Quantum Field Theory for Mathematicians
Spring 2024 Course Notes
Under Construction

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Chapter 1

Introduction

These notes are a work in progress, course notes for a spring 2024 "Topics in Representation Theory" course oriented towards explaining quantum mechanics, quantum field theory, and the Standard Model to mathematicians, emphasizing the relations to representation theory. A sizable part of the early version of these notes is an extract from notes on material covered in a spring 2023 graduate course on Lie groups and representations at Columbia University. For the full version of those notes, see `https://www.math.columbia.edu/~woit/LieGroups-2023/qmnumbertheory.pdf`.

Chapter 2

Classical Mechanics

The classical mechanics description of a physical system involves an "equation of motion", a differential equation which determines the state of the system at later times given its state at some initial time. There are two quite different formalisms used for this purpose, the Hamiltonian and Lagrangian. In this chapter we'll outline the Hamiltonian version, which is closely related to Lie algebras, and then discuss the Lagrangian version.

2.1 Hamiltonian mechanics

In the Hamiltonian formalism, the state of a physical system at a given time is determined by a point in a space called "phase space". The equation of motion is a first order equation in time determined by a function on phase space called the Hamiltonian. One can also think of phase space as the space of solutions of the equation of motion.

In the cases we are most interested in, phase space is an even dimensional vector space $P = \mathbf{R}^{2n}$, with coordinates q_j, p_j for $j = 1, 2, \cdots, n$. Then one can define:

Definition (Poisson bracket). *The Poisson bracket of two functions f_1, f_2 on P is the function*

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f_1}{\partial q_j} \frac{\partial f_2}{\partial p_j} - \frac{\partial f_2}{\partial q_j} \frac{\partial f_1}{\partial p_j} \right)$$

Given a Hamiltonian function h on P, the time dependence of any function f on P will satisfy

$$\frac{df}{dt} = \{f, h\}$$

In particular, for coordinate functions, one gets Hamilton's equations

$$\dot{q}_j = \{q_j, h\} = \frac{\partial h}{\partial p_j}$$

$$\dot{p}_j = \{p_j, h\} = -\frac{\partial h}{\partial q_j}$$

These are the equations of motion in Hamiltonian form. For $h = \frac{1}{2m}|\mathbf{p}|^2 + V(\mathbf{q})$ these give the elementary physics definition of the momentum

$$\mathbf{p} = m\dot{\mathbf{q}}$$

and Newton's second law

$$\dot{\mathbf{p}} = m\ddot{\mathbf{q}} = -\nabla V$$

The Poisson bracket can easily be seen to satisfy the following properties:

- Antisymmetry:
$$\{f_1, f_2\} = -\{f_2, f_1\}$$

- Jacobi identity:
$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$$

- Leibniz rule (derivation property)
$$\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2\{f_1, f_3\}$$

The first two properties imply that the Poisson bracket provides a Lie algebra structure on the space of functions on P. This is an infinite-dimensional Lie algebra.

We'll mainly be interested in the case where P is a linear space, but the whole formalism works equally well for a manifold of the following kind:

Definition (Symplectic manifold)**.** *A symplectic manifold P is a manifold with a two-form $\omega \in \Omega^2(P)$ such that:*

- *ω is non-degenerate. At each $p \in P$ it gives an isomorphism between tangent vectors and cotangent vectors.*

- *ω is closed: $d\omega = 0$*

For the case of $P = \mathbf{R}^{2n}$ and its standard Poisson bracket, one has

$$\omega = \sum_{j=1}^{n}(dq_j \otimes dp_j - dp_j \otimes dq_j) = \sum_{j=1}^{n} dq_j \wedge dp_j$$

Note that, up to a change of coordinates, this is the unique antisymmetric non-degenerate bilinear form on $\mathbf{R}^{2n}$. This is very much analogous to the case of Riemannian geometry, where instead the inner product provides a non-degenerate symmetric bilinear form, and this analogy will play an important role later.

A simple non-linear example of a symplectic manifold is given by $P = S^2$, with ω the area two-form. A large class of examples is given by cotangent bundles $P = T^*M$ of manifolds M, with $\omega = d\theta$ where θ is the canonical one-form on M.

On a symplectic manifold the non-degeneracy condition allows one to associate to a function f a vector field X_f by

$$df = \omega(X_f, \cdot) = i_{X_f}\omega$$

This is a "symplectic gradient", an analog of the usual gradient for a Riemannian manifold, which associates a vector to field to f by using the metric to identify the one-form df with a vector field. Not all vector fields are of the form X_f, those that are are called "Hamiltonian vector fields".

The Poisson bracket can then be defined by

$$\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2})$$

Writing out explicitly the conditon that the three form $d\omega = 0$, one gets the Jacobi identity for the Poisson bracket, and thus a Lie algebra structure on the functions on P. The map

$$f \to X_f$$

is a Lie algebra homomorphism from this Lie algebra of functions to the Lie algebra of vector fields on P.

These are infinite dimensional Lie algebras, which one can locally exponentiate to get a group law (actually a "pseudogroup"). Such a group action preserves ω since the Lie derivative satisfies

$$L_{X_f}\omega = (di_{X_f} + i_{X_f}d)\omega = di_{X_f}\omega = d\omega(X_f, \cdot) = ddf = 0 \tag{2.1}$$

This (pseudo)-group preserving ω is a sub (pseudo)-group of the group of diffeomorphisms of P (the "symplectomorphisms" to mathematicians, "canonical transformations" to physicists).

By 2.1, for any vector field X preserving ω one has $di_X\omega = 0$. When $H^1(M) = 0$ the vector field X will be a Hamiltonian vector field X_f for a function f determined by $i_X\omega = df$. This function f is determined only up to a constant (for P connected).

In the physicist's language the Hamiltonian function h "generates" an action of the Lie group $\mathbf{R}$ on P given by the vector field X_h. This Lie group $\mathbf{R}$ is the group of time translations acting on the physical system. Whenever one has an action of a Lie group G on P that preserves ω, differentiating this gives a Hamiltonian vector field X_L for each $L \in \mathfrak{g}$, the Lie algebra of G. Thus, when $H^1(M) = 0$, for each $L \in \mathfrak{g}$ one can find (ambiguous up to a constant) a function f_L that generates the action infinitesimally given by the action of L. It turns out that when $H^2(\mathfrak{g}) = 0$ (Lie algebra cohomology), the constants can be chosen so that the map $L \to f_L$ is a Lie algebra isomorphism between $\mathfrak{g}$ and a sub-Lie algebra of the Lie algebra of functions on P. This map is known as the "moment map".

When a function f on P Poisson-commutes with the Hamiltonian ($\{f, h\} = 0$), then $\frac{df}{dt} = 0$ and the function f is a constant along the physical trajectories of time evolution generated by the Hamiltonian h. In such a case f is said to be a "conserved quantity". When we have an action of a Lie group G on P preserving ω that commutes with the action of the group $\mathbf{R}$ of time translations, the functions f_L for each $L \in \mathfrak{g}$ will be conserved quantities. This is how conservation laws corresponding to symmetries come about in the Hamiltonian formalism.

Some important examples:

- For a particle in 3 dimensions, $P = \mathbf{R}^6$ with the usual Poisson bracket. There is a Hamiltonian action of $\mathbf{R}^3$ by translation in the q coordinates, generated by the p coordinates. When the hamiltonian h is independent of a coordinate q_j, the corresponding p_j is a conserved quantity: the momentum in the j direction.

