Quantum Field Theory for Mathematicians
Spring 2024 Course Notes
*Under Construction*

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Chapter 1

Introduction

These notes are a work in progress, course notes for a spring 2024 “Topics in Representation Theory” course oriented towards explaining quantum mechanics, quantum field theory, and the Standard Model to mathematicians, emphasizing the relations to representation theory. A sizable part of the early version of these notes is an extract from notes on material covered in a spring 2023 graduate course on Lie groups and representations at Columbia University. For the full version of those notes, see https://www.math.columbia.edu/~woit/LieGroups-2023/qmnumbertheory.pdf.
Chapter 2

Classical Mechanics

The classical mechanics description of a physical system involves an “equation of motion”, a differential equation which determines the state of the system at later times given its state at some initial time. There are two quite different formalisms used for this purpose, the Hamiltonian and Lagrangian. In this chapter we'll outline the Hamiltonian version, which is closely related to Lie algebras, and then discuss the Lagrangian version.

2.1 Hamiltonian mechanics

In the Hamiltonian formalism, the state of a physical system at a given time is determined by a point in a space called “phase space”. The equation of motion is a first order equation in time determined by a function on phase space called the Hamiltonian. One can also think of phase space as the space of solutions of the equation of motion.

In the cases we are most interested in, phase space is an even dimensional vector space \( P = \mathbb{R}^{2n} \), with coordinates \( q_j, p_j \) for \( j = 1, 2, \ldots, n \). Then one can define:

**Definition (Poisson bracket).** The Poisson bracket of two functions \( f_1, f_2 \) on \( P \) is the function

\[
\{f, g\} = \sum_{j=1}^{n} \left( \frac{\partial f_1}{\partial q_j} \frac{\partial f_2}{\partial p_j} - \frac{\partial f_2}{\partial q_j} \frac{\partial f_1}{\partial p_j} \right)
\]

Given a Hamiltonian function \( h \) on \( P \), the time dependence of any function \( f \) on \( P \) will satisfy

\[
\frac{df}{dt} = \{f, h\}
\]

In particular, for coordinate functions, one gets Hamilton’s equations

\[
\dot{q}_j = \{q_j, h\} = \frac{\partial h}{\partial p_j}
\]
\[ \dot{p}_j = \{ p_j, h \} = - \frac{\partial h}{\partial q_j} \]

These are the equations of motion in Hamiltonian form. For \( h = \frac{1}{2m} |p|^2 + V(q) \) these give the elementary physics definition of the momentum

\[ p = m\dot{q} \]

and Newton’s second law

\[ \ddot{p} = m\ddot{q} = -\nabla V \]

The Poisson bracket can easily be seen to satisfy the following properties:

- **Antisymmetry:**
  \[ \{ f_1, f_2 \} = - \{ f_2, f_1 \} \]

- **Jacobi identity:**
  \[ \{ f_1, \{ f_2, f_3 \} \} + \{ f_3, \{ f_1, f_2 \} \} + \{ f_2, \{ f_3, f_1 \} \} = 0 \]

- **Leibniz rule (derivation property):**
  \[ \{ f_1, f_2 f_3 \} = \{ f_1, f_2 \} f_3 + f_2 \{ f_1, f_3 \} \]

The first two properties imply that the Poisson bracket provides a Lie algebra structure on the space of functions on \( P \). This is an infinite-dimensional Lie algebra.

We’ll mainly be interested in the case where \( P \) is a linear space, but the whole formalism works equally well for a manifold of the following kind:

**Definition (Symplectic manifold).** A symplectic manifold \( P \) is a manifold with a two-form \( \omega \in \Omega^2(P) \) such that:

- \( \omega \) is non-degenerate. At each \( p \in P \) it gives an isomorphism between tangent vectors and cotangent vectors.
- \( \omega \) is closed: \( d\omega = 0 \)

For the case of \( P = \mathbb{R}^{2n} \) and its standard Poisson bracket, one has

\[ \omega = \sum_{j=1}^{n} (dq_j \otimes dp_j - dp_j \otimes dq_j) = \sum_{j=1}^{n} dq_j \wedge dp_j \]

Note that, up to a change of coordinates, this is the unique antisymmetric non-degenerate bilinear form on \( \mathbb{R}^{2n} \). This is very much analogous to the case of Riemannian geometry, where instead the inner product provides a non-degenerate symmetric bilinear form, and this analogy will play an important role later.

A simple non-linear example of a symplectic manifold is given by \( P = S^2 \), with \( \omega \) the area two-form. A large class of examples is given by cotangent
bundles \( P = T^*M \) of manifolds \( M \), with \( \omega = d\theta \) where \( \theta \) is the canonical one-form on \( M \).

On a symplectic manifold the non-degeneracy condition allows one to associate to a function \( f \) a vector field \( X_f \) by

\[
df = \omega(X_f, \cdot) = \imath_{X_f}\omega
\]

This is a “symplectic gradient”, an analog of the usual gradient for a Riemannian manifold, which associates a vector to field to \( f \) by using the metric to identify the one-form \( df \) with a vector field. Not all vector fields are of the form \( X_f \), those that are are called “Hamiltonian vector fields”.

The Poisson bracket can then be defined by

\[
\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2})
\]

Writing out explicitly the condition that the three form \( d\omega = 0 \), one gets the Jacobi identity for the Poisson bracket, and thus a Lie algebra structure on the functions on \( P \). The map

\[
f \mapsto X_f
\]

is a Lie algebra homomorphism from this Lie algebra of functions to the Lie algebra of vector fields on \( P \).

These are infinite dimensional Lie algebras, which one can locally exponentiate to get a group law (actually a “pseudogroup”). Such a group action preserves \( \omega \) since the Lie derivative satisfies

\[
L_{X_f}\omega = (di_{X_f} + i_{X_f}d)\omega = di_{X_f}\omega = d\omega(X_f, \cdot) = ddf = 0 \tag{2.1}
\]

This (pseudo)-group preserving \( \omega \) is a sub (pseudo)-group of the group of diffeomorphisms of \( P \) (the “symplectomorphisms” to mathematicians, “canonical transformations” to physicists).

By 2.1, for any vector field \( X \) preserving \( \omega \) one has \( di_{X}\omega = 0 \). When \( H^1(M) = 0 \) the vector field \( X \) will be a Hamiltonian vector field \( X_f \) for a function \( f \) determined by \( i_X\omega = df \). This function \( f \) is determined only up to a constant (for \( P \) connected).

In the physicist’s language the Hamiltonian function \( h \) “generates” an action of the Lie group \( R \) on \( P \) given by the vector field \( X_h \). This Lie group \( R \) is the group of time translations acting on the physical system. Whenever one has an action of a Lie group \( G \) on \( P \) that preserves \( \omega \), differentiating this gives a Hamiltonian vector field \( X_L \) for each \( L \in \mathfrak{g} \), the Lie algebra of \( G \). Thus, when \( H^1(M) = 0 \), for each \( L \in \mathfrak{g} \) one can find (ambiguous up to a constant) a function \( f_L \) that generates the action infinitesimally given by the action of \( L \). It turns out that when \( H^2(\mathfrak{g}) = 0 \) (Lie algebra cohomology), the constants can be chosen so that the map \( L \mapsto f_L \) is a Lie algebra isomorphism between \( \mathfrak{g} \) and a sub-Lie algebra of the Lie algebra of functions on \( P \). This map is known as the “moment map”.

When a function \( f \) on \( P \) Poisson-commutes with the Hamiltonian \( \{f, h\} = 0 \), then \( \frac{df}{dt} = 0 \) and the function \( f \) is a constant along the physical trajectories.
of time evolution generated by the Hamiltonian $h$. In such a case $f$ is said to be a “conserved quantity”. When we have an action of a Lie group $G$ on $P$ preserving $\omega$ that commutes with the action of the group $R$ of time translations, the functions $f_L$ for each $L \in \mathfrak{g}$ will be conserved quantities. This is how conservation laws corresponding to symmetries come about in the Hamiltonian formalism.

Some important examples:

- For a particle in 3 dimensions, $P = \mathbb{R}^6$ with the usual Poisson bracket. There is a Hamiltonian action of $\mathbb{R}^3$ by translation in the $q$ coordinates, generated by the $p$ coordinates. When the Hamiltonian $h$ is independent of a coordinate $q_j$, the corresponding $p_j$ is a conserved quantity: the momentum in the $j$ direction.

- In the same case there is a Hamiltonian action of $SO(3)$, by simultaneous rotation of the $q$ and $p$. The functions that generate rotations about the axes are

\[
\begin{align*}
  l_1 &= q_2p_3 - p_2q_3, \\
  l_2 &= q_3p_1 - p_3q_1, \\
  l_3 &= q_1p_2 - p_1q_2
\end{align*}
\]

These are the components of the angular momentum. When the Hamiltonian is invariant under rotations about the $j$-axis, $l_j$ is a conserved quantity.

For any Lie algebra $\mathfrak{g}$, one can take $P = \mathfrak{g}^*$. Lie algebra elements $X, Y \in \mathfrak{g}$ are linear functions on $P$. On these linear functions the Lie bracket is a Poisson bracket

\[
\{X, Y\} = [X, Y]
\]

and this can be extended using the derivation property to a Poisson bracket on $S^\ast(\mathfrak{g})$, the polynomials on $\mathfrak{g}^* = P$. $P$ is not a symplectic manifold, since this construction does not give a non-degenerate two-form (instead, it’s a “Poisson manifold). A Lie group $G$ acts on its Lie algebra $\mathfrak{g}$ by the adjoint action, and there is a corresponding co-adjoint action on $\mathfrak{g}^*$. On the orbits of the co-adjoint action, one does have a non-degenerate symplectic form, and these orbits are symplectic manifolds. The example of $S^2$ mentioned above is the case of $G = SO(3)$, where the co-adjoint orbits are spheres in $\mathbb{R}^3 = \mathfrak{so}(3)^*$.

### 2.2 Lagrangian mechanics

In the Lagrangian formalism, instead of a phase space $P = \mathbb{R}^{2n}$ of positions $q_j$ and momenta $p_j$, one considers just the position (or configuration) space $M = \mathbb{R}^n$. Instead of a Hamiltonian function $h$ on $P$, one has a functional $S[\gamma]$ of parametrized paths $\gamma$ in $M$ called the “action”. The action is defined by integrating a function of position and velocity called the Lagrangian:

**Definition (Action).** The action $S$ for a path $\gamma$ is

\[
S[\gamma] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt
\]
Here the path is parametrized by \( t \in [t_1, t_2] \) and the Lagrangian \( L \) is a function of \( t \) that depends on the position at \( t \) and its \( t \)-derivative. More generally, one can formulate this for configuration space a manifold \( M \), with \( L(t) \) depending on the velocity vector, which takes values in the tangent space of \( M \).

The fundamental principle of classical mechanics in the Lagrangian formalism is that classical trajectories are given by critical points of the action functional.

