Notes on the Poisson Summation Formula, Theta Functions, and the Zeta Function

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March 22, 2020

1 Introduction

These are notes for the class that was supposed to have been taught on March 9, 2020, will discuss online March 11, 2020.

2 The Poisson summation formula and some applications

Given a Schwarz function $f \in \mathcal{S}(\mathbf{R})$ on the real number line $\mathbf{R}$, you can construct a periodic function by taking the infinite sum

$$F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

This has the properties

- The sum converges for any x, since f is falling off at $\pm\infty$ faster than any power.
- $F_1(x)$ is periodic with period 1

  $$F_1(x+1) = F_1(x)$$

  This is because going from x to $x+1$ just corresponds to a shift of indexing by 1 in the defining sum for $F_1(x)$.

Since $F_1(x)$ is a periodic function, you can treat it by Fourier series methods. Note that the periodicity here is chosen to be 1, not 2π, so you need slightly different formulas. For a function g with period 1 whose Fourier series is pointwise convergent, you have

$$\widehat{g}(n) = \int_0^1 g(x)e^{-i2\pi nx}dx$$

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{i2\pi nx}$$

If you compute the Fourier coefficients of $F_1(x)$ you find

$$\begin{aligned}
\widehat{F_1}(m) &= \int_0^1 \sum_{n=-\infty}^{\infty} f(x+n) e^{-i2\pi mx} dx \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n) e^{-i2\pi mx} dx \\
&= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(x+n) e^{-i2\pi mx} dx \\
&= \int_{-\infty}^{\infty} f(x) e^{-i2\pi mx} dx \\
&= \widehat{f}(m)
\end{aligned}$$

Theorem (Poisson Summation Formula)**.** *If* $f \in \mathcal{S}(\mathbf{R})$

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{i2\pi nx}$$

Proof: The left hand side is the definition of $F_1(x)$, the right hand side is its expression as the sum of its Fourier series.

What most often gets used is the special case $x = 0$, with the general case what you get from this when translating by x:

Corollary. *If* $f \in \mathcal{S}(\mathbf{R})$

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)$$

This is a rather remarkable formula, relating two completely different infinite sums: the sum of the values of f at integer points and the sum of the values of its Fourier transform at integer points.

2.1 The heat kernel

The Poisson summation formula relates the heat kernel on $\mathbf{R}$ and on S^1. Recall that the formula for the heat kernel on $\mathbf{R}$ is

$$H_{t,\mathbf{R}}(x) = \frac{1}{\sqrt{4\pi t}} e^{\frac{x^2}{4t}}$$

with Fourier transform

$$\widehat{H}_{t,\mathbf{R}}(p) = e^{-4\pi^2 p^2 t}$$

Applying the Poisson summation formula to $H_{t,\mathbf{R}}$ gives

$$\sum_{n=-\infty}^{\infty} H_{t,\mathbf{R}}(x+n) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x} = H_{t,S^1}(x) \tag{1}$$

where H_{t,S^1} is the heat kernel on S^1.

Recall that earlier in the class we claimed that H_{t,S^1} was a "good kernel" and thus, for continuous functions $f(\theta)$ on the circle

$$\lim_{t\to 0^+} f * H_{t,S^1}(\theta) = f(\theta)$$

At the time we were unable to prove this, but using the above relation with the simpler $H_{t,\mathbf{R}}$ in equation 1, we can now show that H_{t,S^1} has the desired three properties:

- $$\int_0^1 H_{t,S^1}(x)dx = 1$$

  (the only contribution to the integral is from the $n = 0$ term).

- $$H_{t,S^1}(x) > 0$$

  (since $H_{t,\mathbf{R}} > 0$).

- To show that $H_{t,S^1}(x)$ concentrates at $x = 0$ as $t \to 0^+$, use

  $$H_{t,S^1}(x) = H_{t,\mathbf{R}}(x) + \sum_{|n|\geq 1} H_{t,\mathbf{R}}(x+n)$$

  The first term is a Gaussian that has the concentration property for $t \to 0^+$. The second term goes to 0 as $t \to 0^+$ for $|x| \leq \frac{1}{2}$ (see Stein-Shakarchi, pages 157).