- In the same case there is a Hamiltonian action of $SO(3)$, by simultaneous rotation of the $\mathbf{q}$ and $\mathbf{p}$. The functions that generate rotations about the axes are

$$l_1 = q_2p_3 - p_2q_3, \quad l_2 = q_3p_1 - p_3q_1, \quad l_3 = q_1p_2 - p_1q_2$$

  These are the components of the angular momentum. When the Hamiltonian is invariant under rotations about the j-axis, l_j is a conserved quantity.

For any LIe algebra $\mathfrak{g}$, one can take$P = \mathfrak{g}^*$. Lie algebra elements $X, Y \in \mathfrak{g}$ are linear functions on P. On these linear functions the Lie bracket is a Poisson bracket

$$\{X, Y\} = [X, Y]$$

and this can be extended using the derivation property to a Poisson bracket on $S^*(\mathfrak{g})$, the polynomials on $\mathfrak{g}^* = P$. P is not a symplectic manifold, since this construction does not give a non-degenerate two-form (instead, it's a "Poisson manifold). A Lie group G acts on its LIe algebra $\mathfrak{g}$ by the adjoint action, and there is a corresponding co-adjoint action on $\mathfrak{g}^*$. On the orbits of the co-adjoint action, one does have a non-degenerate symplectic form, and these orbits are symplectic manifolds. The example of S^2 mentioned above is the case of $G = SO(3)$, where the co-adjoint orbits are spheres in $\mathbf{R}^3 = \mathfrak{so}(3)^*$.

2.2 Lagrangian mechanics

In the Lagrangian formalism, instead of a phase space $P = \mathbf{R}^{2n}$ of positions q_j and momenta p_j, one considers just the position (or configuration) space $M = \mathbf{R}^n$. Instead of a Hamiltonian function h on P, one has a functional $S[\gamma]$ of parametrized paths γ in M called the "action". The action is defined by integrating a function of position and velocity called the Lagrangian:

Definition (Action)**.** *The action S for a path γ is*

$$S[\gamma] = \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t))dt$$

Here the path is parametrized by $t \in [t_1, t_2]$ and the Lagrangian L is a function of t that depends on the position at t and its t-derivative. More generally, one can formulate this for configuration space a manifold M, with $L(t)$ depending on the velocity vector, which takes values in the tangent space of M.

The fundamental principle of classical mechanics in the Lagrangian formalism is that classical trajectories are given by critical points of the action functional.

Definition (Critical point for S). *A path γ is a critical point of the functional $S[\gamma]$ if*

$$\delta S(\gamma) \equiv \frac{d}{ds} S(\gamma_s)_{|s=0} = 0$$

where

$$\gamma_s : [t_1, t_2] \to \mathbf{R}^n$$

is a smooth family of paths parametrized by an interval $s \in (-\epsilon, \epsilon)$, with $\gamma_0 = \gamma$.

Critical points will be given by solutions to the Euler-Lagrange equations, which will be the equations of motion for the system:

Theorem (Euler-Lagrange equations). *One has*

$$\delta S[\gamma] = 0$$

for all variations of γ with endpoints $\gamma(t_1)$ and $\gamma(t_2)$ fixed if

$$\frac{\partial L}{\partial q_j}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\right) = 0$$

for $j = 1, \cdots, d$. These are called the Euler-Lagrange equations.

Proof. Ignoring analytical details, the Euler-Lagrange equations follow from the following calculations, which we'll just do for $n = 1$, with the generalization to higher d straightforward. We are calculating the first-order change in S due to an infinitesimal change $\delta\gamma = (\delta q(t), \delta\dot{q}(t))$

$$\begin{aligned}
\delta S[\gamma] &= \int_{t_1}^{t_2} \delta L(q(t), \dot{q}(t))dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}(q(t), \dot{q}(t))\delta q(t) + \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t))\delta\dot{q}(t)\right) dt
\end{aligned}$$

But

$$\delta\dot{q}(t) = \frac{d}{dt}\delta q(t)$$

and, using integration by parts

$$\frac{\partial L}{\partial \dot{q}}\delta\dot{q}(t) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q$$

so

$$\delta S[\gamma] = \int_{t_1}^{t_2} \left(\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \right) dt$$
$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q dt - \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)(t_2) + \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)(t_1) \qquad (2.2)$$

If we keep the endpoints fixed so $\delta q(t_1) = \delta q(t_2) = 0$, then for solutions to

$$\frac{\partial L}{\partial q}(q(t), \dot{q}(t)) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) \right) = 0$$

the integral will be zero for arbitrary variations δq. $\square$

As an example, a particle moving in a potential $V(\mathbf{q})$ will be described by a Lagrangian

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m |\dot{\mathbf{q}}|^2 - V(\mathbf{q})$$

for which the Euler-Lagrange equations will be Newton's second law:

$$-\frac{\partial V}{\partial q_j} = \frac{d}{dt}(m \dot{q}_j) = m \ddot{q}_j$$

Given a Lagrangian classical mechanical system, one would like to be able to find a corresponding Hamiltonian system that will give the same equations of motion. To do this, we proceed by defining for each q_j a corresponding momentum coordinate p_j by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Then, instead of working with trajectories characterized at time t by

$$(\mathbf{q}(t), \dot{\mathbf{q}}(t)) \in \mathbf{R}^{2n}$$

we would like to instead use

$$(\mathbf{q}(t), \mathbf{p}(t)) \in \mathbf{R}^{2n}$$

where $p_j = \frac{\partial L}{\partial \dot{q}_j}$ and identify this $\mathbf{R}^{2n}$ (for example at $t = 0$) as the phase space of the conventional Hamiltonian formalism. In greater generality,

The transformation

$$(q_j, \dot{q}_k) \rightarrow \left(q_j, p_k = \frac{\partial L}{\partial \dot{q}_k} \right)$$

between position-velocity and phase space (in greater generality TM and T^*M) is known as the Legendre transform, and in good cases (for instance when L is quadratic in all the velocities) it is an isomorphism. In general though, this is not an isomorphism, with the Legendre transform often taking position-velocity space to a lower dimensional subspace of phase space. Such cases are not unusual and require a much more complicated formalism, even as classical mechanical systems (this subject is known as "constrained Hamiltonian dynamics").

Besides a phase space, for a Hamiltonian system one needs a Hamiltonian function. Choosing

$$h = \sum_{j=1}^{d} p_j \dot{q}_j - L(\mathbf{q}, \dot{\mathbf{q}})$$

will work, provided the relation

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

can be used to solve for the velocities $\dot{q}_j$ and express them in terms of the momentum variables. In that case, computing the differential of h one finds (for $d = 1$, the generalization to higher d is straightforward)

$$\begin{aligned} dh =& p d\dot{q} + \dot{q} dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial \dot{q}} d\dot{q} \\ =& \dot{q} dp - \frac{\partial L}{\partial q} dq \end{aligned}$$

So one has

$$\frac{\partial h}{\partial p} = \dot{q}, \quad \frac{\partial h}{\partial q} = -\frac{\partial L}{\partial q}$$

but these are precisely Hamilton's equations since the Euler-Lagrange equations imply

$$\frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \dot{p}$$

While the Legendre transform method given above works in some situations, more generally and more abstractly, one can pass from the Lagrangian to the Hamiltonian formalism by taking as phase space the space of solutions of the Euler-Lagrange equations. This is sometimes called the "covariant phase space", and it can often concretely be realized by fixing a time $t = 0$ and parametrizing solutions by their initial conditions at such a $t = 0$. Only for a special class of Lagrangians though will one get a non-degenerate Poisson bracket on a linear phase space and recover the usual properties of the standard Hamiltonian formalism, for greater generality one needs a more complicated formalism to recover the desired features of the Hamiltonian formalism.