**Definition** (Critical point for \( S \)). A path \( \gamma \) is a critical point of the functional \( S[\gamma] \) if

\[
\delta S(\gamma) \equiv \frac{d}{ds} S(\gamma_s)|_{s=0} = 0
\]

where

\[
\gamma_s : [t_1, t_2] \to \mathbb{R}^n
\]

is a smooth family of paths parametrized by an interval \( s \in (-\epsilon, \epsilon) \), with \( \gamma_0 = \gamma \).

Critical points will be given by solutions to the Euler-Lagrange equations, which will be the equations of motion for the system:

**Theorem** (Euler-Lagrange equations). One has

\[
\delta S[\gamma] = 0
\]

for all variations of \( \gamma \) with endpoints \( \gamma(t_1) \) and \( \gamma(t_2) \) fixed if

\[
\frac{\partial L}{\partial q_j}(q(t), \dot{q}(t)) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j}(q(t), \dot{q}(t)) \right) = 0
\]

for \( j = 1, \cdots, d \). These are called the Euler-Lagrange equations.

**Proof.** Ignoring analytical details, the Euler-Lagrange equations follow from the following calculations, which we’ll just do for \( n = 1 \), with the generalization to higher \( d \) straightforward. We are calculating the first-order change in \( S \) due to an infinitesimal change \( \delta \gamma = (\delta q(t), \delta \dot{q}(t)) \)

\[
\delta S[\gamma] = \int_{t_1}^{t_2} \delta L(q(t), \dot{q}(t)) dt
\]

\[
= \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q}(q(t), \dot{q}(t)) \delta q(t) + \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) \delta \dot{q}(t) \right) dt
\]

But

\[
\delta \dot{q}(t) = \frac{d}{dt} \delta q(t)
\]

and, using integration by parts

\[
\frac{\partial L}{\partial \dot{q}} \delta \dot{q}(t) = \frac{d}{dt} \left( \frac{\partial L}{\partial q} \delta q \right) - \left( \frac{d}{dt} \frac{\partial L}{\partial q} \right) \delta q
\]
so
\[ \delta S[\gamma] = \int_{t_1}^{t_2} \left( \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \right) dt \]
\[ = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt - \left( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) (t_2) + \left( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) (t_1) \] (2.2)

If we keep the endpoints fixed so \( \delta q(t_1) = \delta q(t_2) = 0 \), then for solutions to
\[ \frac{\partial L}{\partial q}(q(t), \dot{q}(t)) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) \right) = 0 \]
the integral will be zero for arbitrary variations \( \delta q \).

As an example, a particle moving in a potential \( V(q) \) will be described by a Lagrangian
\[ L(q, \dot{q}) = \frac{1}{2} m |\dot{q}|^2 - V(q) \]
for which the Euler-Lagrange equations will be Newton’s second law:
\[ -\frac{\partial V}{\partial q_j} = \frac{d}{dt} (m \dot{q}_j) = m \ddot{q}_j \]

Given a Lagrangian classical mechanical system, one would like to be able to find a corresponding Hamiltonian system that will give the same equations of motion. To do this, we proceed by defining for each \( q_j \) a corresponding momentum coordinate \( p_j \) by
\[ p_j = \frac{\partial L}{\partial \dot{q}_j} \]

Then, instead of working with trajectories characterized at time \( t \) by
\[ (q(t), \dot{q}(t)) \in \mathbb{R}^{2n} \]
we would like to instead use
\[ (q(t), p(t)) \in \mathbb{R}^{2n} \]
where \( p_j = \frac{\partial L}{\partial \dot{q}_j} \) and identify this \( \mathbb{R}^{2n} \) (for example at \( t = 0 \)) as the phase space of the conventional Hamiltonian formalism. In greater generality,

The transformation
\[ (q_j, \dot{q}_k) \rightarrow \left( q_j, p_k = \frac{\partial L}{\partial \dot{q}_k} \right) \]
between position-velocity and phase space (in greater generality \( TM \) and \( T^* M \)) is known as the Legendre transform, and in good cases (for instance when \( L \) is quadratic in all the velocities) it is an isomorphism. In general though, this is not an isomorphism, with the Legendre transform often taking position-velocity
space to a lower dimensional subspace of phase space. Such cases are not unusual
and require a much more complicated formalism, even as classical mechanical
systems (this subject is known as “constrained Hamiltonian dynamics”).

Besides a phase space, for a Hamiltonian system one needs a Hamiltonian
function. Choosing

\[ h = \sum_{j=1}^{d} p_j \dot{q}_j - L(q, \dot{q}) \]

will work, provided the relation

\[ p_j = \frac{\partial L}{\partial \dot{q}_j} \]

can be used to solve for the velocities \( \dot{q}_j \) and express them in terms of the
momentum variables. In that case, computing the differential of \( h \) one finds
(for \( d = 1 \), the generalization to higher \( d \) is straightforward)

\[ dh = pdq + \dot{q}dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} \]

\[ = \dot{q}dp - \frac{\partial L}{\partial q}dq \]

So one has

\[ \frac{\partial h}{\partial p} = \dot{q}, \quad \frac{\partial h}{\partial q} = -\frac{\partial L}{\partial q} \]

but these are precisely Hamilton’s equations since the Euler-Lagrange equations
imply

\[ \frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \dot{p} \]

While the Legendre transform method given above works in some situations,
more generally and more abstractly, one can pass from the Lagrangian to the
Hamiltonian formalism by taking as phase space the space of solutions of the
Euler-Lagrange equations. This is sometimes called the “covariant phase space”,
and it can often concretely be realized by fixing a time \( t = 0 \) and parametrizing
solutions by their initial conditions at such a \( t = 0 \). Only for a special class
of Lagrangians though will one get a non-degenerate Poisson bracket on a linear
phase space and recover the usual properties of the standard Hamiltonian
formalism, for greater generality one needs a more complicated formalism to
recover the desired features of the Hamiltonian formalism.

\[ \text{2.2.1 Noether’s theorem and symmetries in the Lagrangian formalism} \]

The derivation of the Euler-Lagrange equations given above can also be used to
study the implications of Lie group symmetries of a Lagrangian system. When
a Lie group \( G \) acts on the space of paths, preserving the action \( S \), it will take
classical trajectories to classical trajectories, so we have a Lie group action on the space of solutions to the equations of motion (the Euler-Lagrange equations). On this space of solutions, we have, from equation 2.2 (generalized to multiple coordinate variables),

\[ \delta S[\gamma] = \left( \sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j} \delta q_j(X) \right) (t_1) - \left( \sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j} \delta q_j(X) \right) (t_2) \]

where now \( \delta q_j(X) \) is the infinitesimal change in a classical trajectory coming from the infinitesimal group action by an element \( X \) in the Lie algebra of \( G \). From invariance of the action \( S \) under \( G \) we must have \( \delta S = 0 \), so

\[ \left( \sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j} \delta q_j(X) \right) (t_2) = \left( \sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j} \delta q_j(X) \right) (t_1) \]

This is an example of a more general result known as “Noether’s theorem”. In this context it says that given a Lie group action on a Lagrangian system that leaves the action invariant, for each element \( X \) of the Lie algebra we will have a conserved quantity

\[ \sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j} \delta q_j(X) \]

which is independent of time along the trajectory.

When the Lagrangian \( L \) is translation invariant (depends on \( \dot{q} \), not \( q \)), one recovers by the Noether method the definition of momentum and its conservation law. When \( L \) is rotation invariant, one gets angular momentum and its conservation.

The Lagrangian formalism has the advantage that the dynamics depends only on the choice of action functional on the space of possible trajectories, and it can be straightforwardly generalized to theories where the configuration space is an infinite dimensional space of classical fields. Unlike the usual Hamiltonian formalism for such theories, the Lagrangian formalism allows one to treat space and time symmetrically. For relativistic field theories, this allows one to exploit the full set of space-time symmetries, which can mix space and time directions. In such theories, Noether’s theorem provides a powerful tool for finding the conserved quantities corresponding to symmetries of the system that are due to invariance of the action under some group of transformations.

On the other hand, in the Lagrangian formalism, since Noether’s theorem only considers group actions on configuration space, it does not cover the case of Hamiltonian group actions that mix position and momentum coordinates, something that occurs most notably in the case of the harmonic oscillator.
Chapter 3

Introduction to Quantization

In this chapter we’ll begin our discussion of quantum theory with some basic examples covered in all physics textbooks, followed by some generalities about the role of quantization in representation theory. The three examples here incorporate three important aspects of the quantum field theories we plan to study later in the course.

3.1 Canonical quantization: some examples

What physicists call “canonical quantization” can be understood in terms of the unique non-trivial representation of the Heisenberg group and Lie algebra, which will be described in detail in the next chapter. In this one, we’ll motivate the later representation theory with a standard description of the basic examples of quantum systems.

The space of possible states for a quantum system is a complex vector space $\mathcal{H}$ (generally infinite-dimensional) with Hermitian inner product. For one degree of freedom this space can be taken to be the space of wavefunctions (complex-valued functions $\psi(q)$ of a position variable $q$) in $L^2(\mathbb{R})$. This version of the state space is called the Schrödinger representation and acting on it are powers of the self-adjoint operators

$$Q = q, \quad P = -i\hbar \frac{d}{dq}$$

which satisfy the Heisenberg commutation relations

$$[Q, P] = i\hbar 1$$

Here $\hbar$ is a constant which depends on one’s choice of units, so later we will generally use units in which $\hbar = 1$. 

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The dynamics of the system is determined by specification of an operator (defined in terms of the \( Q, P \) operators), the Hamiltonian \( H \). This operator generates translations in time, with wavefunctions evolving in time according to the Schrödinger equation

\[ i\hbar \frac{d}{dt} \psi = H\psi \]

The connection between this formalism and what one observes, measures and often interprets in a classical picture of the world is given by two principles:

- Self-adjoint operators like \( Q \) and \( P \) correspond to observable quantities, with eigenfunctions of such an operator states with a well-defined measurable value of the observable quantity, given by the eigenvalue.

- If one tries to measure the value of an observable quantity when the state is not an eigenfunction, the result will be one of the eigenvalues, with probability given by the norm-squared of the inner product between the (normalized) state and eigenfunction with that eigenvalue (this is called the “Born rule”).

For a single quantum particle moving in one dimension, subject to a potential \( V(q) \), the Hamiltonian is

\[ H = \frac{1}{2m} P^2 + V(Q) = -\frac{\hbar^2}{2m} \frac{d^2}{dq^2} + V(q) \]

One would like to find the eigenfunctions and eigenvalues of this operator, i.e. find \( E, \psi_E(q) \) such that

\[ (-\frac{\hbar^2}{2m} \frac{d^2}{dq^2} + V(q))\psi_E(q) = E\psi_E(q) \]

and then expand wavefunctions at an initial time \( t = 0 \) in terms of the energy eigenfunctions \( \psi_E(q) \). The Schrödinger equation implies that these evolve in time as

\[ \psi_E(q)e^{-\frac{i}{\hbar}Et} \]

For much more detail about the following basic examples, see any physics textbook on quantum mechanics, or [19].

