections and curvature as for GR.
Cartan (1913), Weyl (1929): spinor geometry

Actually want quantization of the above.

The problem with the unified theory

Main problem with the unified theory

It's too good: no clear examples of data in conflict with the theory

Huge amount of attention devoted to "dark matter" as a potential conflict, and to quantization of gravity as consistency problem, but little progress from this since unified theory came together in 1973.

Absent guidance from experiment, can one do better than current situation?

Currently popular point of view: No, the unified theory is just an environmental artifact of the multiverse
A different point of view: The unified theory is a very special mathematical structure, can we understand the significance of this?

Mathematics as separate fields

Conventionally, mathematics is divided into different fields, each of which ramifies into more specialized subfields.

- Analysis
- Topology
- Geometry
- Algebra
- Number theory

This is still how the subject is taught.
These subfields share a foundation in set theory/mathematical logic.
Less commonly noticed are unifying principles. For example: algebra and geometry are related by thinking of an algebra as some sort of functions on some sort of geometrical space. In studying a specialized subfield one often runs into a connection to another one seemingly far removed.

Unification: the Atiyah-Singer Index Theorem

Some unifying themes in mathematics:

Th Atiyah-Singer index theorem (1963) relates

- Analysis (solutions to differential equations in background fields on a manifold)
- Topology (of the manifold and data describing the equation and background fields)
- Geometry (curvature of the manifold and background fields)

Unification: Representation Theory

A well-known unifying principle in mathematics: **Symmetry**

Usually presented in terms of theory of groups, but more accurately it is about groups and what they act on: representation theory.

Representation theory is a mathematical field which cuts across the usual subfields, often providing a unifying principle tying together different subjects.

It unifies with the Atiyah-Singer index theorem in an "equivariant" version of index theory.

Unification: the Langlands Program

The original Langlands program (1967) unifies number theory and representation theory, relating Galois groups of number fields and their representations with Lie groups and their representations.
Further unification, with geometry, came from the geometric Langlands program (1980s-90s). This is currently a very active field of research.
Edward Frenkel refers to this as **"A Grand Unified Theory of Mathematics"**.

Unifying the Unifications

Each of the above unifying themes in mathematics has significant connections to the unified theory in physics:

Atiyah-Singer index theorem

Differential operators are characterised by their relation to a generator: the Dirac operator. The theorem can then be understood in terms of the physics of a single elementary spin $1/2$ particle.

Representation theory

Study of symmetry in quantum theory is essentially same mathematics as theory of unitary representations.

Langlands Program

Geometric Langlands can be understood in terms of (twisted) $N = 4$ supersymmetric Yang-Mills theory (Witten-Kapustin), similar to standard model.

Recent Developments

A very recent major development (Feb. 2021) unifying number theory Langlands and geometric Langlands:

Fargues-Scholze : *Geometrization of the local Langland correspondence*

In number theory, one thinks of the primes as the points of a "space" (called *Spec*Z), and "local Langlands" is the version of the Langlands program that describes what happens near a prime p. Fargues-Scholze show this can be thought of as geometric Langlands on a curve, the "Fargues-Fontaine" curve.

One can think of *Spec*Z as also having a point corresponding not to a prime, but to the real numbers R. The Fargues-Fontaine curve at the real point is the so-called "twistor P^1": the sphere with its antipodal map. **Twistor theory**: a compelling way of thinking about four-dimensional geometry and conformally invariant physics in four-dimensions. In twistor theory, a point in four-dimensional space is exactly a twistor P^1.
Is this an aspect of deep unification? Is it a random coincidence?

Implications

Philosophy of mathematics: A different perspective on "Foundations of mathematics", which are usually thought of as having nothing to do with physics. On the "Is mathematics discovered or invented?" question, this takes the "discovered" (like physics) side.

Methodological implications for physics: search for an improved unified theory should concentrate on current ideas that unify deeply with mathematics, together with new ideas with this property. Very unlikely to lead anywhere: the idea that the deep geometry of the Standard Model is just a long-distance effective artifact of something fundamentally different.

Methodological implications for mathematics: fundamental physics will remain an ongoing inspiration for new deep mathematical ideas.

A difficult problem: Current doctoral education in theoretical physics gives no training in the mathematics needed to appreciate a fundamental unity of the two subjects. Similarly for doctoral education in mathematics. Few researchers get the training needed to effectively use the insights of both sides.

Last Thoughts

After spending many years among both physicists and mathematicians, I have some idea what many of them are likely to think of this.

A typical physicist's point of view: this is mysticism. Where's the calculation?

A typical mathematician's point of view: this is vague and meaningless. Where's the theorem?

Some reasons to worry:

- I get a lot of emails from people who have a new unified theory. They're mostly quite clearly worthless.
- Deluding oneself by seeing deep connections in unrelated events is a common human problem.

And yet, this sort of unity seems to me very real, since early days as a student, more so the more I've learnt over the years.