2.2.1 Noether's theorem and symmetries in the Lagrangian formalism

The derivation of the Euler-Lagrange equations given above can also be used to study the implications of Lie group symmetries of a Lagrangian system. When a Lie group G acts on the space of paths, preserving the action S, it will take classical trajectories to classical trajectories, so we have a Lie group action on the space of solutions to the equations of motion (the Euler-Lagrange equations). On this space of solutions, we have, from equation 2.2 (generalized to multiple coordinate variables),

$$\delta S[\gamma] = \left(\sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j}\delta q_j(X)\right)(t_1) - \left(\sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j}\delta q_j(X)\right)(t_2)$$

where now $\delta q_j(X)$ is the infinitesimal change in a classical trajectory coming from the infinitesimal group action by an element X in the Lie algebra of G. From invariance of the action S under G we must have $\delta S{=}0$, so

$$\left(\sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j}\delta q_j(X)\right)(t_2) = \left(\sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j}\delta q_j(X)\right)(t_1)$$

This is an example of a more general result known as "Noether's theorem". In this context it says that given a Lie group action on a Lagrangian system that leaves the action invariant, for each element X of the Lie algebra we will have a conserved quantity

$$\sum_{j=1}^{d} \frac{\partial L}{\partial \dot{q}_j}\delta q_j(X)$$

which is independent of time along the trajectory.

When the Lagrangian L is translation invariant (depends on $\dot{q}$, not q), one recovers by the Noether method the definition of momentum and its conservation law. When L is rotation invariant, one gets angular momentum and its conservation.

The Lagrangian formalism has the advantage that the dynamics depends only on the choice of action functional on the space of possible trajectories, and it can be straightforwardly generalized to theories where the configuration space is an infinite dimensional space of classical fields. Unlike the usual Hamiltonian formalism for such theories, the Lagrangian formalism allows one to treat space and time symmetrically. For relativistic field theories, this allows one to exploit the full set of space-time symmetries, which can mix space and time directions. In such theories, Noether's theorem provides a powerful tool for finding the conserved quantities corresponding to symmetries of the system that are due to invariance of the action under some group of transformations.

On the other hand, in the Lagrangian formalism, since Noether's theorem only considers group actions on configuration space, it does not cover the case of Hamiltonian group actions that mix position and momentum coordinates, something that occurs most notably in the case of the harmonic oscillator.

Chapter 3

Introduction to Quantization

In this chapter we'll begin our discussion of quantum theory with some basic examples covered in all physics textbooks, followed by some generalities about the role of quantization in representation theory. The three examples here incorporate three important aspects of the quantum field theories we plan to study later in the course.

3.1 Canonical quantization: some examples

What physicists call "canonical quantization" can be understood in terms of the unique non-trivial representation of the Heisenberg group and Lie algebra, which will be described in detail in the next chapter. In this one, we'll motivate the later representation theory with a standard description of the basic examples of quantum systems.

The space of possible states for a quantum system is a complex vector space $\mathcal{H}$ (generally infinite-dimensional) with Hermitian inner product. For one degree of freedom this space can be taken to be the space of wavefunctions (complex-valued functions $\psi(q)$ of a position variable q) in $L^2(\mathbf{R})$. This version of the state space is called the Schrödinger representation and acting on it are powers of the self-adjoint operators

$$Q = q, \quad P = -i\hbar\frac{d}{dq}$$

which satisfy the Heisenberg commutation relations

$$[Q, P] = i\hbar\mathbf{1}$$

Here $\hbar$ is a constant which depends on one's choice of units, so later we will generally use units in which $\hbar = 1$.

The dynamics of the system is determined by specification of an operator (defined in terms of the Q, P operators), the Hamiltonian H. This operator generates translations in time, with wavefunctions evolving in time according to the Schrödinger equation

$$i\hbar \frac{d}{dt}\psi = H\psi$$

The connection between this formalism and what ones observes, measures and often interprets in a classical picture of the world is given by two principles:

- Self-adjoint operators like Q and P correspond to observable quantities, with eigenfunctions of such an operator states with a well-defined measurable value of the observable quantity, given by the eigenvalue.
- If one tries to measure the value of an observable quantity when the state is not an eigenfunction, the result will be one of the eigenvalues, with probability given by the norm-squared of the inner product between the (normalized) state and eigenfunction with that eigenvalue (this is called the "Born rule).

For a single quantum particle moving in one dimension, subject to a potential $V(q)$, the Hamiltonian is

$$H = \frac{1}{2m}P^2 + V(Q) = -\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + V(q)$$

One would like to find the eigenfunctions and eigenvalues of this operator, i.e. find $E, \psi_E(q)$ such that

$$(-\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + V(q))\psi_E(q) = E\psi_E(q)$$

and then expand wavefunctions at an initial time $t = 0$ in terms of the energy eigenfunctions $\psi_E(q)$. The Schrödinger equation implies that these evolve in time as

$$\psi_E(q)e^{-\frac{i}{\hbar}Et}$$

For much more detail about the following basic examples, see any physics textbook on quantum mechanics, or [18].

3.1.1 The free particle

The case of the free particle is the case $V(q) = 0$. Using Fourier analysis, one finds that the energy eigenvalues and eigenfunctions are parametrized by $p \in \mathbf{R}$ and are given by

$$E_p = \frac{p^2}{2m}, \quad \psi_{E_p}(q) = e^{i\frac{p}{\hbar}q}$$

The spectrum of the Hamiltonian is continuous, all non-negative values in $\mathbf{R}$.

The eigenfunctions of H are also eigenfunctions of the momentum operator P with eigenvalue p. P commutes with H, so if one prepares a state at time 0 with wavefunction $\psi_{E_p}(q)$ and measures its momentum at any later times, one will always get the value p (the momentum is a conserved quantity). Just as H is the generator of time-translations on states, P is the generator of spatial translations.

The eigenfunctions of the operator Q are delta-functions $\delta(q - q')$, with eigenvalue $q' \in \mathbf{R}$. Unlike the case for momentum P, one has $[Q, H] \neq 0$ and these are not energy eigenfunctions. If one prepares a state at time 0 with wavefunction $\delta(q - q')$, so localized at $q = q'$, it will immediately evolve into a linear combination of states with all possible eigenvalues of Q. Measurement of position at later times t may give all possible different values.

Note that the eigenfunctions of Q and P are not functions in $L^2(\mathbf{R})$ and in addition, the operators Q and P don't preserve $L^2(\mathbf{R})$ (multiplying or differentiating by q can take a function that is square-integrable to one that isn't). To deal with these problems simultaneously, one can define the Schwartz space $\mathcal{S}(\mathbf{R})$ of functions such that the function and its derivatives fall off faster than any power at $\pm\infty$. The dual space $\mathcal{S}'(\mathbf{R})$ of continuous linear functionals on $\mathcal{S}(\mathbf{R})$ is called the space of tempered distributions, and includes the eigenfunctions of Q and P. One has the sequence of dense inclusions

$$S(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$$

The Fourier transform takes each term in this sequence to itself.

A problem here is that elements of $\mathcal{S}'(\mathbf{R})$ like the eigenfunctions of Q and P are not in $L^2(\mathbf{R})$. They do not have well-defined norms, so will not be vectors in a unitary representation and the Born rule can't be used for them. However, they are linear functionals on $S(\mathbf{R})$ and one can use this to play the role of their inner products with elements of $S(\mathbf{R})$.

To get a well-defined formalism one has two options:

- Work with states $\psi \in L^2(\mathbf{R})$, taking great care with domains and ranges of operators like P, Q and H that are applied to states. In this case, eigenfunctions of these operators are not in the state space.

- Work with the space $S'(\mathbf{R})$ and distributional states, but be careful to properly pair these only with physical states in $\mathcal{S}(\mathbf{R})$ (sometimes called "wavepackets").