### 3.1.1 The free particle

The case of the free particle is the case \( V(q) = 0 \). Using Fourier analysis, one finds that the energy eigenvalues and eigenfunctions are parametrized by \( p \in \mathbb{R} \) and are given by

\[ E_p = \frac{p^2}{2m}, \quad \psi_{E_p}(q) = e^{i\frac{p}{\hbar}q} \]

The spectrum of the Hamiltonian is continuous, all non-negative values in \( \mathbb{R} \).

The eigenfunctions of \( H \) are also eigenfunctions of the momentum operator \( P \) with eigenvalue \( p \). \( P \) commutes with \( H \), so if one prepares a state at time 0
with wavefunction $\psi_{E_n}(q)$ and measures its momentum at any later times, one will always get the value $p$ (the momentum is a conserved quantity). Just as $H$ is the generator of time-translations on states, $P$ is the generator of spatial translations.

The eigenfunctions of the operator $Q$ are delta-functions $\delta(q - q')$, with eigenvalue $q' \in \mathbb{R}$. Unlike the case for momentum $P$, one has $[Q, H] \neq 0$ and these are not energy eigenfunctions. If one prepares a state at time 0 with wavefunction $\delta(q - q')$, so localized at $q = q'$, it will immediately evolve into a linear combination of states with all possible eigenvalues of $Q$. Measurement of position at later times $t$ may give all possible different values.

Note that the eigenfunctions of $Q$ and $P$ are not functions in $L^2(\mathbb{R})$ and in addition, the operators $Q$ and $P$ don’t preserve $L^2(\mathbb{R})$ (multiplying or differentiating by $q$ can take a function that is square-integrable to one that isn’t). To deal with these problems simultaneously, one can define the Schwartz space $S(\mathbb{R})$ of functions such that the function and its derivatives fall off faster than any power at $\pm \infty$. The dual space $S'(\mathbb{R})$ of continuous linear functionals on $S(\mathbb{R})$ is called the space of tempered distributions, and includes the eigenfunctions of $Q$ and $P$. One has the sequence of dense inclusions

$$S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R})$$

The Fourier transform takes each term in this sequence to itself.

A problem here is that elements of $S'(\mathbb{R})$ like the eigenfunctions of $Q$ and $P$ are not in $L^2(\mathbb{R})$. They do not have well-defined norms, so will not be vectors in a unitary representation and the Born rule can’t be used for them. However, they are linear functionals on $S(\mathbb{R})$ and one can use this to play the role of their inner products with elements of $S(\mathbb{R})$.

To get a well-defined formalism one has two options:

- Work with states $\psi \in L^2(\mathbb{R})$, taking great care with domains and ranges of operators like $P, Q$ and $H$ that are applied to states. In this case, eigenfunctions of these operators are not in the state space.

- Work with the space $S'(\mathbb{R})$ and distributional states, but be careful to properly pair these only with physical states in $S(\mathbb{R})$ (sometimes called “wavepackets”).

### 3.1.2 The harmonic oscillator

The quantum harmonic oscillator is the case of a particle moving in a quadratic potential $V(q) = \frac{1}{2} m \omega^2 q^2$

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 q^2$$

The energy eigenvalues and eigenfunctions are given by

$$E_n = \hbar \omega(n + \frac{1}{2}), \quad \psi_n(q) = H_n \left( \sqrt{\frac{m \omega}{\hbar}} q \right) e^{-\frac{m \omega q^2}{2\hbar}}$$
where \( n = 0, 1, 2, \ldots \) and \( H_n(q) \) are Hermite polynomials. In this case the spectrum of the operator \( H \) is discrete, energy eigenfunctions are in \( L^2(\mathbb{R}) \), and arbitrary \( t = 0 \) wavefunctions in \( L^2(\mathbb{R}) \) can be written as linear combinations of the \( \psi_{E_n}(q) \).

The easiest way to get these results is to work not with \( Q \) and \( P \), but with complex linear combinations of these. For simplicity, rescaling so that \( \hbar = m = \omega = 1 \), one can choose

\[
a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{1}{\sqrt{2}}(q + \frac{d}{dq}), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})
\]

\( a, a^\dagger \) are each others adjoints and satisfy the commutation relation

\[
[a, a^\dagger] = 1
\]

The Hamiltonian is

\[
H = \frac{1}{2}(Q^2 + P^2) = \frac{1}{2}(a^\dagger a + aa^\dagger) = a^\dagger a + \frac{1}{2}
\]

One can easily see (using \([H, a^\dagger] = a^\dagger \) and \([H, a] = -a \)) that \( a^\dagger \) increases the eigenvalue of \( H \) by 1, \( a \) reduces it by 1. To have a spectrum bounded below, one needs a non-zero state \( \psi_0(q) \) satisfying

\[
a\psi_0(q) = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})\psi_0(q) = 0
\]

This state will have energy \( \frac{1}{2} \) and by given by

\[
\psi_0(q) = e^{-\frac{1}{2}q^2}
\]

The other energy eigenstates will have energy \( n + \frac{1}{2} \) for \( n = 1, 2, \cdots \) and can be found explicitly by applying the operator \( a^\dagger \) \( n \)-times to \( \psi_0(q) \), so evaluating

\[
(q - \frac{d}{dq})^n e^{-\frac{1}{2}q^2}
\]

Note that for the harmonic oscillator, \( V(q) \) is not translation invariant, and one has \([P, H] \neq 0 \) as well as \([Q, H] \neq 0 \) so neither position nor momentum are conserved quantities.

For more general potentials one can have both discrete (with eigenfunctions in \( L^2(\mathbb{R}) \)) and continuous (with eigenfunctions not in \( L^2(\mathbb{R}) \)) components of the spectrum. The physical interpretation will involve both “bound states” which correspond to particles localized in some regions of \( \mathbb{R} \) and “scattering states” which correspond to particles with possible positions extending to \( +\infty \) or \( -\infty \).

### 3.2 The spin \( \frac{1}{2} \) quantum system

A very simple and very important example of a quantum system is the spin \( \frac{1}{2} \) system that describes a highly non-classical degree of freedom shared by all
matter particles. Unlike the previous two examples, this one is not in any sense a quantization of a classical Hamiltonian system with phase space \( \mathbb{R}^{2n} \). This system is characterized by

- The state space is \( \mathcal{H} = \mathbb{C}^2 \).
- The operators corresponding to observables (including the Hamiltonian operator \( H \)) are the self-adjoint operators on \( \mathbb{C}^2 \), so (real) linear combinations of

\[
1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Here the \( \sigma_j \) are the Pauli matrices, the physicist’s convention for a basis of self-adjoint two by two matrices.

This same system describes any quantum system with \( \mathbb{C}^2 \), for which \( H \) can be an arbitrary self-adjoint two by two matrix. The solution to the Schrödinger equation will be given by

\[
e^{-iHt}\psi(0)
\]

where \( \psi(0) \in \mathbb{C}^2 \) is the state at \( t = 0 \). Here \( e^{-iHt} \) will be a unitary matrix, so an element of the group \( U(2) \).

This system is described as “spin \( \frac{1}{2} \)” since it is the spinor representation of the group \( Spin(3) = SU(2) \), the double cover of the rotation group \( SO(3) \). It thus describes a degree of freedom which transforms non-trivially under rotations of space. If one normalizes the observable operators by

\[
S_j = \frac{1}{2}\sigma_j
\]

then \( e^{i\theta S_j} \) will give the behavior of a state under a rotation by an angle \( \theta \) about the \( j \)-axis. These are called “spin” operators and have eigenvalues \( \pm \frac{1}{2} \), which by the principles connecting quantum theory to observation should describe the two possible values one can observe for the spin observable. The subtleties of this become apparent once one notes that the different \( S_j \) don’t commute so can’t be simultaneously diagonalized. Unlike in classical mechanics where a system at a given time has three well-defined components of its angular momentum, here something very different is going on.

There are two different ways to think of this system as the “quantization” of something:

- Take phase space to be \( P = S^2 \), a co-adjoint orbit in \( \mathbb{R}^3 = su(2)^* \), and develop a theory of how to “quantize” such symplectic manifolds.
- Take phase space to be \( \mathbb{R}^3 \) but using “anti-commuting” variables, a subject we will develop later.

In either case, it is as a quantum system that there is a very simple description, with any possible quantum analog something much more complicated to describe.
3.3 Quantization and representation theory

3.3.1 Dirac quantization as a Lie algebra representation

In the previous chapter we saw that the polynomial functions on phase space $P = \mathbb{R}^{2n}$ form a Lie algebra, with Lie bracket the Poisson bracket. Very soon after Heisenberg’s 1925 development of quantum theory based upon non-commuting operators corresponding to position and momentum, Dirac proposed a general rule for such operators. If $O_f$ is the quantum operator corresponding to the classical phase space function $f$, then he proposed that

$$O_{\{f,g\}} = -\frac{i}{\hbar}[O_f, O_g] \quad (3.1)$$

generalizing the Heisenberg commutation relations for operators $Q_j, P_j$ corresponding to coordinates $q_j, p_j$. In the language of Lie algebras and representations, this proposal was that quantization is a unitary representation on the state space $H$ of the infinite dimensional Lie algebra of functions on phase space. The passage from classical to quantum is nothing but the passage from a Lie algebra to one of its representations.

Recall that a complex representation $\pi'$ of a Lie algebra $L$ is a Lie algebra homomorphism

$$\pi' : L \to \text{End}(V)$$

Here $\text{End}(V)$ is the Lie algebra of linear operators on $V$, with Lie bracket the commutator. The Lie algebra homomorphism condition is that $\pi'$ preserves Lie brackets:

$$\pi'([X,Y]) = [\pi'(X), \pi'(Y)]$$

Such a representation will be unitary when there is a Hermitian form on $V$ and the $\pi'(X)$ are skew-adjoint operators ($\pi'(X)^\dagger = -\pi'(X)$).

Such a Lie algebra representation may come from a representation $\pi$ of a group $G$ with Lie algebra $L = \text{Lie}(G)$ (in which case it is called “integrable”). Then $\pi$ is a group homomorphism

$$\pi : G \to GL(V)$$

from the group $G$ to the group of invertible linear operators on $V$. $G$ and $GL(V)$ are smooth manifolds, and $\pi'$ will be the derivative of $\pi$, evaluated at the identity. When the representation is unitary $\pi$ takes values in the group $U(V)$ of unitary transformations.

The Dirac quantization rule (setting $\hbar = 1$) says that

$$f \to \pi'(f) = -iO_f$$

is a Lie algebra homomorphism, since the homomorphism property is

$$\pi'([f,g]) = -iO_{\{f,g\}} = [\pi'(f), \pi'(g)] = [-iO_f, -iO_g] = -\hbar^2 [O_f, O_g]$$

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which is Dirac’s 3.1. Note that the operators $O_f$ favored by physicists are self-adjoint (so have real eigenvalues), while the $\pi'(f) = -i\hbar O_f$ are skew-adjoint.

