Topics in Representation Theory: The Borel-Weil Theorem

1 Holomorphic sections of L_λ

We defined the space of sections $\Gamma(L_\lambda)$ of the line bundle L_λ over G/T as the complex-valued functions on G satisfying

$$f(ge^H) = e^{-\lambda(H)} f(g)$$

and showed that this space is infinite dimensional and decomposes into a sum of an infinite number of finite-dimensional irreducibles.

Furthermore, we showed that once one complexifies the Lie algebra $\mathfrak{g}$ of G to get the complexified Lie algebra $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$ and corresponding complexified group $G_{\mathbf{C}}$, the choice of a maximal torus and set of positive roots corresponds to a choice of Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbf{C}}$

$$\mathfrak{b} = \mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{n}^-$$

which has a corresponding Borel subgroup B of $G_{\mathbf{C}}$. As spaces of right-cosets we have

$$G/T = G_{\mathbf{C}}/B$$

so we can think of the flag manifold either in real terms as G/T or in complex terms as $G_{\mathbf{C}}/B$ in which case it is clearly a complex manifold since its tangent space is a quotient of two vector spaces.

We also saw that, by adding in simple positive roots to the definition of $\mathfrak{b}$, one can construct general parabolic subalgebras $\mathfrak{p}$ and corresponding complex parabolic subgroups P of $G_{\mathbf{C}}$.

There is a general definition of what it means for a vector bundle to be a holomorphic vector bundle, in which case one can define the the space of holomorphic sections of the bundle. In the case we are interested in, we can explicitly define the space

$$\Gamma_{\text{hol}}(L_\lambda) \subset \Gamma(L_\lambda)$$

of holomorphic sections of L_λ as the space of holomorphic functions on $G_{\mathbf{C}}$ satisfying an equivariance condition under the right B action:

$$f(gb) = \rho_\lambda(b^{-1}) f(g)$$

From a weight $\lambda \in \mathfrak{t}^*$, by complex linearity one has an element of $\mathfrak{t}^*_{\mathbf{C}}$ and by defining it as 0 on $\mathfrak{n}^-$, an element of $\mathfrak{b}^*$. Exponentiation gives the character ρ_λ.

We will not go into the details of the necessary complex analysis, but functions on G that are restrictions of holomorphic functions on $G_{\mathbf{C}}$ form a dense subspace of the continuous functions on G. For more details about this, see [1], chapter 12.

The difference between $\Gamma(L_\lambda)$ and $\Gamma_{\text{hol}}(L_\lambda)$ as functions on G is that the latter satisfies a stronger equivariance condition, equivariance under B, not just T. The additional condition of B equivariance includes invariance under the action of $\mathfrak{n}^-$ acts from the right. In other words:

For any $Z \in \mathfrak{n}^-$, let $r(Z)$ be infinitesimal right translation by the left-invariant complex vector field Z on G/T (in other words

$$r(Z)f = \frac{d}{dt} f(ge^{tZ})|_{t=0}$$

then

$$\Gamma_{\text{hol}}(L_\lambda) = \{f \in \Gamma(L_\lambda) : r(Z)f = 0 \ \ \forall \ Z \in \mathfrak{n}^-\}$$

2 The Borel-Weil theorem

With the above understanding of what $\Gamma_{\text{hol}}(L_\lambda)$ is, we have

Theorem 1 (Borel-Weil). *As a representation of G, for a dominant weight λ, $\Gamma_{hol}(L_\lambda)$ is a non-zero, irreducible representation of highest weight λ. All irreducible representations of G can be constructed in this way.*

For an outline of the proof from the point of view of complex analysis, see [1] chapter 14. However, the Borel-Weil theorem is really just identical with the highest-weight theorem, except that now one has an explicit construction of the representation and so can prove existence of a the representation with its highest weight vector. The existence of the highest weight vector is just the existence of solutions of a Cauchy-Riemann sort of partial differential equation, which is the condition of infinitesimal right translation invariance by elements of the negative root space.

To see the equivalence with highest-weight theory, recall that using the Peter-Weyl theorem one can show

$$\Gamma(L_\lambda) = \widehat{\bigoplus_i} V_i \otimes (V_i^*)_{-\lambda}$$

where the sum is over all irreducible representations, labelled by i and $(V_i^*)_{-\lambda}$ is the $-\lambda$ weight space of V_γ^*.

$$\Gamma_{\text{hol}}(L_\lambda) \subset \Gamma(L_\lambda)$$

is the subspace that satisfies the extra condition of right invariance under the action of the negative root space, i.e. that $-\lambda$ is a lowest weight, or, equivalently that λ is a highest weight. Explicitly

$$\begin{aligned}
\Gamma_{\text{hol}}(L_\lambda) &= \widehat{\bigoplus_i} V_i \otimes \{v \in V_i^* : \begin{cases} \mathfrak{n}^- v = 0 \\ v \in (V_i^*)_{-\lambda} \end{cases} \} \\
&= \bigoplus_{V_i^* \text{has lowest weight} -\lambda} V_i \otimes (V_i^*)_{-\lambda} \\
&= V_\lambda \otimes \mathbf{C}
\end{aligned}$$

where V_λ is the irreducible representation of highest weight λ.