3.1.2 The harmonic oscillator

The quantum harmonic oscillator is the case of a particle moving in a quadratic potential $V(q) = \frac{1}{2}m\omega^2 q^2$

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2$$

The energy eigenvalues and eigenfunctions are given by

$$E_n = \hbar\omega(n + \frac{1}{2}), \quad \psi_n(q) = H_n\left(\sqrt{\frac{m\omega}{\hbar}}q\right)e^{-\frac{m\omega}{2\hbar}q^2}$$

where $n = 0, 1, 2, \ldots$ and $H_n(q)$ are Hermite polynomials. In this case the spectrum of the operator H is discrete, energy eigenfunctions are in $L^2(\mathbf{R})$, and arbitrary $t = 0$ wavefunctions in $L^2(\mathbf{R})$ can be written as linear combinations of the $\psi_{E_n}(q)$.

The easiest way to get these results is to work not with Q and P, but with complex linear combinations of these. For simplicity, rescaling so that $\hbar = m = \omega = 1$, one can choose

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{1}{\sqrt{2}}(q + \frac{d}{dq}), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})$$

$a, a^\dagger$ are each others adjoints and satisfy the commutation relation

$$[a, a^\dagger] = \mathbf{1}$$

The Hamiltonian is

$$H = \frac{1}{2}(Q^2 + P^2) = \frac{1}{2}(a^\dagger a + aa^\dagger) = a^\dagger a + \frac{1}{2}$$

One can easily see (using $[H, a^\dagger] = a^\dagger$ and $[H, a] = -a$) that $a^\dagger$ increases the eigenvalue of H by 1, a reduces it by 1. To have a spectrum bounded below, one needs a non-zero state $\psi_0(q)$ satisfying

$$a\psi_0(q) = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})\psi_0(q) = 0$$

This state will have energy $\frac{1}{2}$ and by given by

$$\psi_0(q) = e^{-\frac{1}{2}q^2}$$

The other energy eigenstates will have energy $n + \frac{1}{2}$ for $n = 1, 2, \cdots$ and can be found explicitly by applying the operator $a^\dagger$ n-times to $\psi_0(q)$, so evaluating

$$(q - \frac{d}{dq})^n e^{-\frac{1}{2}q^2}$$

Note that for the harmonic oscillator, $V(q)$ is not translation invariant, and one has $[P, H] \neq 0$ as well as $[Q, H] \neq 0$ so neither position nor momentum are conserved quantities.

For more general potentials one can have both discrete (with eigenfunctions in $L^2(\mathbf{R})$) and continuous (with eigenfunctions not in $L^2(\mathbf{R})$) components of the spectrum. The physical interpretation will involve both "bound states" which correspond to particles localized in some regions of $\mathbf{R}$ and "scattering states" which correspond to particles with possible positions extending to $+\infty$ or $-\infty$.

3.2 The spin $\frac{1}{2}$ quantum system

A very simple and very important example of a quantum system is the spin $\frac{1}{2}$ system that describes a highly non-classical degree of freedom shared by all matter particles. Unlike the previous two examples, this one is not in any sense a quantization of a classical Hamiltonian system with phase space $\mathbf{R}^{2n}$. This system is characterized by

- The state space is $\mathcal{H} = \mathbf{C}^2$.
- The operators corresponding to observables (including the Hamiltonian operator H) are the self-adjoint operators on $\mathbf{C}^2$, so (real) linear combinations of

$$\mathbf{1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  Here the σ_j are the Pauli matrices, the physicist's convention for a basis of self-adjoint two by two matrices.

This same system describes any quantum system with $\mathbf{C}^2$, for which H can be an arbitrary self-adjoint two by two matrix. The solution to the Schrödinger equation will be given by

$$e^{-iHt}\psi(0)$$

where $\psi(0) \in \mathbf{C}^2$ is the state at $t = 0$. Here e^{-iHt} will be a unitary matrix, so an element of the group $U(2)$.

This system is described as "spin $\frac{1}{2}$" since it is the spinor representation of the group $Spin(3) = SU(2)$, the double cover of the rotation group $SO(3)$. It thus describes a degree of freedom which transforms non-trivially under rotations of space. If one normalizes the observable operators by

$$S_j = \frac{1}{2}\sigma_j$$

then $e^{i\theta S_j}$ will give the behavior of a state under a rotation by an angle θ about the j-axis. These are called "spin" operators and have eigenvalues $\pm\frac{1}{2}$, which by the principles connecting quantum theory to observation should describe the two possible values one can observe for the spin observable. The subtleties of this become apparent once one notes that the different S_j don't commute so can't be simultaneously diagonalized. Unlike in classical mechanics where a system at a given time has three well-defined components of its angular momentum, here something very different is going on.

There are two different ways to think of this system as the "quantization" of something:

- Take phase space to be $P = S^2$, a co-adjoint orbit in $\mathbf{R}^3 = \mathfrak{su}(2)^*$, and develop a theory of how to "quantize" such symplectic manifolds.
- Take phase space to be $\mathbf{R}^3$ but using "anti-commuting" variables, a subject we will develop later.

In either case, it is as a quantum system that there is a very simple description, with any possible quantum analog something much more complicated to describe.

3.3 Quantization and representation theory

3.3.1 Dirac quantization as a Lie algebra representation

In the previous chapter we saw that the polynomial functions on phase space $P = \mathbf{R}^{2n}$ form a Lie algebra, with Lie bracket the Poisson bracket. Very soon after Heisenberg's 1925 development of quantum theory based upon non-commuting operators corresponding to position and momentum, Dirac proposed a general rule for such operators. If O_f is the quantum operator corresponding to the classical phase space function f, then he proposed that

$$O_{\{f,g\}} = -\frac{i}{\hbar}[O_f, O_g] \tag{3.1}$$

generalizing the Heisenberg commutation relations for operators Q_j, P_j corresponding to coordinates q_j, p_j. In the language of Lie algebras and representations, this proposal was that quantization is a unitary representation on the state space $\mathcal{H}$ of the infinite dimensional Lie algebra of functions on phase space. The passage from classical to quantum is nothing but the passage from a Lie algebra to one of its representations.

Recall that a complex representation π' of a Lie algebra L is a Lie algebra homomorphism

$$\pi' : L \to End(V)$$

Here $End(V)$ is the Lie algebra of linear operators on V, with Lie bracket the commutator. The Lie algebra homomorphism condition is that π' preserves Lie brackets:

$$\pi'([X,Y]) = [\pi'(X), \pi'(Y)]$$

Such a representation will be unitary when there is a Hermitian form on V and the $\pi'(X)$ are skew-adjoint operators ($\pi'(X)^\dagger = -\pi'(X)$).

Such a Lie algebra representation may come from a representation π of a group G with Lie algebra $L = Lie(G)$ (in which case it is called "integrable"). Then π is a group homomorphism

$$\pi : G \to GL(V)$$

from the group G to the group of invertible linear operators on V. G and $GL(V)$ are smooth manifolds, and π' will be the derivative of π, evaluated at the identity. When the representation is unitary π takes values in the group $U(V)$ of unitary transformations.

The Dirac quantization rule (setting $\hbar = 1$) says that

$$f \to \pi'(f) = -i\hbar O_f$$

is a Lie algebra homomorphism, since the homomorphism property is

$$\pi'(\{f,g\}) = -i\hbar O_{\{f,g\}} = [\pi'(f), \pi'(g)] = [-i\hbar O_f, -i\hbar O_g] = -\hbar^2[O_f, O_g]$$

which is Dirac's 3.1. Note that the operators O_f favored by physicists are self-adjoint (so have real eigenvalues), while the $\pi'(f) = -i\hbar O_f$ are skew-adjoint.