It turns out that Dirac’s proposal is flawed. In the next chapter we will see that there is a representation $\pi'$ which has the right properties for polynomials of degree up to two (so, for $n = 1$, the Lie subalgebra with basis $1, q, p, q^2, p^2, qp$), but this cannot be extended consistently to higher order polynomials. This is a theorem (called the Groenewold-van Hove no-go theorem) and well-known to physicists in the form of the existence of “operator-ordering ambiguities” occurring when one tries to implement Dirac’s proposal.

### 3.3.2 Some generalities about quantization and representation theory

We will study in the next chapter “canonical quantization” which is the general case of a representation of the Lie algebra of polynomials of degree up to two on $\mathbb{R}^{2n}$. More generally, if one starts with a general classical Hamiltonian system with $P$ a general symplectic manifold, one will still have a Poisson bracket and can ask for a more general notion of quantization that gives a state space with operators satisfying commutation relations corresponding to the Poisson bracket relations. The subject of “geometric quantization” attempts to provide such generalization, but turns out to have limited applicability, especially in not being able to provide the full range of observable operators one would like. Mathematicians studying the representation theory of Lie groups and Lie algebras draw inspiration from quantum systems studied by physicists. These often are “quantizations” of some classic system, potentially providing an example of a new way to construct representations.

A special case is the one we discussed earlier of co-adjoint orbits in $P = \mathfrak{g}^*$. In this case the question of how to get operators has a compelling answer: the algebra of operators in the quantization of $\mathfrak{g}^*$ should be the universal enveloping algebra $U(\mathfrak{g})$. The problem is that one has to represent these operators on a complex vector space $V$, and this is precisely the general problem of representation theory for Lie algebras, that of how to classify and construct all possible representations. It is a well-known principle in this field that a fruitful way to approach this problem is the “orbit method”. Here one uses the decomposition of $\mathfrak{g}$ into co-adjoint orbits and tries to associate to each co-adjoint orbit an irreducible representation, by “quantizing” the classical Hamiltonians system with phase space that orbit. This returns one to the problem of quantizing phase spaces $P$ that are not linear, but for examples that have a great deal of extra structure governed by the Lie algebra $\mathfrak{g}$. The subject of “geometric quantization” has been very much motivated by efforts to solve this problem of quantizing co-adjoint orbits.

In what follows we will stick to a very special case of this general problem, using the Lie algebra of Heisenberg group, for which the co-adjoint orbits are exactly the linear phase spaces $P = \mathbb{R}^{2n}$.

For a summary of the orbit philosophy and how it mostly (but not always) leads to constructions of irreducible representations, see [7].
Chapter 4

Canonical quantization: bosons

4.1 The Heisenberg group and its representations

Quantum mechanics as we know it was born in 1925 in a series of conceptual breakthroughs which began with Heisenberg’s creation of a theory involving non-commuting quantities, soon reformulated (by Max Born) in terms of position and momentum operators $Q$ and $P$ satisfying the commutation relations

$$[Q, P] = i\hbar 1$$

(now known as the Heisenberg commutation relations). We are for now considering just one degree of freedom. $\hbar$ is a constant that depends on units used to measure position and momentum. We will choose units such that $\hbar = 1$. The mathematician Hermann Weyl soon recognized these relations as those of a unitary representation of a Lie algebra now known as the Heisenberg Lie algebra, and described the corresponding Heisenberg group.

Late in 1925, Schrödinger formulated a seemingly different version of quantum mechanics, in terms of wave-functions satisfying a differential equation. What Schrödinger had found was a construction of a representation of the Heisenberg Lie algebra on the vector space of functions $\psi(q)$ of a position variable $q$, with $Q$ the multiplication by $q$ operator and $P$ the differential operator

$$P = -i \frac{d}{dq}$$

We’ll begin with the Lie algebra corresponding to the Heisenberg commutation relations, then find the group with this Lie algebra and show that Schrödinger’s wave-functions give an irreducible unitary representation of the Lie algebra and group. It turns out that any irreducible unitary representation
of the Heisenberg group is essentially equivalent to this one (Stone-von Neumann theorem), but the family of different ways of constructing these representations carries an intricate structure.

4.1.1 The Heisenberg Lie algebra and Lie group

The Lie algebra spanned by $1, q, p$ will be the three-dimensional Lie algebra with a basis $X, Y, Z$ and Lie bracket relations

$$[X, Z] = [Y, Z] = 0, \quad [X, Y] = Z$$

This Lie algebra can be identified with the Lie algebra of three by three strictly upper-triangular matrices by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is called the Heisenberg Lie algebra by mathematicians, and we’ll use the notation $h_3$.

A unitary representation (which we’ll call $\pi'$) will be given by three skew-adjoint operator $\pi'(X), \pi'(Y), \pi'(Z)$ satisfying

$$\quad [\pi'(X), \pi'(Y)] = \pi'(Z), \quad [\pi'(X), \pi'(Z)] = 0, \quad [\pi'(Y), \pi'(Z)] = 0$$

These become the Heisenberg commutation relations if we identify

$$\quad \pi'(X) = -iQ, \quad \pi'(Y) = -iP, \quad \pi'(Z) = -i1$$

Note that factors of $i$ are appearing here just because physicists like to work with self-adjoint operators (since their eigenvalues are real), but for unitary representations the Lie algebra representation operators are skew-adjoint.

In terms of matrices, exponentiating elements of $h_3$ as in

$$\exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

gives the elements of the Heisenberg group $H_3$ (physicists often call this the “Weyl group”, but this means something different to mathematicians). This is the group of upper triangular matrices with 1s on the diagonal. Using $x, y, z$ as (“exponential”) coordinates on the group, $H_3$ is the space $\mathbb{R}^3$ with multiplication law

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$$

For computations with the Heisenberg group it is often convenient to use the Baker-Campbell-Hausdorf formula, which simplifies greatly in this case since all Lie brackets except $[X, Y] = Z$ vanish. As a result, for $A, B \in h_3$ one has

$$e^{A} e^{B} = e^{A + B + \frac{1}{2}[A, B]}$$
This group is a central extension
\[ 0 \to (\mathbb{R}, +) \to H_3 \to (\mathbb{R}^2, +) \to 0 \]
of the additive group of \( \mathbb{R}^2 \) by the additive group of \( \mathbb{R} \) (which is the center of the group).

A slightly different version of the Heisenberg group (which we’ll call \( H_{3,\text{red}} \)) that is sometimes used takes a quotient by \( \mathbb{Z} \) and replaces the central \( \mathbb{R} \) with a central \( U(1) \), so is a central extension
\[ 0 \to U(1) \to H_{3,\text{red}} \to (\mathbb{R}^2, +) \to 0 \]
Elements are labeled by \((x, y, u)\) where \( x \) and \( y \) are in \( \mathbb{R} \) and \( u \in U(1) \), and the group law is
\[(x, y, u)(x', y', u') = (x + x', y + y', uu'e^{\frac{i}{2}(xy' - x'y)})\]

### 4.1.2 The Schrödinger representation

The Schrödinger representation \( \pi_S \) will be a representation on a vector space \( \mathcal{H} \) of complex valued functions \( \psi(q) \) on \( \mathbb{R} \), with derivative the Lie algebra representation
\[
\pi'_S(X) = -iQ = -iq, \quad \pi'_S(Y) = -iP = -\frac{d}{dq}, \quad \pi'_S(Z) = -i1
\]
Exponentiating these operators gives unitary operators that generate \( \pi_S \)
\[
\pi_S(x) = e^{-ixq}, \quad \pi_S(y) = e^{-y\frac{d}{dq}}, \quad \pi_S(z) = e^{-iz1} \tag{4.1}
\]
Note that \( \pi_S(y) \) acts on the representation space by translation
\[
\pi_S(y)\psi(q) = \psi(q - y)
\]
**Definition** (Schrödinger representation). The Schrödinger representation of the Heisenberg group \( H \) is given by
\[
\pi_S(x, y, z)\psi(q) = e^{-i\frac{z}{2}(xy - x'y)}e^{-izq}\psi(q - y) \tag{4.2}
\]
for \((x, y, z) \in H\).

One can easily check that this is a representation, since it satisfies the homomorphism property
\[
\pi_S(x, y, z)\pi_S(x', y', z') = \pi_S(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))
\]
Taking as representation space \( \mathcal{H} = L^2(\mathbb{R}) \), for the Lie algebra representation \( \pi'_S \) there will be domain (functions on which operators not defined) and range (operators take something in \( L^2(\mathbb{R}) \) to something not in \( L^2(\mathbb{R}) \)) problems. As
an alternative, one can take $\mathcal{H} = \mathcal{S}(\mathbb{R})$ so that the representation operators are well-defined (but then the dual space is something different, the tempered distributions $\mathcal{S}'(\mathbb{R})$). For the group representation, the operators $\pi_{S}$ are well defined on $\mathcal{H} = L^2(\mathbb{R})$. Giving up on a well-defined inner-product and unitarity, one can take $\mathcal{H} = \mathcal{S}'(\mathbb{R})$ and have both a Lie algebra and Lie group representation.

This multiplicity of closely related versions of the representation is a general phenomenon for infinite-dimensional representations of non-compact Lie groups, where one has inequivalent representations on a sequence of dense inclusions of representation spaces, here

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$$

### 4.1.3 The Stone-von Neumann theorem

The remarkable fact about representations of the Heisenberg group is that there is essentially only one representation (once one has specified the constant by which $Z$ acts, but non-zero choices are related by a rescaling). More specifically, any irreducible representation of $H_3$ will be unitarily equivalent to the Schrödinger representation. One has the following theorem

**Theorem** (Stone-von Neumann). *For any irreducible unitary representation $\pi$ of $H_3$ (with action of the center $\pi(0, 0, z) = e^{-iz}$) on a Hilbert space $\mathcal{H}$, there is a unitary operator $U : \mathcal{H} \to L^2(\mathbb{R})$ such that*

$$U \pi U^{-1} = \pi_{S}$$

We will not give a proof here, since the analysis is somewhat involved, but what follows should make clear some problems that any proof needs to overcome and motivate the strategy for an actual proof.

Recall that one can define the adjoint pair of operators

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{1}{\sqrt{2}}(q + \frac{d}{dq}), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})$$

and for the harmonic oscillator Hamiltonian the lowest energy eigenspace is the one-dimensional space of solutions in $L^2(\mathbb{R})$ of

$$a \psi_0(q) = 0$$

These are all proportional to

$$\psi_0 = e^{-\frac{1}{2}q^2}$$

The rest of the state space can be generated by repeatedly applying the operator $a^\dagger$ to $\psi_0$.

