Topics in Representation Theory: Lie Groups, Lie Algebras and the Exponential Map

Most of the groups we will be considering this semester will be matrix groups, i.e. subgroups of $G = Aut(V)$, the group of invertible linear transformations from V to itself for V an n-dimensional vector space over a field F. Once a basis for V has been chosen then elements of G are invertible n by n matrices with entries in F and $G = Gl(n, F)$. Group multiplication is just matrix multiplication.

Associated to the group $Aut(V)$ is the Lie algebra $\mathfrak{g} = End(V)$ of linear endomorphisms of V, i.e linear maps from V to itself, not necessarily invertible. Again, once one has a basis of V, $\mathfrak{g} = M(n, F)$, all n by n matrices with entries in F (not necessarily invertible). The matrix exponential gives a map

$$exp : X \in \mathfrak{g} \rightarrow exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \in G$$

On $\mathfrak{g} = End(V)$ there is a non-associative bilinear skew-symmetric product given by taking commutators

$$(X, Y) \in \mathfrak{g} \times \mathfrak{g} \rightarrow [X, Y] = XY - YX \in \mathfrak{g}$$

While matrix groups and their subgroups comprise most examples of Lie groups that one is interested in, we will be defining Lie groups in geometrical terms for several reasons

- There are important examples of Lie groups that are not matrix groups. One that we will consider later on in this course is the metaplectic group.
- The geometry of Lie groups is fundamental to the entire modern approach to geometry. Understanding the geometry of Lie groups is crucial to understanding geometry in general.
- We would like to use a geometrical approach to the construction of Lie group representations.

The geometrical framework means that we will be restricting attention to Lie groups that are manifolds and thus locally like real vector spaces. If one is interested in groups defined over more general fields, to work geometrically one needs to enter the realm of algebraic geometry and algebraic groups, something that we will not do in this course.

The amount of geometry we will need is relatively minimal. You should be familiar with the following basic geometrical concepts, there is a short review of them in section VII.1 of [1], a good textbook for a detailed exposition is [2]. What follows is a short review, emphasizing some results we will be using in this course.

1 Differential Geometry, a Review

An n-dimensional smooth or (C^∞) manifold is a space M covered by open sets U_α together with "coordinate maps"

$$\phi_\alpha : U_\alpha \to \mathbf{R}^n$$

such that ϕ_α is a homeomorphism of U_α and its range, and

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a C^∞ map. In this course, maps should be assumed to be smooth unless otherwise stated.

Definition 1 (Vector Field). *A vector field on a smooth manifold M is a derivation*

$$X : C^\infty(M) \to C^\infty(M)$$

i.e. a linear map such that on a product of functions f and g

$$X(fg) = f(Xg) + (Xf)g$$

Locally one can choose coordinates so that such a derivation is a linear combination of the derivatives with respect to the coordinates

$$X = \sum_{i=1}^{n} a_i(x_1, \cdots, x_n) \frac{\partial}{\partial x_i}$$

and with respect to this choice of coordinates, a vector field is given at each point by the n-vector $(a_1, \cdots, a_n)$. Sometimes we'll also refer to the value of a vector field at a point $m \in M$ as X_m, this can be thought of in terms of derivations acting on germs of functions at m. The space of such X_m makes up the tangent space to M at m, called $T_m(M)$.

On the space of vector fields there is an anticommutative bilinear operation:

Definition 2 (Lie Bracket). *The Lie bracket of two vector fields X and Y is defined by*

$$[X, Y] = X \circ Y - Y \circ X$$

The Lie bracket of vector fields satisfies the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

Vector fields behave "covariantly" under smooth maps, i.e. for a smooth map of smooth manifolds

$$\psi : M_1 \to M_2$$

there is a "push-forward" map ψ_*, the differential of ψ, defined by:

$$\psi_* X(g) = X(g \circ \psi)$$

that takes vector fields on M_1 to vector fields on M_2 (here $g \in C^\infty(M_2)$).

We will also need to use differential forms on M. Recall that at each point on a manifold M, the differential k-forms, $\Omega^k(M)$ are multilinear, anti-symmetric, maps taking k vectors to $\mathbf{R}$. There is an exterior multiplication map

$$(\omega_1, \omega_2) \in \Omega^k(M) \otimes \Omega^l(M) \rightarrow \omega_1 \wedge \omega_2 \in \Omega^{k+l}(M)$$

and an exterior derivative

$$d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

Under smooth maps differential forms behave "contravariantly", i.e. for a map ψ as above there is a "pull-back" map

$$\psi^* : \Omega^k(N) \rightarrow \Omega^k(M)$$

A vector field X on M provides "interior product" operation on differential forms:

$$i_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$$

One can integrate a vector field to get a transformation of M as follows:

Definition 3 (Exponential Map). *Given a vector field X on M, a map*

$$\Phi(t, m) : \mathbf{R} \times M \rightarrow M$$

such that

$$\lim_{t \to 0} \frac{f(\Phi(t, m)) - f(m)}{t} = Xf(m)$$

(for $f \in C^\infty(M)$) is called the "flow" or "exponential map" of X, and also written

$$exp(tX)(m) = \Phi(t, m)$$

For an open neighborhood of $m \in M$, the fundamental existence and uniqueness theorem for first-order differential equations gives existence and uniqueness of the exponential map for t in some interval about $t = 0$, and that $exp(tX)$ is a diffeomorphism where it is defined.

