Topics in Representation Theory: The Metaplectic Representation

The metaplectic representation can be constructed in a way quite analogous to the construction of the spin representation. In this case instead of dealing with a finite dimensional exterior algebra one has to use an infinite dimensional function space, so the analysis becomes non-trivial. The discussion we'll give here is very sketchy, for a rigorous and detailed treatment, see [1].

1 The Bargmann-Fock Representation

We have seen that, after complexification, the Heisenberg algebra can be identified with the CCR algebra generated by $2n$ operators $a_i, a_i^\dagger$ satisfying the relations

$$[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, [a_j, a_k^\dagger] = \delta_{jk}$$

Just as the CAR algebra has a representation on the exterior algebra, the CCR algebra has a representation on the symmetric algebra, i.e. the polynomial algebra $\mathbf{C}[w_1, \cdots, w_n]$, with

$$a_j^\dagger \to w_j \cdot$$

and

$$a_j \to \frac{\partial}{\partial w_j}$$

satisfy the commutation relations since all commutator are zero except

$$[\frac{\partial}{\partial w_j}, w_j]w_j^n = (n+1)w_j^n - nw_j^n = w_j^n$$

We would like a representation on a Hilbert space $\mathcal{H}$ and want a_j and $a_j^\dagger$ to be adjoint operators with respect to the inner product on $\mathcal{H}$. We can choose $\mathcal{H}$ to be the following function space:

Definition 1 (Fock Space). *Given an identification $\mathbf{R}^{2n} = \mathbf{C}^n$, Fock space is the space of entire functions on $\mathbf{C}^n$, with finite norm using the inner product*

$$(f_1(\mathbf{w}), f_2(\mathbf{w})) = \frac{1}{\pi^n} \int_{\mathbf{C}^n} \overline{f_1(\mathbf{w})} f_2(\mathbf{w}) e^{-\overline{\mathbf{w}} \cdot \mathbf{w}}$$

An orthonormal basis of $\mathcal{H}$ is given by apropriately normalized monomials. For the case $n = 1$, one can calculate

$$\begin{aligned}
(w^m, w^n) &= \frac{1}{\pi} \int_{\mathbf{C}} \bar{w}^m w^n e^{-|w|^2} \\
&= \frac{1}{\pi} \int_0^\infty (\int_0^{2\pi} e^{i\theta(n-m)} d\theta) r^{n+m} e^{-r^2} r dr \\
&= n! \delta_{n,m}
\end{aligned}$$

and see that the functions $\frac{w^n}{\sqrt{n!}}$ are orthornormal.

We'll leave it as an exercise to show that, with respect to this inner product, a_j and $a_j^\dagger$ are adjoint operators. One can show that $\mathcal{H}$ with this action of the a_j and $a_j^\dagger$ is an irreducible representation of the Heisenberg group and we will often denote this representation by M. By the Stone-von Neumann theorem it is the only one up to unitary equivalence.

2 The Metaplectic Representation

The action of the Heisenberg group on M is given by the action of linear functions of a_j and $a_j^\dagger$ on the Fock space. In the case of the spin representation, we saw that the Lie algebra of the spin group consisted of quadratic elements of the Clifford algebra, with the elements $\frac{1}{2}e_ie_j$ satisfying the same commutation relations as L_{ij}, the generators of rotations of in the $i-j$ plane. We won't take the time here to go into the details of what the commutation relations are for $\mathfrak{sp}(2n,\mathbf{R})$, but just as in the orthogonal group case, one can construct operators with these commutation relations using the quadratic elements, now in $\mathfrak{h}_n$ rather than $C(n)$.

$\mathfrak{sp}(2n,\mathbf{R})$ has a subalgebra $\mathfrak{u}(n)$, and the operators

$$\frac{1}{2}(a_i^\dagger a_j + a_j a_i^\dagger) = \frac{1}{2}(w_i\frac{\partial}{\partial w_j} + \frac{\partial}{\partial w_j}w_i)$$

are a representation of this subalgebra on M. The other elements of $\mathfrak{sp}(2n,\mathbf{R})$ can be represented by operators

$$a_i^\dagger a_j^\dagger = w_iw_j$$

and

$$a_ia_j = \frac{\partial^2}{\partial w_i\partial w_j}$$

A choice of elements of $\mathfrak{sp}(2n,\mathbf{R})$ that correspond to a maximal torus (this is a subalgebra of $\mathfrak{u}(n)$) are

$$\frac{1}{2}(a_i^\dagger a_i + a_i a_i^\dagger) = a_i^\dagger a_i + \frac{1}{2} \quad i = 1,\cdots,n$$

where we have used the relation

$$a_ia_i^\dagger = 1 + a_i^\dagger a_i$$

coming from the CCR.