It turns out that Dirac's proposal is flawed. In the next chapter we will see that there is a representation π' which has the right properties for polynomials of degree up to two (so, for $n = 1$, the Lie subalgebra with basis $1, q, p, q^2, p^2, qp$), but this cannot be extended consistently to higher order polynomials. This is a theorem (called the Groenewold-van Hove no-go theorem) and well-known to physicists in the form of the existence of "operator-ordering ambiguities" occurring when one tries to implement Dirac's proposal.

3.3.2 Some generalities about quantization and representation theory

We will study in the next chapter "canonical quantization" which is the general case of a representation of the Lie algebra of polynomials of degree up to two on $\mathbf{R}^{2n}$. More generally, if one starts with a general classical Hamiltonian system with P a general symplectic manifold, one will still have a Poisson bracket and can ask for a more general notion of quantization that gives a state space with operators satisfying commutation relations corresponding to the Poisson bracket relations. The subject of "geometric quantization" attempts to provide such generalization, but turns out to have limited applicability, especially in not being able to provide the full range of observable operators one would like. Mathematicians studying the representation theory of Lie groups and Lie algebras draw inspiration from quantum systems studied by physicists. These often are "quantizations" of some classic system, potentially providing an example of a new way to construct representations.

A special case is the one we discussed earlier of co-adjoint orbits in $P = \mathfrak{g}^*$. In this case the question of how to get operators has a compelling answer: the algebra of operators in the quantization of $\mathfrak{g}^*$ should be the universal enveloping algebra $U(\mathfrak{g})$. The problem is that one has to represent these operators on a complex vector space V, and this is precisely the general problem of representation theory for Lie algebras, that of how to classify and construct all possible representations. It is a well-known principle in this field that a fruitful way to approach this problem is the "orbit method". Here one uses the decomposition of $\mathfrak{g}^*$ into co-adjoint orbits and tries to associate to each co-adjoint orbit an irreducible representation, by "quantizing" the classical Hamiltonians system with phase space that orbit. This returns one to the problem of quantizing phase spaces P that are not linear, but for examples that have a great deal of extra structure governed by the Lie algebra $\mathfrak{g}$. The subject of "geometric quantization" has been very much motivated by efforts to solve this problem of quantizing co-adjoint orbits.

In what follows we will stick to a very special case of this general problem, using the Lie algebra of Heisenberg group, for which the co-adjoint orbits are exactly the linear phase spaces $P = \mathbf{R}^{2n}$.

For a summary of the orbit philosophy and how it mostly (but not always) leads to constructions of irreducible representations, see [7].

Chapter 4

Canonical quantization: bosons

4.1 The Heisenberg group and its representations

Quantum mechanics as we know it was born in 1925 in a series of conceptual breakthroughs which began with Heisenberg's creation of a theory involving non-commuting quantities, soon reformulated (by Max Born) in terms of position and momentum operators Q and P satisfying the commutation relations

$$[Q, P] = i\hbar\mathbf{1}$$

(now known as the Heisenberg commutation relations). We are for now considering just one degree of freedom. $\hbar$ is a constant that depends on units used to measure position and momentum. We will choose units such that $\hbar = 1$. The mathematician Hermann Weyl soon recognized these relations as those of a unitary representation of a Lie algebra now known as the Heisenberg Lie algebra, and described the corresponding Heisenberg group.

Late in 1925, Schrödinger formulated a seemingly different version of quantum mechanics, in terms of wave-functions satisfying a differential equation. What Schrödinger had found was a construction of a representation of the Heisenberg Lie algebra on the vector space of functions $\psi(q)$ of a position variable q, with Q the multiplication by q operator and P the differential operator

$$P = -i\frac{d}{dq}$$

We'll begin with the Lie algebra corresponding to the Heisenberg commutation relations, then find the group with this Lie algebra and show that Schrödinger's wave-functions give an irreducible unitary representation of the Lie algebra and group. It turns out that any irreducible unitary representation of the Heisenberg group is essentially equivalent to this one (Stone-von Neumann theorem), but the family of different ways of constructing these representations carries an intricate structure.

4.1.1 The Heisenberg Lie algebra and Lie group

The Lie algebra spanned by $1, q, p$ will be the three-dimensional Lie algebra with a basis X, Y, Z and Lie bracket relations

$$[X, Z] = [Y, Z] = 0, \quad [X, Y] = Z$$

This Lie algebra can be identified with the Lie algebra of three by three strictly upper-triangular matrices by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad , Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is called the Heisenberg Lie algebra by mathematicians, and we'll use the notation $\mathfrak{h}_3$.

A unitary representation (which we'll call π') will be given by three skew-adjoint operator $\pi'(X), \pi'(Y), \pi'(Z)$ satisfying

$$[\pi'(X), \pi'(Y)] = \pi'(Z), \quad [\pi'(X), \pi'(Z)] = 0, \quad [\pi'(Y), \pi'(Z)] = 0$$

These become the Heisenberg commutation relations if we identify

$$\pi'(X) = -iQ, \quad \pi'(Y) = -iP, \quad \pi'(Z) = -i\mathbf{1}$$

Note that factors of i are appearing here just because physicists like to work with self-adjoint operators (since their eigenvalues are real), but for unitary representations the Lie algebra representation operators are skew-adjoint.

In terms of matrices, exponentiating elements of $\mathfrak{h}_3$ as in

$$exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

gives the elements of the Heisenberg group H_3 (physicists often call this the "Weyl group", but this means something different to mathematicians). This is the group of upper triangular matrices with 1s on the diagonal. Using x, y, z as ("exponential") coordinates on the group, H_3 is the space $\mathbf{R}^3$ with multiplication law

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$$

For computations with the Heisenberg group it is often convenient to use the Baker-Campbell-Hausdorf formula, which simplifies greatly in this case since all Lie brackets except $[X, Y] = Z$ vanish. As a result, for $A, B \in \mathfrak{h}_3$ one has

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

This group is a central extension

$$0 \to (\mathbf{R}, +) \to H_3 \to (\mathbf{R}^2, +) \to 0$$

of the additive group of $\mathbf{R}^2$ by the additive group of $\mathbf{R}$ (which is the center of the group).

A slightly different version of the Heisenberg goup (which we'll call $H_{3,red}$) that is sometimes used takes a quotient by $\mathbf{Z}$ and replaces the central $\mathbf{R}$ with a central $U(1)$, so is a central extension

$$0 \to U(1) \to H_{3,red} \to (\mathbf{R}^2, +) \to 0$$

Elements are labeled by (x, y, u) where x and y are in $\mathbf{R}$ and $u \in U(1)$, and the group law is

$$(x, y, u)(x', y', u') = (x + x', y + y', uu'e^{i\frac{1}{2}(xy'-x'y)})$$

4.1.2 The Schrödinger representation

The Schrödinger representation π_S will be a representation on a vector space $\mathcal{H}$ of complex valued functions $\psi(q)$ on $\mathbf{R}$, with derivative the Lie algebra representation

$$\pi'_S(X) = -iQ = -iq, \quad \pi'_S(Y) = -iP = -\frac{d}{dq}, \quad \pi'_S(Z) = -i1$$

Exponentiating these operators gives unitary operators that generate π_S

$$\pi_S(x) = e^{-ixq}, \quad \pi_S(y) = e^{-y\frac{d}{dq}}, \quad \pi_S(z) = e^{-iz}\mathbf{1} \tag{4.1}$$

Note that $\pi_S(y)$ acts on the representation space by translation

$$\pi_S(y)\psi(q) = \psi(q - y)$$

Definition (Schrödinger representation)**.** *The Schrödinger representation of the Heisenberg group H is given by*

$$\pi_S(x, y, z)\psi(q) = e^{-iz}e^{i\frac{1}{2}xy}e^{-ixq}\psi(q - y) \tag{4.2}$$

for $(x, y, z) \in H$.