On its interval of definition, $exp(tX)$ satisfies the condition

$$exp(tX) \circ exp(sX) = exp((t + s)X)$$

so it can be thought of as a one-parameter group of diffeomorphisms of M, i.e. a homomorphism

$$\mathbf{R} \rightarrow Diff(M)$$

(although it may only be partially defined).

The derivative with respect to t of the differential of the exponential map is just the Lie bracket:

$$\frac{d}{dt}(exp(tX)_*(Y))|_{t=0} = [X, Y]$$

Later on in the course we may use some of the language of principal and vector bundles and their sections, but we will go over those concepts at that time.

2 Lie Groups, Lie Algebras and the Exponential Map

Basically a Lie group is a smooth manifold whose points can be (smoothly) multiplied together

Definition 4 (Lie Group). *A* **Lie group** *is a smooth manifold G together with a smooth multiplication map*

$$(g_1, g_2) \in G \times G \rightarrow g_1 g_2 \in G$$

and a smooth inverse map

$$g \in G \rightarrow g^{-1} \in G$$

that satisfy the group axioms.

For each element $g \in G$, there are two maps of G to itself, given by right and left multiplication.

$$L_g(h) = gh, \quad R_g(h) = hg^{-1}$$

(the inverse is there so that $R_g \circ R_h = R_{gh}$).

As on any manifold, there's an infinite dimensional space of vector fields on G, but in this case we can restrict attention to invariant ones.

Definition 5 (Lie Algebra). *The Lie algebra $\mathfrak{g}$ of G is the space of all left-invariant vector fields on G, i.e. vector fields satisfying*

$$X_{gh} = (L_g)_*(X_h)$$

The Lie bracket of of two left-invariant vector fields is left invariant, so it defines an antisymmetric bilinear product $[X, Y]$ on $\mathfrak{g}$ satisfying the Jacobi identity. Vector fields are first-order differential operators, the universal enveloping algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/(X \otimes Y - Y \otimes X - [X, Y])$$

is the space of all left-invariant differential operators on G.

One can identify $\mathfrak{g}$ more explicitly with the following second way of characterizing the Lie algebra:

Theorem 1. *The map*

$$X \in \mathfrak{g} \rightarrow X_e \in T_e(G)$$

given by restriction of a vector field to its value at the identity is a bijection.

Finally, there is a third way to characterize the Lie algebra using the exponential map.

Theorem 2. *For $X \in \mathfrak{g}$ the exponential map*

$$exp(tX) : G \to G$$

is defined for all t, and gives a smooth homomorphism

$$t \in \mathbf{R} \to exp(tX)(e) \in G$$

Any such homomorphism is of this form for some X.

For proofs of these theorems, see [1], section VII.2. We will interchangeably use these three characterizations of $\mathfrak{g}$.

We'll now go back to our main examples, matrix groups. $GL(n, \mathbf{R})$ is a Lie group, it is the open set of $\mathbf{R}^{n^2}$ given by $det(A) \neq 0$. It is a manifold with a global set of coordinates (the matrix entries), and the multiplication map is clearly a differentiable function of these coordinates. The Lie algebra $\mathfrak{gl}(n, \mathbf{R})$ is the tangent space at the identity and can be identified with the n by n real matrices. The same story hold for $GL(n, \mathbf{C})$.

It is a general fact that any closed subgroup of a Lie group is a Lie group, for a proof, see [1] Theorem VII.2.5. As a result, all matrix groups are Lie groups. For matrix groups the exponential map is given explicitly by the standard matrix power series:

$$exp(tX) = 1 + tX + \frac{t^2X^2}{2!} + \cdots$$

One can prove this is the exponential map discussed above by showing that it associates to X a one-parameter subgroup of G (homomorphism from $\mathbf{R}$ to G) and recalling that the exponential map is precisely the map identifying elements of $\mathfrak{g}$ with such homomorphisms.

Recall the following basic facts about groups that are subgroups of $GL(n, \mathbf{R})$, these will be the ones we will mostly be concerned with in this course:

$SL(n, \mathbf{R})$ is the group of matrices of determinant one. Its Lie algebra $\mathfrak{sl}(n, \mathbf{R})$ consists of matrices with trace zero.

$O(n)$ is the group of orthogonal matrices, i.e. satisfying $g^Tg = 1$, its Lie algebra $\mathfrak{o}(n)$ consists of anti-symmetric real matrices. The rows (or columns) of an orthogonal matrix make up an orthonormal basis of $\mathbf{R}^n$, so constructing an arbitrary element of $O(n)$ involves picking first an arbitrary unit vector in $\mathbf{R}^n$, then choosing one of the unit vectors perpendicular to it, then one perpendicular to the first two, etc.

$U(n)$ is the group of unitary matrices. These are complex n by n matrices satisfying $g^*g = 1$, with Lie algebra consisting of skew-adjoint matrices. The subgroup $SU(n)$ is those that have determinant 1, the Lie algebra $\mathfrak{su}(n)$ is those with trace zero. The rows (and columns)of $U(n)$ are orthogonal vectors of length one, using the hermitian inner product.

A very important example to keep in mind is that of $SU(2)$. Here all elements are of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

for some complex numbers α and β satisfying

$$|\alpha|^2 + |\beta|^2 = 1$$

Note that this is the equation for $S^3 \subset \mathbf{C}^2$.

A basis for the Lie algebra $\mathfrak{su}(2)$ is given by taking the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and multiplying them by i.

References

[1] Simon, B., *Representations of Finite and Compact Groups,*American Mathematical Society, 1996.

[2] Warner, F., *Foundations of Differentiable Manifolds and Lie Groups,* Springer, 1983.