**Exercise.** *Use this basis to prove that the Schrödinger representation is irreducible.*
For some motivation for why the Stone-von Neumann theorem might be true, for $\pi'$ one can construct analogs of the $a, a^\dagger$

$$b = \frac{1}{2} (i \pi'(X) - \pi'(Y))$$

and its adjoint $b^\dagger$. These will satisfy $[b, b^\dagger] = 1$ and by the argument given for the harmonic oscillator state space, there should be a state $|0_b\rangle$ satisfying $b |0_b\rangle = 0$, which together with the $(b^\dagger)^k |0_b\rangle$ should give an orthonormal basis of the state space in the $\pi'$ representation. There will be a unitary operator $U : \mathcal{H} \to L^2(\mathbb{R})$ taking the basis constructing using the $b, b^\dagger$ operators to the standard basis of harmonic oscillator energy eigenstates in the Schrödinger representation. A possible approach to the Stone-von Neumann theorem would be to note that

$$UbU^{-1} = a, \quad Ub^\dagger U^{-1} = a^\dagger$$

that $b$ has a one-dimensional kernel (irreducibility), and that the rest of the representation is given by repeated applications of $b^\dagger$. The $U$ would then give the desired unitary equivalence.

Unfortunately, this can’t work, since there is no guarantee that vectors in the range of $b^\dagger$ will be in its domain, so one can’t generate the representation by repeatedly applying $b^\dagger$ (it is not hard to construct examples of this using wave-functions with specific boundary conditions). It turns out that the Stone-von Neumann theorem is not true for general Lie algebra representations of $\mathfrak{h}_3$, only works for Lie algebra representations that integrate to give a group representation. To get a proof that does work, one needs to work not with $b, b^\dagger$ and $a, a^\dagger$, but with their exponentiated versions. For details, see [?], chapter 14.

An important example of an irreducible representation unitarily equivalent to the Schrödinger representation is given by using the Fourier transform $\mathcal{F}$

$$\psi(q) \rightarrow \tilde{\psi}(p) = (\mathcal{F}\psi)(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ipq} \psi(q) dq$$

This is a unitary transformation on $L^2(\mathbb{R})$, with inverse $\mathcal{F}^{-1}$ given by Fourier inversion

$$\tilde{\psi}(p) \rightarrow (\mathcal{F}^{-1}\tilde{\psi})(q) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ipq} \tilde{\psi}(p) dp$$

The Stone-von Neumann theorem applies, with $U = \mathcal{F}, \quad U^{-1} = \mathcal{F}$.

Note that we will generically refer to the essentially unique representation of the Heisenberg using $\mathcal{H}$ for the representation space and $\pi$ for the homomorphism from the group to operators on $\mathcal{H}$, with $\pi'$ for the Lie algebra representation. When we want to specify a specific construction, the $\pi$ may acquire a subscript (e.g. $\pi_S$ for the Schrödinger construction) and $\mathcal{H}$ may get further specified (e.g. $L^2(\mathbb{R})$). Terminology in this subject can be a bit confusing, since instead of the usual multiple representations to keep track of, here there is only one, but with multiple quite different constructions.
4.1.4 The Bargmann-Fock representation

The Stone-von Neumann theorem also applies to very different constructions of representations on other versions of Hilbert space. In particular, it is clear from looking at the harmonic oscillator calculations that energy eigenstates can be identified with monomials in a complex variable, with $a$ and $a^\dagger$ decreasing and increasing the degree. To find a construction of the Heisenberg group irreducible representation on $\mathbb{C}[w]$, one needs a Hilbert space structure, which one can define as follows:

**Definition** (Fock Space). Fock space $\mathcal{H}_F$ is the space of entire functions on $\mathbb{C}$, with finite norm in the inner product

$$\langle f(w), g(w) \rangle = \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(w)}g(w)e^{-|w|^2}$$

An orthonormal basis of $\mathcal{H}_F$ is given by appropriately normalized monomials. Since

$$\langle w^m, w^n \rangle = \frac{1}{\pi} \int_{\mathbb{C}} \overline{w^m}w^n e^{-|w|^2}$$

$$= \frac{1}{\pi} \int_0^\infty \left( \int_0^{2\pi} e^{i\theta(n-m)}d\theta \right) r^{n+m} e^{-r^2} dr$$

$$= n!\delta_{n,m}$$

we see that the functions $\frac{w^n}{\sqrt{n!}}$ are orthonormal.

To get a representation of the (complexified) Heisenberg Lie algebra on this space, define

$$a = \frac{d}{dw}, \quad a^\dagger = w$$

**Exercise.** Show that these operators are each other’s adjoints with respect to the inner product on Fock space.

On the real Heisenberg Lie algebra, this representation exponentiates to a representation of the Heisenberg group. By the Stone-von Neumann theorem it is unitarily equivalent to the Schrödinger representation on $L^2(\mathbb{R})$.

To explicitly write the Bargmann-Fock representation of the Heisenberg Lie algebra, one can complexify and work with operators that depend on complex linear combinations of the real basis $X, Y, Z$. If one does this first in the Schrödinger representation one has

$$\pi'_S(iX) = Q, \quad \pi'_S(iY) = P, \quad \pi'_S(iZ) = 1$$

and so

$$\pi'_S\left(\frac{1}{\sqrt{2}}(iX + iY)\right) = a = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$$

The Bargmann-Fock representation

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and so

$$\pi'_S\left(\frac{1}{\sqrt{2}}(iX + iY)\right) = a = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$$
(with at similar formula for $a^\dagger$). To get Bargmann-Fock one wants a $\pi'_{BF}$ that takes the same linear combinations to $\frac{d}{dw}$ and $w$, acting on $\mathcal{H}_F$. Thus

$$\pi'_{BF}(\frac{1}{\sqrt{2}}(iX + i(iY))) = a = \frac{d}{dw}, \quad \pi'_{BF}(\frac{1}{\sqrt{2}}(iX - i(iY))) = a^\dagger = w, \quad \pi'_{BF}(iZ) = 1$$

We won’t work this out here, but these operators can be exponentiated to get operators for a Heisenberg Lie group representation. By Stone-von Neumann, there will be a unitary operators

$$U : \mathcal{H}_F \rightarrow L^2(\mathbb{R}), \quad U^{-1} : L^2(\mathbb{R}) \rightarrow \mathcal{H}_F$$

These operators are quite non-trivial and interesting in analysis, giving unitary isomorphisms between two very different kinds of function spaces. The explicit form for $U^{-1}$ is often called the Bargmann transform and is given by

$$(U^{-1}\psi)(w) = \left(\frac{1}{\pi}\right)^\frac{1}{4} e^{-\frac{1}{2}w^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}q^2} e^{\sqrt{2}wq} \psi(q) dq$$

The relation between the Schrödinger and Bargmann-Fock operators will be given by

$$U \frac{d}{dw} U^{-1} = \frac{1}{\sqrt{2}}(q + \frac{d}{dq}), \quad UwU^{-1} = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})$$

Note that what we have been calling the “Fock representation” is defined in terms of polynomials on a complex vector space of dimension $n$. Using the isomorphism between polynomials on a vector space $V$ and the symmetric tensor product $S^*(V^*)$, one can instead define this in terms of tensor products. We will later write this out in detail, since it becomes important in quantum field theory, where one deals with the case of $V$ infinite-dimensional.

For more on the Bargmann-Fock representation and the Bargmann transform a good source is Chapter 1, Section 6 of [2].

### 4.1.5 The Weyl algebra

A closely related algebra to the Heisenberg Lie algebra is the Weyl algebra, which can be defined as the non-commutative algebra of polynomial coefficient differential operators for a complex variable $w$. The generators of the algebra are

- Multiplication by $w$.  
- Differentiation by $w$: $\frac{d}{dw}$

These satisfy the same commutation relations as $a, a^\dagger$

$$[\frac{d}{dw}, w] = 1$$
since
\[ \frac{d}{dw}(wf) - w \frac{df}{dw} = f \]

Recall that one can think of representations of a Lie algebra \( g \) as modules for the associative algebra \( U(g) \) (the universal enveloping algebra of \( g \)). It is convenient here also to complexify, and for any Lie algebra \( g \) we’ll use the notation \( U(g) \) to refer to \( U(g) \otimes \mathbb{C} = U(g \otimes \mathbb{C}) \). For the Heisenberg Lie algebra \( h_3 \), \( U(h_3) \) is given by all complex linear combinations of products of basis elements \( X, Y, Z \), modulo the relations
\[
[X, Z] = [Y, Z] = 0, 
[X, Y] = Z
\]
The center of \( U(h_3) \) (denoted here \( Z(h_3) \)) is the commutative algebra \( \mathbb{C}[Z] \) of polynomials in \( Z \). In any irreducible representation \( \pi' \) of a Lie algebra \( g \), by Schur’s lemma elements of the center \( Z(g) \) act by scalars. This gives a homomorphism
\[ \chi_{\pi'} : Z(g) \to \mathbb{C} \]
called the infinitesimal character of the representation. In the case of \( g = h_3 \), since \( Z(h_3) \) is an algebra of the polynomial functions in one variable, the infinitesimal character is evaluation of the polynomial at some \( c \in \mathbb{C} \). This \( c \) is the scalar given by the action of \( \pi'(Z) \) on the representation space. The Schrödinger representation as we have defined it is an irreducible representation with \( c = -i \).

For general Lie algebra representations of the complexified Lie algebra \( h_3 \otimes \mathbb{C} \), for each \( c \neq 0 \) we have the irreducible representation unitarily equivalent to the Schrödinger representation (rescaled from \( c = -i \)). These will be unitary for \( c \) imaginary.

\( Z \) acts by a scalar \( c_{\pi'} \). Polynomials in \( Z \) also act by a scalar, the evaluation of the polynomial at \( c_{\pi'} \). The Schrödinger representation as we have defined it is an irreducible representation with \( c_{\pi'} = -i \). Restricting attention to Lie algebra representations for which \( \pi'(Z) = c \mathbf{1} \) for a chosen \( c \in \mathbb{C} \), these will be modules for the quotient algebra
\[ U(h_3)/(Z - c) \]
By rescaling \( X \) and \( Y \), for \( c \neq 0 \), we get the Weyl algebra, and so an irreducible Heisenberg algebra representation will be a module for the Weyl algebra. Among these modules is the standard one on polynomials on \( w \), which corresponds to the one we have studying, which is integrable to a unitary Heisenberg group representation. But there are many different modules for the Weyl algebra, with the study of these modules the beginning of the subject of D-modules in algebraic geometry (see for instance [?]).

### 4.1.6 The Heisenberg group and symplectic geometry

The three-dimensional Heisenberg group that we have been studying has a simple generalization that behaves in much the same way. For any \( n \), define the
2n + 1 dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ to be the Lie algebra with basis $X_j, Y_j, Z$ ($j = 1, 2, \cdots, n$) and all Lie brackets zero except

$$[X_j, Y_k] = \delta_{jk} Z$$

One can easily get a corresponding Heisenberg Lie group $H_{2n+1}$ generalizing the $n = 1$ case by exponentiating.