On the Fock space $a_i^\dagger a_i$ has eigenvalues the non-negative integers since

$$a_i^\dagger a_i w^n = w\frac{d}{dw}w^n = nw^n$$

The eigenvalues of the generators of the maximal torus will be of the form $n+\frac{1}{2}$. Upon exponentiation to the group $Sp(2n,\mathbf{R})$ we see that M will only be a projective representation: as one takes one of the angles in the maximal torus from 0 to 2π, its action on M will go from I to $-I$ and a double covering of the group is required to make the representation single valued. This double covering is called the metaplectic group and is various written as $Mp(2n,\mathbf{R})$ or $\widetilde{Sp}(2n,\mathbf{R})$.

The representation of $Mp(2n,\mathbf{R})$ on M is called the metaplectic representation. Note that since $Mp(2n,\mathbf{R})$ is constructed out of quadratic elements it leaves the parity of elements of M invariant, so M actually breaks up into two irreducible components

$$M = M^{+} + M^{-}$$

corresponding to the even and odd polynomials in the Fock space construction of M.

Physically M can be thought of as the Hilbert space of a harmonic oscillator with n degrees of freedom and Hamiltonian operator

$$H=\sum_{i}^{n}\frac{1}{2}(P_i^2+Q_i^2)=\sum_{i}^{n}\frac{1}{2}(a_i^{\dagger}a_i+a_ia_i^{\dagger})$$

Specializing to the case $n=1$ of a harmonic oscillator with a single degree of freedom we get a projective representation of $Sp(2,\mathbf{R})=SL(2,\mathbf{R})$ on the space of polynomials in a single variable w. The Hamiltonian eigenvalues are the energy levels and equal to

$$\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\cdots$$

The lowest energy state has energy not 0 but $\frac{1}{2}$, sometimes known as the "zero-point energy" and physically explained by the idea the since the only way to have zero energy would be to simultaneous have zero momentum and zero position, that would violate the uncertainty principle. The Hamiltonian operator generates a $U(1)$ maximal torus subgroup of $SL(2,\mathbf{R})$, but there are two other generators of the Lie algebra of this group that also act on the Hilbert space M. Since they don't commute with the Hamiltonian, while they in some sense generate symmetries of the system, states with well-defined energies are not eigenstates for these operators.

3 The Siegel Space of Complex Structures

The construction we have given of the metaplectic representation M using Fock space depends upon an identification $\mathbf{R}^{2n}=\mathbf{C}^n$ and thus upon a choice of complex structure J which determines a decomposition $\mathbf{R}^{2n}\otimes\mathbf{C}=W_J\oplus\bar{W}_J$. M varies with a change in J and we should perhaps better think of it as M_J, generated by a set of operators $(a_J)_i^{\dagger}$, corresponding to a basis of W_J, acting on a "vacuum vector" Ω_J.

The set of possible complex structures J that we want to consider is different than in the orthogonal case. Here everything is based not upon an underlying symmetric form, but instead an underlying non-degenerate antisymmetric two-form $S(\cdot,\cdot)$. Such a two-form S is called a symplectic form. A choice of complex structure J should preserve S, i.e.:

$$S(v,w) = S(Jv, Jw)$$

We will call such a complex structure a symplectic complex structure.

Claim 1.

$$(v,w)_J = S(Jv,w)$$

is a non-degenerate symmetric form.

This is true because

$$(v,w)_J = S(Jv,w) = S(JJv,Jw) = -S(v,Jw) = S(Jw,v) = (w,v)_J$$

Definition 2. *For each choice of symplectic complex structure J, there is a hermitian form*

$$< v,w >_J = S(Jv,w) + iS(v,w)$$

This is hermitian since

$$< w,v >_J = \overline{< v,w >_J}$$

Note that we could have done the same thing in the orthogonal case where we started with an inner product $< \cdot,\cdot >$ and could have constructed a hermitian form

$$< v,w >_J = < v,w > + i < Jv,w >$$

In the orthogonal case, if we started with a positive definite inner product, the hermitian form would be positive definite. In the symplectic case, if we want our hermitian form to be positive definite, we need to impose that as a separate condition. The set of symplectic complex structures corresponding to positive definite hermitian forms is called the Siegel space, i.e.

Definition 3. *The Siegel space associated to $(\mathbf{R}^{2n}, S(\cdot,\cdot))$ is the space of linear maps*

$$J : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$$

such that

1.
$$J^2 = -1$$

2.
$$S(Jv,Jw) = S(v,w)$$

3.

$$< v, v >_J = S(Jv, v) \geq 0, \;\; v \neq 0$$

The Siegel space can be identified with $Sp(2n, \mathbf{R})/U(n)$ since each J is in $Sp(2n, \mathbf{R})$ and different elements of $Sp(2n, \mathbf{R})$ that give the same decomposition $\mathbf{R}^{2n} \otimes \mathbf{C} = W_J \oplus \bar{W}_J$ are related by a $U(n) \subset Sp(2n, \mathbf{R})$.