2.2 The Poisson kernel

Recall that, given a Schwarz function f on $\mathbf{R}$, one could construct a harmonic function $u(x,y)$ on the upper half plane with that boundary condition by taking

$$u(x,y) = f * P_{y,\mathbf{R}}(x)$$

where the Poisson kernel in this case is

$$P_{y,\mathbf{R}} = \frac{1}{\pi}\frac{y}{x^2+y^2}$$

which has Fourier transform

$$\widehat{P}_{y,\mathbf{R}}(p) = e^{-2\pi|p|y}$$

For continuous functions f on the circle bounding the unit disk, a unique harmonic function $u(r,\theta)$ on the disk with that boundary condition is given by

$$u(r,\theta) = f * P_{r,S^1}(\theta)$$

where the Poisson kernel in this case is

$$P_{r,S^1}(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

Applying Poisson summation to $P_{y,\mathbf{R}}$ one gets a relation between these two kernels

$$\begin{aligned}\sum_{n=-\infty}^{\infty} P_{y,\mathbf{R}}(x+n) &= \sum_{n=-\infty}^{\infty} \widehat{P}_{y,\mathbf{R}}(n) e^{2\pi i n x} \\ &= \sum_{n=-\infty}^{\infty} e^{-2\pi|n|y} e^{2\pi i n x} \\ &= P_{e^{-2\pi y},S^1}(2\pi x)\end{aligned}$$

which is the Poisson kernel on the disk, with $r = e^{-2\pi y}$ and $\theta = 2\pi x$.

3 Theta functions

One can define a fascinating class of functions called "theta functions", with the simplest example

$$\theta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$$

Here s is real and the sum converges for $s > 0$, but one can also take $s \in \mathbf{C}$, with $\mathrm{Re}(s) > 0$. Applying the Poisson summation formula to the Schwarz function

$$f(x) = e^{-\pi s x^2}, \quad \widehat{f}(p) = \frac{1}{\sqrt{s}} e^{-\pi \frac{p^2}{s}}$$

gives

$$\begin{aligned}\theta(s) &= \sum_{n=-\infty}^{\infty} f(n) \\ &= \sum_{n=-\infty}^{\infty} \widehat{f}(n) \\ &= \frac{1}{\sqrt{s}} \sum_{n=-\infty}^{\infty} e^{-\pi \frac{n^2}{s}} \\ &= \frac{1}{\sqrt{s}} \theta(\frac{1}{s})\end{aligned}$$

which is often called the “functional equation” of the theta function. We will later see that this can be used to understand the properties of the zeta function in number theory.

3.1 The Jacobi theta function

May add more detail to this section

This section is about a more general theta function, called the Jacobi theta function. It is getting a bit far from the material of this course, but I wanted to write it up here so that you can see the connection to the heat and Schrödinger equations on the circle.

Definition (Jacobi theta function)**.** *The Jacobi theta function is the function of two complex variables given by*

$$\Theta(z,\tau)=\sum_{n=-\infty}^{\infty}e^{i\pi n^2\tau}e^{i2\pi nz}$$

This sum converges for $z,\tau\in\mathbf{C}$, with τ in the upper half plane. Note that the heat kernel is given by

$$H_{t,\mathbf{R}}(x)=\Theta(x,i4\pi t)$$

and is well-defined for $t>0$. The Schrödinger kernel is given by

$$S_t(x)=\Theta(x,\frac{\hbar}{2m}\frac{t}{\pi})$$

which (to stay in the upper half plane of definition), really should be defined as

$$S_t(x)=\lim_{\epsilon\to 0^+}\Theta(x,\frac{\hbar}{2m}\frac{t+i\epsilon}{\pi})$$

The Jacobi theta function has the following properties:

- Two-fold periodicity in z (up to a phase, for fixed τ). Clearly

  $$\Theta(z+1,\tau)=\Theta(z,\tau)$$

  One also has

  $$\Theta(z+\tau,\tau)=\Theta(z,\tau)e^{-\pi i\tau}e^{-2\pi iz}$$

since

$$\begin{aligned}\Theta(z+\tau,\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau} e^{2\pi i n(z+\tau)}\\ &= \sum_{n=-\infty}^{\infty} e^{\pi i(n^2+2n)\tau} e^{2\pi i n z}\\ &= \sum_{n=-\infty}^{\infty} e^{\pi i(n+1)^2\tau} e^{-\pi i\tau} e^{2\pi i n z}\\ &= \sum_{n=-\infty}^{\infty} e^{\pi i(n+1)^2\tau} e^{-\pi i\tau} e^{2\pi i(n+1)z} e^{-2\pi i z}\\ &= \Theta(z,\tau)e^{-\pi i\tau}e^{-2\pi i z}\end{aligned}$$

- Periodicity in τ

$$\Theta(z,\tau+2)=\Theta(z,\tau)$$

since

$$\begin{aligned}\Theta(z,\tau+2) &= \sum_{n=-\infty}^{\infty} e^{\pi i n^2(\tau+2)} e^{2\pi i n z}\\ &= \sum_{n=-\infty}^{\infty} e^{2\pi i n^2} e^{\pi i n^2\tau} e^{2\pi i n z}\\ &= \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau} e^{2\pi i n z}\\ &= \Theta(z,\tau)\end{aligned}$$

- The following property under inversion in the τ plane, which follows from the functional equation for $\theta(s)$.

$$\Theta(z,-\frac{1}{\tau})=\sqrt{\frac{\tau}{i}}e^{\pi i\tau z^2}\Theta(z\tau,\tau)$$

4 The Riemann zeta function

Recall that we have shown that one can evaluate the sums

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\quad \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

using Fourier series methods. A central object in number theory is

Definition (Riemann zeta function). *The Riemann zeta function is given by*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

For $s \in \mathbf{R}$, this converges for $s > 1$.

One can evaluate $\zeta(s)$ not just at $s = 2, 4$, but at s any even integer (see problem sets) with result

$$\zeta(2n)=\frac{(-1)^{n+1}}{2(2n)!}B_{2n}(2\pi)^{2n}$$

Here B_n are the Bernoulli numbers, which can be defined as the coefficients of the power series expansion

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}$$

There is no known formula for the values of $\zeta(s)$ at odd integers.

The zeta function contains a wealth of information about the distribution of prime numbers. Using the unique decomposition of an integer into primes, one can show

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty}\frac{1}{n^s}\\ &= \prod_{\text{primes p}}(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\\ &= \prod_{\text{primes p}}\frac{1}{1-p^{-s}}\end{aligned}$$

One can consider the zeta function for complex values of s, in which case the sum defining it converges in the half-plane $\text{Re}(s) > 1$. This function of s can be uniquely extended as a complex valued function to parts of the complex plane with $\text{Re}(s) \leq 1$ by the process of "analytic continuation". This can be done by finding a solution of the Cauchy-Riemann equations which matches the values of $\zeta(s)$ for values of s where the sum in the definition converges, but also exists for other values of s. This analytically extended zeta function then has the following properties:

- $\zeta(s)$ has a well-defined analytic continuation for all s except $s = 1$. There is a pole at $s = 1$ with residue 1, meaning $\zeta(s)$ behaves like $\frac{1}{1-s}$ near $s = 1$.
- $$\zeta(0)=-\frac{1}{2}$$

Note that if you tried to define $\zeta(0)$ using the sum, this would imply

$$-\frac{1}{2} = 1 + 1 + 1 + \cdots$$

- At negative integers

$$\zeta(-n) = \begin{cases} 0 & n \text{ even} \\ (-1)^n \frac{B_{n+1}}{n+1} & n \text{ odd} \end{cases}$$

In particular, $\zeta(-1) = -\frac{1}{12}$ which motivates claims one sometimes sees that

$$-\frac{1}{12} = 1 + 2 + 3 + 4 + \cdots$$

4.1 The Mellin transform

To prove the functional equation for the zeta function, we need to relate it to the theta function, and will do this using

Definition (Mellin transform)**.** *The Mellin transform of a function $f(x)$ is the function*

$$(\mathcal{M}f)(s) = \int_0^\infty f(x) x^s \frac{dx}{x}$$

Note that the Mellin transform is the analog of the Fourier transform one gets when one replaces the additive group $\mathbf{R}$ with the multiplicative group of positive real numbers. The analogy goes as follows:

- For the Fourier transform, we are using behavior of functions under the transformation

$$f(x) \to f(x + a)$$

where $a \in \mathbf{R}$.