Instead of working with a basis like this, one can define this Lie group in a more coordinate-invariant way, starting with any symplectic form on $M = \mathbb{R}^{2n}$ (note that $M$ corresponds to $P^*$, the dual of phase space, since coordinates on phase space are a basis of $M$), where

**Definition (Symplectic form).** A symplectic form $\Omega$ on a vector space $M$ is a non-degenerate anti-symmetric bilinear form

$$(v_1, v_2) \in M \times M \to \Omega(v_1, v_2) \in \mathbb{R}$$
on $M$.

This is the same definition as that of an inner product on a vector space $V$, with “symmetric” replaced by “antisymmetric.” For any even-dimensional real vector space $M$ with a symplectic form $\Omega$, one can define a Lie algebra structure on $M \oplus \mathbb{R}$ by taking the Lie bracket to be

$$[(v, z), (v', z')] = (0, \Omega(v, v'))$$

where $(v, z)$ are elements of $M \oplus \mathbb{R}$. One gets a corresponding Lie group by taking as group law on $M \oplus \mathbb{R}$

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2} \Omega(v, v'))$$

In the inner product case, by Gram-Schmidt orthonormalization one can always find an orthonormal basis of $V$, with any other basis related to this one by an element of $GL(V)$. The subgroup of $GL(V)$ preserving the inner product and thus taking orthonormal bases to orthonormal bases is the orthogonal group $O(V)$. In the symplectic case, $M$ has to be even-dimensional (to have a non-degenerate $\Omega$).

**Exercise.** Show that one can always find a “symplectic basis”: $X_j$ and $Y_j$ for $j = 1, 2, \cdots, n$ satisfying

$$\Omega(X_j, X_k) = \Omega(Y_j, Y_k) = 0, \quad \Omega(X_j, Y_k) = \delta_{jk}$$

and that in this basis one recovers the earlier definition of the Heisenberg Lie algebra and Lie group of dimension $2n + 1$.

The subgroup of $GL(M)$ preserving $\Omega$ and taking symplectic bases to symplectic bases is by definition the symplectic group $Sp(M)$. Choosing a basis, this group will be a matrix group that can be denoted $Sp(2n, \mathbb{R})$. Note that this
is different than the group often written as $Sp(n)$, the group of $n$ by $n$ quaternionic matrices preserving the standard hermitian form on $\mathbb{H}^n$. The groups $Sp(n)$ and $Sp(2n,\mathbb{R})$ are different real forms of the group $Sp(2n,\mathbb{C})$ of linear transformations preserving a non-degenerate anti-symmetric bilinear form on $\mathbb{C}^{2n}$.

4.2 The symplectic group and the oscillator representation

The irreducible representation of the Heisenberg group we have been studying provides a projective representation of the symplectic group, which we'll construct in this section. This has various names, of which we'll choose Roger Howe's “oscillator representation.” For more details, a good source is [2].

4.2.1 The Poisson bracket and the Lie algebras $\mathfrak{h}_{2n+1}$ and $\mathfrak{sp}(2n,\mathbb{R})$

In the last section we studied the Lie algebra of the Heisenberg group, which is $2n + 1$ dimensional. As a Lie subalgebra of the functions on phase space $P$, it has basis $1, q_j, p_j$ for $j = 1, \ldots, n$, with non-zero Lie brackets the Poisson brackets

$$\{q_j, p_k\} = \delta_{jk}.$$ 

In this section we'll extend this to the Lie algebra of monomials of degree up to two.

The space of degree two monomials on $P$ has as basis elements $q_j p_k$ for all $j, k$ and $q_j q_k, p_j p_k$ for $j \leq k$. The Poisson bracket of two of these is a linear combination of degree two monomials, so these provide a real Lie algebra of dimension $2n^2 + n$. This will turn out to be the Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$ of the symplectic group $Sp(2n,\mathbb{R})$.

Here we will work out explicitly what happens for $n = 1$. The symplectic Lie algebra $\mathfrak{sp}(2,\mathbb{R})$ has basis $q^2, p^2, qp$ with non-zero Lie brackets

$$\{q^2/2, p^2/2\} = qp, \quad \{qp, p^2\} = 2p^2, \quad \{qp, q^2\} = -2q^2.$$ 

This is isomorphic to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $2$ by $2$ traceless real matrices, with bracket the commutator, where a conventional basis is

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

The isomorphism is explicitly given by

$$\frac{q^2}{2} \leftrightarrow E, \quad -\frac{p^2}{2} \leftrightarrow F, \quad -qp \leftrightarrow G.$$ 

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or by

\[-aqp + \frac{bq^2}{2} - \frac{cp^2}{2} \iff \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \]

The semi-direct product of $H_3$ and $SL(2, \mathbb{R})$ puts the above two Lie algebras together, with the action of $SL(2, \mathbb{R})$ on $H_3$ by automorphisms reflected in the non-zero Lie brackets.

Putting together the Lie algebras $h_3$ and $sp(2, \mathbb{R})$, we get not the direct sum of the Lie algebras but something more interesting, due to the non-zero Poisson brackets between degree two and degree one monomials:

\[
\{qp, q\} = -q, \quad \{qp, p\} = p
\]

\[
\{p^2/2, q\} = -p, \quad \{q^2/2, p\} = q
\]

These are the infinitesimal expression of the fact that we are looking not at the product group $H_3 \times Sp(2, \mathbb{R})$, but at a semidirect product $H_3 \rtimes Sp(2, \mathbb{R})$, which uses the fact that the action of $Sp(2, \mathbb{R})$ on phase space gives an action on $H_3$ by automorphisms.

From these relations one can see that

\[-qp \iff \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

generates a group $\mathbb{R}$ acting on the $q$ direction in the $qp$ plane by $e^t$, on the $p$ direction by $e^{-t}$. The element

\[
\frac{1}{2}(q^2 + p^2) \iff \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

generates an $SO(2)$ subgroup of rotations in the $qp$ plane.

### 4.2.2 The Schrödinger model for the oscillator representation

We have seen that the Schrödinger representation is given as a representation of $h_3$ by the operators

\[
\pi'_S(q) = -iQ = -iq, \quad \pi'_S(p) = -iP = -\frac{d}{dq}, \quad \pi'_S(1) = -i1
\]

Dirac’s original definition of “quantization” asked for an extension of this representation from linear functions to all functions on phase space, i.e. a choice of operators that would take any polynomial in $q$ and $p$ to an operator, with Poisson bracket of functions going to commutator of operators, so a Lie algebra homomorphism. But going from functions of $q$ and $p$ to operators built out of $Q$ and $P$, one runs into “operator-ordering” ambiguities since $Q$ and $P$ do not
commute. It turns out that one can get a Lie algebra homomorphism for polynomials up to degree two, but this is impossible in higher degree (Groenewold-van Hove theorem).

What works in degree two is to extend the Schrödinger representation to a representation of $\mathfrak{sl}(2, \mathbb{R})$ (and of the semi-direct product with $\mathfrak{h}_3$) by taking

$$
\pi'_S(q^2) = -iQ^2 = -iq^2, \quad \pi'_S(p^2) = -iP^2 = i \frac{d^2}{dq^2}
$$

and making the choice

$$
\pi'_S(qp) = -i \frac{1}{2} (QP + PQ) = -i \frac{1}{2} (2QP - iI) = -q \frac{d}{dq} - \frac{1}{2} I
$$

(which gives a skew-adjoint operator).

These operators will satisfy the commutation relations given by the Lie bracket of $\mathfrak{sl}(2, \mathbb{R})$, so give a representation, which is the oscillator representation. (it has many other names, including the “Weil representation”). The representation will be on the same space as the Schrödinger representation, extending the action of the Heisenberg Lie algebra, so we will often denote it by the same symbol $\pi_S$.

One would like to exponentiate the Lie algebra representation operators to get a representation of the Lie group $SL(2, \mathbb{R})$. In the case of $\pi'_S(qp)$ the operator exponentiates to an operator on functions which rescales in the $q$ variable. It is though not so easy to exponentiate the second order differential operator

$$
-iP^2 = i \frac{d^2}{dq^2}
$$

If one takes a Fourier transform to turn derivatives in $q$ into multiplication operators, the problem just moves to the operator $-iQ^2$ which changes from a multiplication operator to a second-order differential operator.

The problem is best thought of as having to do with exponentiating the Lie algebra element

$$
\frac{1}{2} (q^2 + p^2)
$$

which generates the $SO(2) \subset SL(2, \mathbb{R})$ subgroup of rotations in the $qp$ plane. So, for the oscillator representation, we need to explicitly construct the operator

$$
e^{\theta \pi'_S(\frac{1}{2} (q^2 + p^2))}
$$

where

$$
\pi'_S \left( \frac{1}{2} (q^2 + p^2) \right) = -\frac{1}{2} (Q^2 + P^2) = -\frac{1}{2} (q^2 - \frac{d^2}{dq^2})
$$

Changing notation from $\theta$ to $t$, this is just the standard physics problem of solving the Schrödinger equation for the Hamiltonian $H = \frac{1}{2} (Q^2 + P^2)$ and so constructing the unitary operator

$$
U(t) = e^{-it \frac{1}{2} (Q^2 + P^2)}
$$

(4.3)
With some effort (see for instance exercises 4 and 5 of chapter III of [6]), one can derive a formula for the kernel $K_t(q, q')$ (known in physics as the “propagator”) where

$$K_t(q, q') = \frac{1}{\sqrt{2\pi\sin t}} \exp \left( -\frac{1}{2} \left( q - q' \right) \left( \begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array} \right) \left( \begin{array}{c} q \\ q' \end{array} \right) \right)$$

One finds

$$K_t(q, q') = \frac{1}{\sqrt{2\pi\sin t}} \exp \left( -\frac{1}{2} \left( q - q' \right) \left( \begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array} \right) \left( \begin{array}{c} q \\ q' \end{array} \right) \right)$$

This expression requires interpretation as a distribution defined as a boundary value of a holomorphic function, replacing $t$ by $t - i\epsilon$ and taking the limit as positive $\epsilon$ vanishes.

One can show that

$$\lim_{\epsilon \to 0^+} U(t) \longrightarrow e^{i\pi F}$$

This is the oscillator representation operator for an element of the symplectic group corresponding to a $\frac{\pi}{2}$ rotation in the $q,p$ plane, interchanging the role of $q$ and $p$. As expected from the Stone-von Neumann theorem, one gets the Fourier transform, up to a phase factor. The calculation of the propagator fixes the phase factor. In some sense, rotations by arbitrary values of $t$ will give “fractional Fourier transforms.”