One can easily check that this is a representation, since it satisfies the homomorphism property

$$\pi_S(x, y, z)\pi_S(x', y', z') = \pi_S(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$$

Taking as representation space $\mathcal{H} = L^2(\mathbf{R})$, for the Lie algebra representation π'_S there will be domain (functions on which operators not defined) and range (operators take something in $L^2(\mathbf{R})$ to something not in $L^2(\mathbf{R})$) problems. As an alternative, one can take $\mathcal{H} = \mathcal{S}(\mathbf{R})$ so that the representation operators are well-defined (but then the dual space is something different, the tempered distributions $S'(\mathbf{R})$). For the group representation, the operators π_S are well defined on $\mathcal{H} = L^2(\mathbf{R})$. Giving up on a well-defined inner-product and unitarity, one can take $\mathcal{H} = \mathcal{S}'(\mathbf{R})$ and have both a Lie algebra and Lie group representation.

This multiplicity of closely related versions of the representation is a general phenomenon for infinite-dimensional representations of non-compact Lie groups, where one has inequivalent representations on a sequence of dense inclusions of representation spaces, here

$$\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$$

4.1.3 The Stone-von Neumann theorem

The remarkable fact about representations of the Heisenberg group is that there is essentially only one representation (once one has specified the constant by which Z acts, but non-zero choices are related by a rescaling). More specifically, any irreducible representation of H_3 will be unitarily equivalent to the Schrödinger representation. One has the following theorem

Theorem (Stone-von Neumann)**.** *For any irreducible unitary representation π of H_3 (with action of the center $\pi(0,0,z) = e^{-iz}$) on a Hilbert space $\mathcal{H}$, there is a unitary operator $U : \mathcal{H} \to L^2(\mathbf{R})$ such that*

$$U\pi U^{-1} = \pi_S$$

We will not give a proof here, since the analysis is somewhat involved, but what follows should make clear some problems that any proof needs to overcome and motivate the strategy for an actual proof.

Recall that one can define the adjoint pair of operators

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{1}{\sqrt{2}}(q + \frac{d}{dq}), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})$$

and for the harmonic oscillator Hamiltonian the lowest energy eigenspace is the one-dimensional space of solutions in $L^2(\mathbf{R})$ of

$$a\psi_0(q) = 0$$

These are all proportional to

$$\psi_0 = e^{-\frac{1}{2}q^2}$$

The rest of the state space can be generated by repeatedly applying the operator $a^\dagger$ to ψ_0.

Exercise. *Use this basis to prove that the Schrödinger representation is irreducible.*

For some motivation for why the Stone-von Neumann theorem might be true, for π' one can construct analogs of the $a, a^\dagger$

$$b = \frac{1}{2}(i\pi'(X) - \pi'(Y))$$

and its adjoint $b^\dagger$. These will satisfy $[b, b^\dagger] = 1$ and by the argument given for the harmonic oscillator state space, there should be a state $|0_b\rangle$ satisfying $b\,|0_b\rangle = 0$, which together with the $(b^\dagger)^k\,|0_b\rangle$ should give an orthonormal basis of the state space in the π' representation. There will be a unitary operator $U : \mathcal{H} \to L^2(\mathbf{R})$ taking the basis constructing using the $b, b^\dagger$ operators to the standard basis of harmonic oscillator energy eigenstates in the Schrödinger representation. A possible approach to the Stone-von Neumann theorem would be to note that

$$UbU^{-1} = a, \quad Ub^\dagger U^{-1} = a^\dagger$$

that b has a one-dimensional kernel (irreducibility), and that the rest of the representation is given by repeated applications of $b^\dagger$. The U would then give the desired unitary equivalence.

Unfortunately, this can't work, since there is no guarantee that vectors in the range of $b^\dagger$ will be in its domain, so one can't generate the representation by repeatedly applying $b^\dagger$ (it is not hard to construct examples of this using wave-functions with specific boundary conditions).. It turns out that the Stone-von Neumann theorem is not true for general Lie algebra representations of $\mathfrak{h}_3$, only works for Lie algebra representations that integrate to give a group representation. To get a proof that does work, one needs to work not with $b, b^\dagger$ and $a, a^\dagger$, but with their exponentiated versions. For details, see [**?**], chapter 14.

An important example of an irreducible representation unitarily equivalent to the Schrödinger representation is given by using the Fourier transform $\mathcal{F}$

$$\psi(q) \to \widetilde{\psi}(p) = (\mathcal{F}\psi)(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ipq}\psi(q)dq$$

This is a unitary transformation on $L^2(\mathbf{R})$, with inverse $\widetilde{\mathcal{F}}$ given by Fourier inversion

$$\widetilde{\psi}(p) \to (\widetilde{\mathcal{F}}\widetilde{\psi})(q) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ipq}\widetilde{\psi}(p)dp$$

The Stone-von Neumann theorem applies, with $U = \widetilde{\mathcal{F}}, \quad U^{-1} = \mathcal{F}$.

Note that we will generically refer to the essentially unique representation of the Heisenberg using $\mathcal{H}$ for the representation space and π for the homomorphism from the group to operators on $\mathcal{H}$, with π' for the Lie algebra representation. When we want to specify a specific construction, the π may acquire a subscript (e.g. π_S for the Schrödinger construction) and $\mathcal{H}$ may get further specified (e.g. $L^2(\mathbf{R})$). Terminology in this subject can be a bit confusing, since instead of the usual multiple representations to keep track of, here there is only one, but with multiple quite different constructions.

4.1.4 The Bargmann-Fock representation

The Stone-von Neumann theorem also applies to very different constructions of representations on other versions of Hilbert space. In particular, it is clear from looking at the harmonic oscillator calculations that energy eigenstates can be identified with monomials in a complex variable, with a and $a^\dagger$ decreasing and increasing the degree. To find a construction of the Heisenberg group irreducible representation on $\mathbf{C}[w]$, one needs a Hilbert space structure, which one can define as follows:

Definition (Fock Space)**.** *Fock space $\mathcal{H}_F$ is the space of entire functions on $\mathbf{C}$, with finite norm in the inner product*

$$\langle f(w), g(w)\rangle = \frac{1}{\pi}\int_{\mathbf{C}} \overline{f(w)}g(w)e^{-|w|^2}$$

An orthonormal basis of $\mathcal{H}_F$ is given by apropriately normalized monomials. Since

$$\begin{aligned}
\langle w^m, w^n\rangle &= \frac{1}{\pi}\int_{\mathbf{C}} \overline{w}^m w^n e^{-|w|^2} \\
&= \frac{1}{\pi}\int_0^\infty (\int_0^{2\pi} e^{i\theta(n-m)}d\theta) r^{n+m}e^{-r^2}rdr \\
&= n!\delta_{n,m}
\end{aligned}$$

we see that the functions $\frac{w^n}{\sqrt{n!}}$ are orthonormal.

To get a representation of the (complexified) Heisenberg Lie algebra on this space, define

$$a = \frac{d}{dw}, \quad a^\dagger = w$$

Exercise. *Show that these operators are each other's adjoints with respect to the inner product on Fock space.*

On the real Heisenberg Lie algebra, this representation exponentiates to a representation of the Heisenberg group. By the Stone-von Neumann theorem it is unitarily equivalent to the Schrödinger representation on $L^2(\mathbf{R})$.