For the Mellin transform, we are using behavior of functions under the transformation

$$f(x) \to f(ax)$$

where $a \in \mathbf{R}$ is positive.

- In the Fourier transform case, the function e^{ipx} behaves simply (multiplication by a scalar) under the transformation:

$$e^{ipx} \to e^{ip(x+a)} = e^{ipa} e^{ipx}$$

In the Mellin transform case, the function x^s behaves simply (multiplication by a scalar) under the transformation:

$$x^s \to (ax)^s = a^s x^s$$

- In the Fourier transform case, the integral

$$\int_{-\infty}^{\infty} (\cdot) dx$$

  is invariant under the transformation.

  In the Mellin transform case, the integral

$$\int_{0}^{\infty} (\cdot) \frac{dx}{x}$$

  is invariant under the transformation, since

$$\frac{d(ax)}{ax} = \frac{dx}{x}$$

Using the Mellin transform, one can define the gamma function by

Definition (Gamma function)**.** *The gamma function is*

$$\Gamma(s) = (\mathcal{M}e^x)(s) = \int_0^\infty e^{-x}x^{s-1}dx$$

The gamma function generalizes the factorial, satisfying $\Gamma(n) = n!$ for n a positive integer. This is because one has

$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty e^{-x}x^s dx \\ &= -x^s e^{-x} \Big|_0^\infty + s\int_0^\infty e^{-x}x^{s-1}dx \\ &= s\Gamma(s)\end{aligned}$$

and

$$\Gamma(1) = \int_0^\infty e^{-x}dx = -e^{-x} \Big|_0^\infty = 1$$

If one allows s to be a complex variable, the definition of $\Gamma(s)$ as an integral converges for $\mathrm{Re}(s) > 0$. One can extend the region of definition of $\Gamma(s)$ to the entire complex plane using the relation

$$\Gamma(s) = \frac{1}{s}\Gamma(s+1)$$

This satisfies the Cauchy-Riemann equations and is the analytic continuation of $\Gamma(s)$ from the region of $\mathbf{C}$ where it is defined by an integral. This definition does imply poles at the non-positive integers $s = 0, -1, -2, \ldots$.

For another example of the Mellin transform, one can take the transform of a Gaussian and get a gamma function:

$$
\begin{aligned}
(\mathcal{M}e^{-x^2})(s) &= \int_0^\infty e^{-x^2} x^{s-1} dx \\
&= \int_0^\infty e^{-y} (y^{\frac{1}{2}})^{s-1} \frac{dy}{2\sqrt{y}} \\
&= \frac{1}{2} \int_0^\infty e^{-y} y^{\frac{s}{2}-1} dy \\
&= \frac{1}{2}\Gamma(\frac{s}{2})
\end{aligned}
$$

where in the second line we have made the substitution

$$
y = x^2, \quad dy = 2xdx, \quad dx = \frac{dy}{2\sqrt{y}}
$$

4.2 The zeta function and the Mellin transform of the theta function

It turns out that the zeta function is closely related to the Mellin transform of the theta function. In this section we will show this, in the next use the functional equation of the theta function to give a functional equation for the zeta function.

Recall the definition of the theta function

$$
\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}
$$

We will work with a slight variant of this, defining

$$
w(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x} = \frac{1}{2}(\theta(x) - 1)
$$

Taking the Mellin transform one finds

$$
\begin{aligned}
(\mathcal{M}w)(s) &= \int_0^\infty w(x) x^{s-1} dx \\
&= \sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 x} x^{s-1} dx
\end{aligned}
$$

One can do these integrals with the substitution

$$
u = \pi n^2 x, \quad dx = \frac{du}{\pi n^2}
$$

with the result

$$\int_0^\infty e^{-\pi n^2 x} x^{s-1} dx = \int_0^\infty e^{-u} (\frac{u}{\pi n^2})^{s-1} \frac{du}{\pi n^2}$$
$$= (\frac{1}{\pi n^2})^{s-1} \frac{1}{\pi n^2} \int_0^\infty e^{-u} u^{s-1} du$$
$$= \frac{1}{\pi^s n^{2s}} \Gamma(s)$$

and one finally gets for the Mellin transform of $w(x)$

$$(\mathcal{M}w)(s) = \sum_{n=1}^{\infty} \frac{1}{\pi^s} \frac{1}{n^{2s}} \Gamma(s) \tag{2}$$
$$= \frac{1}{\pi^s} \Gamma(s) \zeta(2s+1) \tag{3}$$