Rotation by $\pi$ in the $q,p$ plane is given by

$$iF^2$$

The $F^2$ is as expected since $F^2$ acts on functions by

$$\psi(q) \rightarrow F^2 \psi(q) = \psi(-q)$$

corresponding to a rotation by $\pi$ taking $q$ to $-q$. Rotation by $2\pi$ is given by $-F^4 = -1$ rather than the $1$ expected if $U(t)$ is to be a true (rather than up to $\pm 1$) representation of $SO(2) \subset SL(2,\mathbb{R})$. This is a precise analog of what happens when we take the spinor Lie algebra representation of $SO(3)$ and exponentiate: we find that rotating around an axis by $2\pi$ gives a factor of $-1$. The representation is only a projective (up to sign) representation of $SO(3)$. To get a true representation, one needs the double cover $Spin(3) = SU(2)$. Here again we have a representation up to sign and need a double cover of $Sp(2,\mathbb{R})$. This will be the metaplectic group $Mp(2,\mathbb{R})$, which is not a matrix group.

### 4.2.3 The Bargmann-Fock model for the oscillator representation

The best way to calculate the phase factors in the exponentiated version of the oscillator representation is not to use the Schrödinger version of the representation and the complicated formula 4.4 for the propagator, but to instead use the
Bargmann-Fock version. Here the representation is on the space of polynomials $C[w]$ (with the Bargmann-Fock inner product) and the operators

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{d}{dw}, \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = \frac{d}{dw}$$

provide a representation of the complexified Heisenberg Lie algebra (which is the standard one on the real Lie algebra).

As in the Schrödinger case, one can extend this representation to the oscillator representation of $\mathfrak{sp}(2n, \mathbb{R})$ by taking quadratic combinations of the Heisenberg Lie algebra operators. In particular, using

$$\frac{1}{2}(Q^2 + P^2) = \frac{1}{2}(a^\dagger a + aa^\dagger) = a^\dagger a + \frac{1}{2}$$

one has (writing elements of $\mathfrak{sl}(2, \mathbb{R})$ both as quadratic polynomials and as matrices)

$$\pi'_{BF}(\frac{1}{2}(q^2 + p^2)) = \pi'_{BF}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = -i(a^\dagger a + \frac{1}{2}) = -i(w \frac{d}{dw} + \frac{1}{2})$$

This operator can easily be exponentiated:

$$e^{\theta \pi'_{BF}(\frac{1}{2}(q^2+p^2))}$$

acts on $C[w]$ by multiplying the monomial $w^n$ by $e^{-i\theta(n + \frac{1}{2})}$. This gives the minus sign previously discussed for $\theta = 2\pi$.

In this representation the other two basis elements of $\mathfrak{sl}(2, \mathbb{R})$ are

$$\pi'_{BF}(-qp) = \pi'_{BF}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = -\frac{1}{2}((a^\dagger)^2 - a^2)$$

$$\pi'_{BF}(q^2 - p^2) = \pi'_{BF}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = -\frac{i}{2}((a^\dagger)^2 + a^2)$$

Note that these operators do not change the parity of monomials they act on, and you can get from any monomial of a given parity to any other of the same parity by applying these operators repeatedly. So, the oscillator representations we have constructed here is the sum of two irreducibles (all polynomials of even degree, and all polynomials of odd degree).

4.2.4 The symplectic group and automorphisms of the Heisenberg Lie group

Since the definition of the Heisenberg Lie algebra and Lie group only depend on the antisymmetric bilinear form $\Omega$ on $M = \mathbb{R}^{2n}$, the group $Sp(2n, \mathbb{R})$ of linear maps preserving $\Omega$ acts on this Lie algebra and group as automorphisms. Using $(v, z) \in V \oplus \mathbb{R}$ as coordinates on $H_{2n+1}$, the action of $g \in Sp(2n, \mathbb{R})$ on the Heisenberg group is

$$\Phi_g(v, z) = (gv, z)$$
Using this automorphism, one can construct the semi-direct product
\[ H_{2n+1} \rtimes Sp(2n, \mathbb{R}) \]
which is sometimes called the “Jacobi group.”

We also can use these automorphisms to act on the set of representations of \( H_{2n+1} \), taking
\[ \pi \to \pi_g \]
where
\[ \pi_g(v, z) = \pi(\Phi_g(v, z)) \]
If \( \pi \) is irreducible, \( \pi_g \) will also be irreducible, and by the Stone-von Neumann theorem there will be unitary operators \( U_g \) such that
\[ \pi_g = U_g \pi U_g^{-1} \]
By Schur’s lemma, these operators will be unique up to a phase factor. They will then provide a representation of \( Sp(2n, \mathbb{R}) \) up to a phase factor (a projective representation)
\[ U_{g_1} U_{g_2} = e^{i\theta(g_1, g_2)} U_{g_1 g_2} \]
By changing the \( U_g \) by a phase factor
\[ U_g \to V(g) = e^{i\phi(g)} U(g) \]
one can try and remove the projective factor from the multiplication law. As we have seen explicitly in the case \( n = 1 \), this can only be done up to sign, a problem much like that which occurs in the case of the spin representation of the rotation group. As in the case of the rotation group, one can get a true representation by going to a double cover of \( Sp(2n, \mathbb{R}) \), which we’ll denote \( Mp(2n, \mathbb{R}) \) and call the “metaplectic group.” Two differences from the rotation group case are:

- In the rotation group case \( \pi_1(SO(n)) = \mathbb{Z}_2 \) and the double cover \( Spin(n) \) is the universal cover. In the symplectic case \( \pi_1(Sp(2n, \mathbb{R})) = \mathbb{Z} \) and the metaplectic double cover is just one of many possible covering groups.

- \( Spin(n) \) can be identified with a group of finite-dimensional matrices. This is not true for \( Mp(2n, \mathbb{R}) \), a group which has no finite-dimensional faithful representations. It provides a very unusual example of where thinking of Lie theory just in terms of matrix groups is inadequate.

We will refer to the representation of \( Mp(2n, \mathbb{R}) \) as the “oscillator representation (it goes by many other names, including Weil representation, Segal-Shale-Weil representation, etc.). The representation will be on the same space \( \mathcal{H} \) as the Schrödinger representation, extending the action of the Heisenberg Lie group, so we will often denote it by the same symbol \( \pi_S \) and also call the representation of the Heisenberg group by the same name. We will also describe this representation as being “essentially unique”, meaning that all versions of it are the same up to unitary transformations, possible rescaling, and differences in the definition of \( \mathcal{H} \) related by dense inclusions.
4.3 Choice of polarization

4.3.1 Real polarizations and the Schrödinger representation

From the discussion in section 4.1.6, the real symplectic vector space $M$ can be written as

$$M = L \oplus L^*$$

where $L$ is an $n$-dimensional vector space with basis $X_j$ and $L^*$ is the dual vector space with basis elements $Y_j$ dual to the $X_j$ (i.e. $Y_j(X_k) = \delta_{jk}$). Note that for any vectors $x, x' \in L \subset M$ one has $\Omega(x, x') = 0$. A subspace with this property is called "isotropic". The maximal dimension of a subspace of $M$ on which $\Omega$ is zero is $n$, and such isotropic subspaces are called "Lagrangian". $L^*$ is also Lagrangian.

Since the definition of the Heisenberg Lie algebra and Lie group depend only on the symplectic form $\Omega$, and by Stone-von Neumann there is only one irreducible representation, one might expect that the definition of this irreducible representation should depend just on $\Omega$. It turns out though that all constructions of this representation depend upon a choice of additional structure. We have seen that the construction of the Schrödinger representation depends on a choice of $n$ position coordinates $q_j$, corresponding to the basis elements $X_j$ of the Lie algebra, which span a Lagrangian subspace of $\mathbb{R}^{2n}$. The Fourier transform takes this construction to a different one, depending on $n$ momentum coordinates $p_j$, corresponding to the basis elements $Y_j$ of the Lie algebra, which span a complementary Lagrangian subspace of $\mathbb{R}^{2n}$.

More generally, one can construct a version of the Schrödinger representation for any choice of Lagrangian subspace $\ell \subset \mathbb{R}^{2n}$. By the Stone-von Neumann theorem, for each $\ell$ there will be an operator $U_\ell$ giving a unitary equivalence with the construction for the standard Schrödinger choice of $\ell = L$ spanned by the $X_j$. For $\ell = L^*$ spanned by the $Y_j$, $U_\ell$ will be the Fourier transform, but for more general $\ell$ its construction is rather non-trivial. A choice of a Lagrangian $\ell$ and thus a decomposition $M = \ell \oplus \ell^*$ is called a "real polarization" of $M$.

**Exercise.** Show that the choices of Lagrangian subspace $\ell$ are parametrized by the space $U(n)/O(n)$.

For the case $n = 1$, $U(1)/O(1) = \mathbb{R}P^1$, which is a circle, so real polarizations $\ell$ are parametrized by an angle $\theta$. The operators $U_\ell$ are the operators $U(\theta)$ of equation 4.3, going once around $\mathbb{R}P^1$ as $\theta$ goes from 0 to $\pi$.

4.3.2 Complex polarizations

The Bargmann-Fock construction involves a different sort of polarization, called a "complex polarization." Here one complexifies $M$ and asks for Lagrangian subspaces $W$ and $\overline{W}$ such that

$$M \otimes \mathbb{C} = W \oplus \overline{W}$$
where $W$ and $\bar{W}$ are interchanged by the conjugation map on $C$.

Such a decomposition is equivalent to the choice of a compatible complex structure on $M$, where

**Definition (Complex structure).** A complex structure on a real vector space $M$ is a (real)-linear map

$$J : M \to M$$

satisfying $J^2 = -1$.

and

**Definition (Compatible complex structure).** A complex structure on $M$ is compatible with a symplectic form $\Omega$ on $M$ when

$$\Omega(Jv_1, Jv_2) = \Omega(v_1, v_2)$$

Such $J$ only exist if the dimension of $M$ is even and one can think of them as ways of making $M$ a complex vector space (so identifying $\mathbb{R}^{2n} = \mathbb{C}^n$), with multiplication by $i$ given by $J$. $J$ has no eigenvectors in $M$, but it does have complex eigenvalues $\pm i$, giving a decomposition

$$V \otimes \mathbb{C} = M^+_J \oplus M^-_J$$

into $\pm i$ eigenspaces for $J$. This will be a polarization of $M$ when $J$ is compatible with $\Omega$ since then $M^+_J$ and $M^-_J$ are Lagrangian subspaces. To see this, note that for $w_1, w_2 \in V^+_J$

$$\Omega(w_1, w_2) = \Omega(Jw_1, Jw_2) = \Omega(iw_1, iw_2) = -\Omega(w_1, w_2)$$

so must be zero.

Given both a symplectic form $\Omega$ and a compatible complex structure $J$ on $M$, $M$ becomes not just a complex vector space, but a complex vector space with Hermitian inner product, defined by

$$\langle v_1, v_2 \rangle_J = \Omega(v_1, Jv_2) + i\Omega(v_1, v_2)$$

One can easily check that this is Hermitian, but it is not necessarily positive. To get a positive Hermitian structure one needs to impose an additional condition on $J$, that, for non-zero $v \in M$ one has

$$\Omega(v, Jv) > 0$$

The possible choices of general complex structure $J$ are parametrized by

$$GL(2n, \mathbb{R})/GL(n, \mathbb{C})$$

The compatibility condition implies that $J \in Sp(2n, \mathbb{R})$.