To any choice of J in the Siegel space, there is an associated Fock space

Definition 4. *$\mathcal{H}_J$ is the space of entire functions on W_J such that*

$$(f_1(\mathbf{w}), f_2(\mathbf{w}))_J = \frac{1}{\pi^n} \int_{W_J} \overline{f_1(\mathbf{w})} f_2(\mathbf{w}) e^{-<\mathbf{w} \cdot \mathbf{w}>_J}$$

In a similar fashion to the spin case, one can pick a single $J = J_0$ and then represent the metaplectic representation for other choices of J on $\mathcal{H}_{J_0}$ by choosing a varying vacuum vector $\Omega_J \in \mathcal{H}_{J_0}$, which will be given by a Gaussian function. The complex lines generated by Ω_J define a complex line bundle over the Siegel space and an invariant definition of the metaplectic representation is as holomorphic sections of this line bundle. This is quite analogous to the Borel-Weil construction of the spin representation, but the non-compactness of the Siegel space makes the analysis trickier and the representation is infinite dimensional.

4 The Schrödinger Representation

The Fock representation of Heisenberg commutation relations is not the first one generally considered in quantum mechanics texts. One can also represent the Heisenberg commutation relations on $L^2(\mathbf{R}^n)$ with q_i coordinates on $\mathbf{R}^n$ and the generators represented as operators as follows

$$Q_i = \text{multiplication by } q_i$$

$$P_i = -i \frac{\partial}{\partial q_i}$$

By the Stone-von Neumann theorem, this representation must be unitarily related to the Fock representation. The unitary transformation that takes an L^2 function of the q_i to a holomorphic function of n complex variables is called the Bargmann transform. For details about it, see [1]. It can be interpreted as the transform that takes

$$f \rightarrow \text{Analytic continuation of } e^{-\Delta/2} f$$

5 A Detailed Analogy

The analogous constructions of the spin and metaplectic representations that we have been discussing are often dealt with by physicists as analogous constructions using "odd" (anti-commuting) and "even" (commuting) variables. We haven't pursued this sort of language, but some of the details of the analogy between the spin and the metapectic cases are contained in the following table:

Spin	Metaplectic
$\mathbf{R}^{2n}$ "odd" variables	$\mathbf{R}^{2n}$ "even" variables
Q: Symmetric Quadratic Form on $\mathbf{R}^{2n}$	S: Antisymmetric 2-form on $\mathbf{R}^{2n}$
Clifford Algebra $C(2n)$	Heisenberg Algebra $\mathfrak{h}_n$
CAR algebra	CCR algebra
–	Heisenberg group H_n
Exterior algebra $\Lambda^*(\mathbf{R}^{2n})$	Symmetric algebra $S^*(\mathbf{R}^{2n})$
$SO(2n)$: Automorphisms of $C(2n)$	$Sp(2n,\mathbf{R})$: Automorphisms of $\mathfrak{h}_n$
$Spin(2n)$: Double cover of $SO(2n)$	$Mp(2n,\mathbf{R})$: Double cover of $Sp(2n,\mathbf{R})$
$\mathfrak{spin}(2n)$: quadratic elements of $C(2n)$	$\mathfrak{mp}(2n)$: quadratic elements of $\mathfrak{h}_n$
J: Orthogonal complex structure	J: Symplectic complex structure
$\mathbf{R}^{2n} = W_J \oplus \bar{W}_J$	$\mathbf{R}^{2n} = W_J \oplus \bar{W}_J$
Irreducible $C(n)$ module: $\Lambda^*(W_J)$	Irreducible H_n representation: $S^*(W_J)$
$S = \Lambda^*(W_J) \times (\Lambda^n(W_J))^{-\frac{1}{2}}$	$M = S^*(W_J) \times (\Lambda^n(W_J))^{\frac{1}{2}}$
S: Projective Rep. of $SO(2n)$	M: Projective Rep. of $Sp(2n,\mathbf{R})$
S: Spinor Rep., Rep. of $Spin(2n)$	M: Metaplectic Rep., as Rep. of $Mp(2n,\mathbf{R})$
$S = S^+ + S^-$	$M = M^+ + M^-$
$SO(2n)/U(n)$: Orthogonal complex structures	$Sp(2n,\mathbf{R})/U(n)$: Symplectic complex structures
Vacuum vector $\Omega_J \in S$	Vacuum vector $\Omega_J \in M$
Line bundle L, $L \otimes L = (\det)^{-1}$	Line bundle L, $L \otimes L = \det$
$S = \Gamma_{hol}(L)$	$M = \Gamma_{hol}(L)$
Particle with spin $\frac{1}{2}$ in $2n$ dimensions	Harmonic oscillator with n degrees of freedom

References

[1] Folland, G., *Harmonic Analysis in Phase Space,* Princeton University Press, 1989.