4.3 The functional equation for the zeta function

Finally we would like to use the relation between the zeta function and the Mellin transform of the theta function, together with the functional equation of the theta function to show

Theorem (Functional equation of the zeta function)**.** *If we define*

$$\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$$

then

$$\Lambda(s) = \Lambda(1-s)$$

This theorem allows us to extend our definition of $\Lambda(s)$ (and thus $\zeta(s)$) from the region $\mathrm{Re}(s) > 1$ where it is defined by a convergent infinite sum to the region $\mathrm{Re}(s) < 0$, giving us an analytic continuation of $\Lambda(s)$ to this region. The most mysterious behavior of $\Lambda(s)$ is in the "critical strip" $0 < \mathrm{Re}(s) < 1$ where has no definition other than as an analytic continuation exists. Perhaps the most famous unproved conjecture in mathematics is

Conjecture (The Riemann Hypothesis)**.** *The zeros of $\Lambda(s)$ all lie on the centerline of the critical strip, where the real part of s is $\frac{1}{2}$.*

We will of course not prove the Riemann hypothesis, but will prove the functional equation for the zeta function, by showing that the functional equation of the theta equation implies

$$(\mathcal{M}w)(s) = (\mathcal{M}w)(\frac{1}{2} - s) \tag{4}$$

This implies (using equation 2) that

$$\pi^{-s} \Gamma(s) \zeta(2s) = \pi^{s-\frac{1}{2}} \Gamma(\frac{1}{2} - s) \zeta(1-2s)$$

which implies (changing s to $\frac{s}{2}$) that

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-2}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

and thus

$$\Lambda(s) = \Lambda(1-s)$$

To show equation 4, we will use the fact that the functional equation for the theta function

$$\theta(s) = \frac{1}{\sqrt{s}}\theta(\frac{1}{s})$$

implies

$$\begin{aligned}
w(x) &= \frac{1}{2}(\theta(x)-1)\\
&= \frac{1}{2}(\frac{1}{\sqrt{x}}\theta(\frac{1}{x})-1)\\
&= \frac{1}{2}(\frac{1}{\sqrt{x}}(2w(\frac{1}{x})+1)-1)\\
&= \frac{1}{\sqrt{x}}w(\frac{1}{x})+\frac{1}{2}\frac{1}{\sqrt{x}}-\frac{1}{2}
\end{aligned}$$

Breaking the integral for the Mellin transform of $w(x)$ into two parts and using the above relation for $w(x)$ in the first part we find

$$\begin{aligned}
(\mathcal{M}w)(s) &= \int_0^1 w(x)x^{s-1}dx + \int_1^\infty w(x)x^{s-1}dx\\
&= \int_0^1 (\frac{1}{\sqrt{x}}w(\frac{1}{x})+\frac{1}{2}\frac{1}{\sqrt{x}}-\frac{1}{2})dx + \int_1^\infty w(x)x^{s-1}dx\\
&= \int_1^\infty u^{-1-s}(u^{\frac{1}{2}}w(u)-\frac{1}{2}+\frac{1}{2}u^{\frac{1}{2}})du + \int_1^\infty w(x)x^{s-1}dx\\
&= \int_1^\infty u^{-\frac{1}{2}-s}w(u)du - \frac{1}{2s} - \frac{1}{1-2s} + \int_1^\infty w(x)x^{s-1}dx\\
&= \int_1^\infty w(u)(u^{-\frac{1}{2}-s}+u^{s-1})du - \frac{1}{2s} - \frac{1}{2(\frac{1}{2}-s)}
\end{aligned}$$

This is symmetric under the interchange of s and $\frac{1}{2}-s$, which gives equation 4. In the second step we used the substitution

$$u = \frac{1}{x}, \quad du = -x^{-2}dx = -u^2dx$$

5 References