**Exercise.** Show that the space of possible positive complex structures compatible with $\Omega$ is $Sp(2n, \mathbb{R})/U(n)$. This is called the Siegel upper half space.
The $n = 1$ case

For the case $n = 1$, the geometry of the space $SL(2, \mathbb{R})/U(1)$ is best understood in terms of the geometry of $\mathbb{CP}^1$, the space of complex lines in $\mathbb{C}^2$. This is also the best way to understand the holomorphic line bundles on $SL(2, \mathbb{R})/U(1)$ and how representations of $SL(2, \mathbb{C})$ and its subgroups can be constructed geometrically (see Appendix A).

$SL(2, \mathbb{C})$ acts linearly on $\mathbb{C}^2$ and transitively on the the space $\mathbb{CP}^1$. The space $\mathbb{CP}^1$ is a complex manifold, the Riemann version of the sphere $S^2$, and the action of $SL(2, \mathbb{C})$ is holomorphic and thus an action by conformal transformations. One can choose the coordinate of the line in $\mathbb{C}^2$ generated by

\[
\left( \frac{z_1}{z_2} \right)
\]

to be $z = z_1/z_2$. This gives a good coordinate system away from one point, that of the line generated by $z_1 = 1, z_2 = 0$.

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \in SL(2, \mathbb{C})
\]

acts on this coordinate by the fractional linear transformation

\[
z \to \frac{\alpha z + \beta}{\gamma z + \delta}
\]

The subgroup $SL(2, \mathbb{R})$ of real matrices acts in the coordinate $z$ preserving the sign of $\text{Im } z$ and so does not act transitively. There are three orbits of the action: the upper and lower open half planes, and the real line. On $\mathbb{CP}^1$, the three orbits are two open hemispheres and the equator separating them. The correspondence of the three orbits in the $z$ coordinate with the three orbits on $\mathbb{CP}^1$ is that the point where $z$ is not a good coordinate is on the equator orbit, and approached as one goes off to infinity in any direction in the $z$-plane.

Picking the point $z = i$ in the upper half plane, the subgroup of elements of $SL(2, \mathbb{R})$ of elements stabilizing the point is the an $SO(2) = U(1)$ subgroup given by elements of the form

\[
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]

We can identify the upper half plane (which we’ll denote $\mathcal{H}$) with $SL(2, \mathbb{R})/U(1)$.

The Cayley transform

\[
z \to z' = \frac{z - i}{z + i}
\]

takes the upper half plane to the unit disk. Conjugating an element of $SL(2, \mathbb{R})$ by this transformation gives a matrix of the form

\[
\begin{pmatrix}
\alpha & \beta \\
\beta & \alpha
\end{pmatrix}
\]

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where $\alpha, \beta$ are complex numbers satisfying $|\alpha|^2 - |\beta|^2 = 1$. Such matrices give the subgroup $SU(1, 1)$ of $SL(2, \mathbb{C})$ preserving a $(1, 1)$ signature Hermitian form. It is isomorphic to $SL(2, \mathbb{R})$ by the conjugation map. At each point in the open unit disk, $SU(1, 1)$ acts with stabilizer a $U(1)$ subgroup. The Cayley transform takes $z = i$ to $z' = 0$, which is stabilized by elements of the form

$$
\begin{pmatrix}
    e^{i\theta} & 0 \\
    0 & e^{-i\theta}
\end{pmatrix}
$$

The subgroup of such elements acts by rotation of the unit disk about its center.

In this $n = 1$ case, changing complex polarization corresponds to changing the linear combinations of $Q$ and $P$ that define annihilation and creation operators. One gets an analog of the Bargmann-Fock construction for any $\tau \in \mathbb{C}$ with positive imaginary part by changing

$$
a = \frac{1}{\sqrt{2}}(Q - iP) \rightarrow a_\tau = \frac{1}{\sqrt{2} \sqrt{\text{Im} \tau}}(Q - \frac{1}{\tau}P)$$

$$
a^\dagger = \frac{1}{\sqrt{2}}(Q + iP) \rightarrow a^\dagger_\tau = \frac{1}{\sqrt{2} \sqrt{\text{Im} \tau}}(Q - \frac{1}{\tau}P)
$$

$a_\tau$ and $a^\dagger_\tau$ are adjoint operators satisfying the commutation relation

$$[a_\tau, a^\dagger_\tau] = 1$$

and the representation is constructed by starting with a distinguished vector annihilated by $a_\tau$ and generating the rest of the representation by applying powers of $a^\dagger_\tau$.

The unitary transformation to the Schrödinger representation will then take the distinguished vector to a solution of

$$a_\tau \psi(q) = \frac{1}{\sqrt{2} \sqrt{\text{Im} \tau}}(Q - \frac{1}{\tau}P)\psi(q) = \frac{1}{\sqrt{2} \sqrt{\text{Im} \tau}}(Q + \frac{i}{\tau} \frac{d}{dq})\psi(q) = 0$$

Solutions will be proportional to

$$\psi(q) = e^{\frac{i}{2} \tau q^2}$$

and normalizable for $\text{Im} \tau > 0$.

To visualize the entire space of possible choices of polarization that give constructions of the oscillator representation for $n = 1$, one should think of the unit disk, with interior points corresponding to complex polarizations and the Bargmann-Fock construction for different $\tau$ given above. As one approaches the boundary, the distinguished vectors annihilated by $a_\tau$ become non-normalizable and leave the space $L^2(\mathbb{R})$ (they will still be distributions in $S'(\mathbb{R})$).

For more details and to see how this picture generalizes to $n \geq 1$, see Graeme Segal’s notes on *Symplectic manifolds and quantization* [15].

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4.4 Representations and holomorphic line bundles

While the oscillator representation is essentially unique, any construction of the representation requires specification of an additional structure. For complex polarizations, this additional structure is a complex subspace

$$W \subset M \otimes \mathbb{C} \subset \mathfrak{h}_{2n+1} \otimes \mathbb{C}$$

There are corresponding operators on the oscillator representation, the annihilation operators, and a distinguished vector

$$|0\rangle_{\tau} \subset \mathcal{H}$$

annihilated by these operators. Here the notation reflects that in the physical interpretation in which annihilation and creation operators annihilate and create quanta, this is the state with zero quanta. The subscript $\tau$ in general labels points in the Siegel upper half space. In the $n=1$ case, $\tau$ is a complex number with positive imaginary part, and in the Schrödinger representation one has explicitly

$$|0\rangle_{\tau} = e^{i\tau q^2}$$

More precisely, what the choice of $\tau$ picks out is the one-dimensional complex line in $\mathcal{H}$ generated by $|0\rangle_{\tau} \subset \mathcal{H}$. The space of these complex lines gives a complex line bundle $\mathcal{L}$ over $Sp(2n, \mathbb{R})/U(n)$. This is not quite an $Sp(2n, \mathbb{R})$ equivariant line bundle (i.e. with an action of $Sp(2n, \mathbb{R})$ on $\mathcal{L}$ that projects to the action by left multiplication on the base $Sp(2n, \mathbb{R})$), since the only the double cover $Mp(2n, \mathbb{R})$ of $Sp(2n, \mathbb{R})$ acts on $\mathcal{H}$. $\mathcal{L}$ is a $Mp(2n, \mathbb{R})$ equivariant bundle over the Siegel upper half space, described as $Mp(2n, \mathbb{R})/\widehat{U(n)}$ where $\widehat{U(n)}$ is a double-cover of $U(n)$.

In the case $n=1$, recall that $SL(2, \mathbb{C})$ acts on $\mathbb{CP}^1$, with a subgroup $SU(2)$ acting transitively, identifying $SU(2)/U(1) = \mathbb{CP}^1$. Two ways to form $SU(2)$ equivariant line bundles over $\mathbb{CP}^1$ are

- Consider the product

$$SU(2) \times \mathbb{C}$$

and quotient by the action

$$(g, w) \rightarrow (gh_\theta, e^{ik\theta} w)$$

of $U(1)$, where

$$h_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

This will give a line bundle we’ll call $L^k$, with sections

$$\Gamma(L^k) = \{ \phi : SU(2) \rightarrow \mathbb{C}, \phi(gh_\theta) = e^{ik\theta}\phi(g) \}$$
Since a point in $\mathbb{C}P^1$ is a complex line, one tautologically gets a line bundle (the fiber above a point is the point).

One can show that the tautological line bundle is isomorphic with $L^{-1}$, which is the dual line bundle to $L = L^1$. The bundle of holomorphic one-forms $\mathbb{C}P^1$ is the line bundle $L^2$, so in some sense $L$ is the bundle of $1/2$-forms (the spinors in this two-dimensional geometry).

The $L^k$ for all $k \in \mathbb{Z}$ can be thought of as tensor powers of $L$ or $L^{-1}$. It turns out that these are holomorphic line bundles and one can consider their holomorphic sections

$$\Gamma_{\text{hol}}(L^k)$$

(in the algebraic geometer’s notation, this is $H^0(\mathbb{C}P^1, \mathcal{O}(k))$). The action of $SU(2)$ takes holomorphic sections to holomorphic sections and one finds (this is a simple example of the Borel-Weil theorem) that

$$\Gamma_{\text{hol}}(L^k) = \begin{cases} V^k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

where $V^k$ is the irreducible representation of $SU(2)$ of dimension $k + 1$ (in physicist’s language, the spin $\frac{k}{2}$ representation).

For the subgroup $SU(1,1) \subset SL(2,\mathbb{C})$, the story is quite different, since the action of $SU(1,1)$ on $\mathbb{C}P^1$ is not transitive. Instead there are three orbits: two hemispheres and the equator between them. On a hemisphere $D$, one can use the same definition of the line bundle $L^k$ as a quotient given above (replacing $SU(2)$ by $U(1)$) and get an irreducible representation of $SU(1,1)$ on $\Gamma_{\text{hol}}(L^k)$ but this space of sections is now infinite dimensional. This representation will be the discrete series representation $D^+_k$.

The even irreducible component of the oscillator representation can be realized as holomorphic sections of the line bundle $\mathcal{L}$, and one can show that $\mathcal{L} \otimes \mathcal{L} = L$. $L$ is a square root of $L$, and a fourth-root of the holomorphic one-forms.

On the subset $D \subset \mathbb{C}P^1$ the line bundle $L$ is the trivial bundle $D \times \mathbb{C}$, so one can choose coordinates on $D$ and work with the first description of $L$ given above, in which sections are holomorphic functions on $D$. $L$ is an equivariant bundle under the action of $SL(2,\mathbb{R})$ and one wants to choose coordinates that transform simply under $SL(2,\mathbb{R})$.

To add some day: explicit description in coordinates
Bibliography