To explicitly write the Bargmann-Fock representation of the Heisenberg Lie algebra, one can complexify and work with operators that depend on complex linear combinations of the real basis X, Y, Z. If one does this first in the Schrödinger representation one has

$$\pi'_S(iX) = Q, \quad \pi'_S(iY) = P, \quad \pi'_S(iZ) = \mathbf{1}$$

and so

$$\pi'_S(\frac{1}{\sqrt{2}}(iX + i(iY))) = a = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$$

(with at similar formula for $a^\dagger$). To get Bargmann-Fock one wants a π'_{BF} that takes the same linear combinations to $\frac{d}{dw}$ and w, acting on $\mathcal{H}_F$. Thus

$$\pi'_{BF}(\frac{1}{\sqrt{2}}(iX+i(iY)) = a = \frac{d}{dw},\ \ \pi'_{BF}(\frac{1}{\sqrt{2}}(iX-i(iY)) = a^\dagger = w,\ \ \pi'_{BF}(iZ) = \mathbf{1}$$

We won't work this out here, but these operators can be exponentiated to get operators for a Heisenberg Lie group representation. By Stone-von Neumann, there will be a unitary operators

$$U : \mathcal{H}_F \rightarrow L^2(\mathbf{R}),\ \ U^{-1} : L^2(\mathbf{R}) \rightarrow \mathcal{H}_F$$

These operators are quite non-trivial and interesting in analysis, giving unitary isomorphisms between two very different kinds of function spaces. The explicit form for U^{-1} is often called the Bargmann transform and is given by

$$(U^{-1}\psi)(w) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}w^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}q^2} e^{\sqrt{2}wq}\psi(q)dq$$

The relation between the Schrödinger and Bargmann-Fock operators will be given by

$$U\frac{d}{dw}U^{-1} = \frac{1}{\sqrt{2}}(q + \frac{d}{dq}),\ \ UwU^{-1} = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})$$

Note that what we have been calling the "Fock representation" is defined in terms of polynomials on a complex vector space of dimension n. Using the isomorphism between polynomials on a vector space V and the symmetric tensor product $S^*(V^*)$, one can instead define this in terms of tensor products. We will later write this out in detail, since it becomes important in quantum field theory, where one deals with the case of V infinite-dimensional.

For more on the Bargmann-Fock representation and the Bargmann transform a good source is Chapter 1, Section 6 of [3].

4.1.5 The Weyl algebra

A closely related algebra to the Heisenberg Lie algebra is the Weyl algebra, which can be defined as the non-commutative algebra of polynomial coefficient differential operators for a complex variable w. The generators of the algebra are

- Multiplication by w.
- Differentiation by w: $\frac{d}{dw}$

These satisfy the same commutation relations as $a, a^\dagger$

$$[\frac{d}{dw}, w] = 1$$

since

$$\frac{d}{dw}(wf) - w\frac{df}{dw} = f$$

Recall that one can think of representations of a Lie algebra $\mathfrak{g}$ as modules for the associative algebra $U(\mathfrak{g})$ (the universal enveloping algebra of $\mathfrak{g}$). It is convenient here also to complexify, and for any Lie algebra we'll use the notation $U(\mathfrak{g})$ to refer to $U(\mathfrak{g}) \otimes \mathbf{C} = U(\mathfrak{g} \otimes \mathbf{C})$. For the Heisenberg Lie algebra $\mathfrak{h}_3$, $U(\mathfrak{h}_3)$ is given by all complex linear combinations of products of basis elements X, Y, Z, modulo the relations

$$[X, Z] = [Y, Z] = 0, \quad [X, Y] = Z$$

The center of $U(\mathfrak{h}_3)$ (denoted here $\mathcal{Z}(\mathfrak{h}_3)$) is the commutative algebra $\mathbf{C}[Z]$ of polynomials in Z. In any irreducible representation π' of a Lie algebra $\mathfrak{g}$, by Schur's lemma elements of the center $\mathcal{Z}(\mathfrak{g})$ act by scalars. This gives a homomorphism

$$\chi_{\pi'} : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbf{C}$$

called the infinitesimal character of the representation. In the case of $\mathfrak{g} = \mathfrak{h}_3$, since $\mathcal{Z}(\mathfrak{h}_3)$ is an algebra of the polynomial functions in one variable, the infinitesimal character is evaluation of the polynomial at some $c \in \mathbf{C}$. This c is the scalar given by the action of $\pi'(Z)$ on the representation space. The Schrödinger representation as we have defined it is an irreducible representation with $c = -i$.

For general Lie algebra representations of the complexified Lie algebra $\mathfrak{h}_3 \otimes \mathbf{C}$, for each $c \neq 0$ we have the irreducible representation unitarily equivalent to the Schrödinger representation (rescaled from $c = -i$). These will be unitary for c imaginary.

Z acts by a scalar we'll call $c_{\pi'}$. Polynomials in Z also act by a scalar, the evaluation of the polynomial at $c_{\pi'}$. The Schrödinger representation as we have defined it is an irreducible representation with $c_{\pi'_S} = -i$. Restricting attention to Lie algebra representations for which $\pi'(Z) = c\mathbf{1}$ for a chosen $c \in \mathbf{C}$, these will be modules for the quotient algebra

$$U(\mathfrak{h}_3)/(Z - c)$$

By rescaling X and Y, for $c \neq 0$, we get the Weyl algebra, and so an irreducible Heisenberg algebra representation will be a module for the Weyl algebra. Among these modules is the standard one on polynomials on w, which corresponds to the one we have studying, which is integrable to a unitary Heisenberg group representation. But there are many different modules for the Weyl algebra, with the study of thes modules the beginning of the subject of D-modules in algebraic geometry (see for instance [**?**]).

4.1.6 The Heisenberg group and symplectic geometry

The three-dimensional Heisenberg group that we have been studying has a simple generalization that behaves in much the same way. For any n, define the

$2n+1$ dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ to be the Lie algebra with basis X_j, Y_j, Z ($j = 1, 2, \cdots, n$) and all Lie brackets zero except

$$[X_j, Y_k] = \delta_{jk} Z$$

One can easily easily get a corresponding Heisenberg Lie group H_{2n+1} generalizing the $n = 1$ case by exponentiating.

Instead of working with a basis like this, one can define this Lie group in a more coordinate-invariant way, starting with any symplectic form on $M = \mathbf{R}^{2n}$ (note that M corresponds to P^*, the dual of phase space, since coordinates on phase space are a basis of M), where

Definition (Symplectic form)**.** *A symplectic form Ω on a vector space M is a non-degenerate anti-symmetric bilinear form*

$$(v_1, v_2) \in M \times M \to \Omega(v_1, v_2) \in \mathbf{R}$$

on M.

This is the same definition as that of an inner product on a vector space V, with "symmetric" replaced by "antisymmetric." For any even-dimensional real vector space M with a symplectic form Ω, one can define a Lie algebra structure on $M \oplus \mathbf{R}$ by taking the LIe bracket to be

$$[(v, z), (v', z')] = (0, \Omega(v, v'))$$

where (v, z) are elements of $M \oplus \mathbf{R}$. One gets a corresponding Lie group by taking as group law on $M \oplus \mathbf{R}$

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\Omega(v, v'))$$

In the inner product case, by Gram-Schmidt orthonormalization one can always find an orthonormal basis of V, with any other basis related to this one by an element of $GL(V)$. The subgroup of $GL(V)$ preserving the inner product and thus taking orthonormal bases to orthonormal bases is the orthogonal group $O(V)$. In the symplectic case, M has to be even-dimensional (to have a non-degenerate Ω).