3 The Borel-Weil theorem: Examples

Recall that for the case of $G = SU(2)$, we had an explicit construction of irreducible representations in terms of homogeneous polynomials in two variables. Such a construction can be interpreted in the Borel-Weil language by identifying holomorphic sections explicitly in terms of homogeneous polynomials. We will begin by working this out for the SU(2) case. For this case

$$G/T = SU(2)/U(1) = SL(2, \mathbf{C})/B = \mathbf{C}P^1$$

the space of complex lines in $\mathbf{C}^2$. Elements of $SL(2, \mathbf{C})$ are of the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1$$

and elements of the subgroup B are of the form

$$b = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$$

B can also be defined as the subgroup that stabilizes a standard complex line in $\mathbf{C}^2$, and one can check that for $b \in B$

$$b\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The subgroup N of B in this case is the matrices of the form

$$n = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

and the subgroup $T_{\mathbf{C}}$ is elements of the form

$$t = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

The space of holomorphic sections $\Gamma_{\text{hol}}(L_k)$ will be functions on $SL(2, \mathbf{C})$ such the subgroup N acts trivially from the right and the subgroup $T_{\mathbf{C}}$ acts via a character of T, which corresponds to an integer k. More explicitly

$$\Gamma_{\text{hol}}(L_k) = \{f : SL(2, \mathbf{C}) \rightarrow \mathbf{C}, f(gb) = \alpha^k f(g) \ \forall\ b \in B\}$$

We'll analyze what this equivariance condition says in two parts. First choosing $b \in N$, since

$$gb = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & \beta' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta'\alpha + \beta \\ \gamma & \beta'\gamma + \delta \end{pmatrix}$$

the condition $f(gb) = f(g)$ means that f depends only on the first column of the matrix.

Secondly, choosing $b \in T_{\mathbf{C}}$,

$$gb = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ 0 & (\alpha')^{-1} \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & (\alpha')^{-1}\beta \\ \gamma\alpha' & (\alpha')^{-1}\delta \end{pmatrix}$$

so the equivariance condition $f(gb) = (\alpha')^k f(g)$ implies that

$$f(\alpha' \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}) = (\alpha')^k f(\begin{pmatrix} \alpha \\ \gamma \end{pmatrix})$$

so our homogeneous polynomials of degree k in two variables (α, γ) provide holomorphic sections in $\Gamma_{\text{hol}}(L_k)$ and it turns out these are all such sections.

Another way of thinking about how to produce the appropriate holomorphic function on $SL(2, \mathbf{C})$ out of a homogeneous polynomial $P(z_1, z_2)$ is by the map

$$P \to f(g) = P(g \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = P(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = P(\begin{pmatrix} \alpha \\ \gamma \end{pmatrix})$$

In the more general case of $G = SU(n)$, representations on polynomials in n variables correspond to sections of a line bundle over

$$SU(n)/U(n-1) = SL(n, \mathbf{C})/P = \mathbf{CP}^{n-1}$$

the space of complex lines in $\mathbf{C}^n$. The parabolic subgroup P in this case can be taken to be the set of all matrices with zero in the first column, except for the diagonal element in the first row. The equivariance condition defining line bundles over this space is the same as in the $SU(2)$ case and the relation of holomorphic sections and homogeneous polynomials is much the same.

Note that for $G = U(n)$, $G/B = Fl(n)$, the space of flags in $\mathbf{C}$ has an obvious map to any of the partial flag manifolds G/P such as $G/P = \mathbf{CP}^{n-1}$, given by just forgetting some parts of the flag. In the case of $G/P = \mathbf{CP}^{n-1}$, the map just forgets all parts of the flag except for the complex line. The line bundle on G/B of the Borel-Weil theorem is just the pull-back under this forgetting map of the one we constructed on $\mathbf{CP}^{n-1}$ with homogeneous polynomials as its holomorphic sections.

The fundamental representations of $SU(n)$ include the $k = 1$ case above which is just the defining representation on $\mathbf{C}^n$, but also include the representations on the higher degree parts of $\Lambda^*(\mathbf{C}^n)$. The representation on $\Lambda^k(\mathbf{C}^n)$ corresponds to holomorphic sections of a certain line bundle over the Grassmanian

$$Gr(k, n) = SL(n, \mathbf{C})/P = \frac{U(n)}{U(k) \times U(n-k)}$$

For more details about this, the Borel-Weil theorem and its relation to the examples discussed here, see Chapter 11 and 14 of [1].

4 Other Topics

There are other perspectives on the geometry we have been discussing here which we do not have time to go into. An important one is that one can also think of flag manifolds and partial flag manifolds by picking a highest weight vector v in an irreducible representation V and looking a the orbit of v under $G_{\mathbf{C}}$. Projectivizing and just looking at the orbit as a set of complex lines in $P(V)$, it turns out the the stabilizer of the line defined by a highest weight vector will be a parabolic subgroup P so our orbit gives a map

$$G_{\mathbf{C}}/P \rightarrow P(V)$$

This gives a projective embedding of $G_{\mathbf{C}}/P$ and shows that these are projective algebraic varieties in the sense of algebraic geometry.

In the language of sheaf cohomology in algebraic geometry, we have been looking at the zero degree cohomology of the sheaf of sections of a line bundle

$$\Gamma_{\mathrm{hol}}(L_\lambda) = H^0(G/T, \mathcal{O}(L_\lambda))$$

There are also higher degree cohomology groups which also provide irreducible representations of G. For λ a non-dominant weight, there will be no holomorphic sections, but there will be non-zero higher dimensional cohomology. The Borel-Weil-Bott theorem describes what happens in this case. I hope to return to it, but first we will next consider the topics of Clifford algebras and spinors.

References

[1] Segal, G. Lectures on Lie groups, in *Lectures on Lie Groups and Lie Algebras,* Cambridge University Press, 1995.