Exercise. *Show that one can always find a "symplectic basis": X_j and Y_j for $j = 1, 2, \cdots, n$ satisfying*

$$\Omega(X_j, X_k) = \Omega(Y_j, Y_k) = 0, \quad \Omega(X_j, Y_k) = \delta_{jk}$$

and that in this basis one recovers the earlier definition of the Heisenberg Lie algebra and Lie group of dimension $2n + 1$.

The subgroup of $GL(M)$ preserving Ω and taking symplectic bases to symplectic bases is by definition the symplectic group $Sp(M)$. Choosing a basis, this group will be a matrix group that can be denoted $Sp(2n, \mathbf{R})$. Note that this is different than the group often written as $Sp(n)$, the group of n by n quaternionic matrices preserving the standard hermitian form on $\mathbf{H}^n$. The groups $Sp(n)$ and $Sp(2n, \mathbf{R})$ are different real forms of the group $Sp(2n, \mathbf{C})$ of linear transformations preserving a non-degenerate anti-symmetric bilinear form on $\mathbf{C}^{2n}$.

4.2 The symplectic group and the oscillator representation

The irreducible representation of the Heisenberg group we have been studying provides a projective representation of the symplectic group, which we'll construct in this section. This has various names, of which we'll choose Roger Howe's "oscillator representation." For more details, a good source is [3].

4.2.1 The Poisson bracket and the Lie algebras $\mathfrak{h}_{2n+1}$ and $\mathfrak{sp}(2n, \mathbf{R})$

In the last section we studied the Lie algebra of the Heisenberg group, which is $2n+1$ dimensional. As a Lie subalgebra of the functions on phase space P, it has basis $1, q_j, p_j$ for $j = 1, \ldots, n$, with non-zero Lie brackets the Poisson brackets

$$\{q_j, p_k\} = \delta_{jk}$$

In this section we'll extend this to the Lie algebra of monomials of degree up to two.

The space of degree two monomials on P has as basis elements $q_j p_k$ for all j, k and $q_j q_k, p_j p_k$ for $j \leq k$. The Poisson bracket of two of these is a linear combination of degree two monomials, so these provide a real Lie algebra of dimension $2n^2 + n$. This will turn out to be the Lie algebra $\mathfrak{sp}(2n, \mathbf{R})$ of the symplectic group $Sp(2n, \mathbf{R})$.

Here we will work out explicitly what happens for $n = 1$. The symplectic Lie algebra $\mathfrak{sp}(2, \mathbf{R})$ has basis q^2, p^2, qp with non-zero Lie brackets

$$\{\frac{q^2}{2}, \frac{p^2}{2}\} = qp, \quad \{qp, p^2\} = 2p^2, \quad \{qp, q^2\} = -2q^2$$

This is isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbf{R})$ of 2 by 2 traceless real matrices, with bracket the commutator, where a conventional basis is

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The isomorphism is explicitly given by

$$\frac{q^2}{2} \leftrightarrow E, \quad -\frac{p^2}{2} \leftrightarrow F, \quad -qp \leftrightarrow G$$

or by

$$-aqp + \frac{bq^2}{2} - \frac{cp^2}{2} \leftrightarrow \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

The semi-direct product of H_3 and $SL(2, \mathbf{R})$ puts the above two Lie algebras together, with the action of $SL(2, \mathbf{R})$ on H_3 by automorphisms reflected in the non-zero Lie brackets

Putting together the Lie algebras $\mathfrak{h}_3$ and $\mathfrak{sp}(2, \mathbf{R})$, we get not the direct sum of the Lie algebras but something more interesting, due to the non-zero Poisson brackets between degree two and degree one monomials:

$$\{qp, q\} = -q. \quad \{qp, p\} = p$$

$$\{\frac{p^2}{2}, q\} = -p, \quad \{\frac{q^2}{2}, p\} = q$$

These are the infinitesimal expression of the fact that we are looking not at the product group $H_3 \times Sp(2, \mathbf{R})$, but at a semidirect product $H_3 \rtimes Sp(2, \mathbf{R})$, which uses the fact that the action of $Sp(2, \mathbf{R})$ on phase space gives an action on H_3 by automorphisms.

From these relations one can see that

$$-qp \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

generates a group $\mathbf{R}$ acting on the q direction in the qp plane by e^t, on the p direction by e^{-t}. The element

$$\frac{1}{2}(q^2 + p^2) \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generates an $SO(2)$ subgroup of rotations in the qp plane.

4.2.2 The Schrödinger model for the oscillator representation

We have seen that the Schrödinger representation is given as a representation of $\mathfrak{h}_3$ by the operators

$$\pi'_S(q) = -iQ = -iq, \quad \pi'_S(p) = -iP = -\frac{d}{dq}, \quad \pi'_S(1) = -i\mathbf{1}$$

Dirac's original definition of "quantization" asked for an extension of this representation from linear functions to all functions on phase space, i.e. a choice of operators that would take any polynomial in q and p to an operator, with Poisson bracket of functions going to commutator of operators, so a Lie algebra homomorphism. But going from functions of q and p to operators built out of Q and P, one runs into "operator-ordering" ambiguities since Q and P do not commute. It turns out that one can get a Lie algebra homomorphism for polynomials up to degree two, but this is impossible in higher degree (Groenewold-van Hove theorem).

What works in degree two is to extend the Schrödinger representation to a representation of $\mathfrak{sl}(2,\mathbf{R})$ (and of the semi-direct product with $\mathfrak{h}_3$) by taking

$$\pi'_S(q^2) = -iQ^2 = -iq^2, \quad \pi'_S(p^2) = -iP^2 = i\frac{d^2}{dq^2}$$

and making the choice

$$\pi'_S(qp) = -i\frac{1}{2}(QP+PQ) = -i\frac{1}{2}(2QP - i\mathbf{1}) = -q\frac{d}{dq} - \frac{1}{2}\mathbf{1}$$

(which gives a skew-adjoint operator).

These operators will satisfy the commutation relations given by the Lie bracket of $\mathfrak{sl}(2,\mathbf{R})$, so give a representation, which is the oscillator representation. (it has many other names, including the "Weil representation"). The representation will be on the same space as the Schrödinger representation, extending the action of the Heisenberg Lie algebra, so we will often denote it by the same symbol π_S.

One would like to exponentiate the Lie algebra representation operators to get a representation of the Lie group $SL(2,\mathbf{R})$. In the case of $\pi'_S(qp)$ the operator exponentiates to an operator on functions which rescales in the q variable. It is though not so easy to exponentiate the second order differential operator

$$-iP^2 = i\frac{d^2}{dq^2}$$

If one takes a Fourier transform to turn derivatives in q into multiplication operators, the problem just moves to the operator $-iQ^2$ which changes from a multiplication operator to a second-order differential operator.

The problem is best thought of as having to do with exponentiating the Lie algebra element

$$\frac{1}{2}(q^2+p^2)$$

which generates the $SO(2) \subset SL(2,\mathbf{R})$ subgroup of rotations in the qp plane. So, for the oscillator representation, we need to explictly construct the operator

$$e^{\theta\pi'_S\left(\frac{1}{2}(q^2+p^2)\right)}$$

where

$$\pi'_S\left(\frac{1}{2}(q^2+p^2)\right) = -i\frac{1}{2}(Q^2+P^2) = -i\frac{1}{2}(q^2 - \frac{d^2}{dq^2})$$

Changing notation from θ to t, this is just the standard physics problem of solving the Schrödinger equation for the Hamiltonian $H = \frac{1}{2}(Q^2+P^2)$ and so constructing the unitary operator

$$U(t) = e^{-it\frac{1}{2}(Q^2+P^2)} \tag{4.3}$$

With some effort (see for instance exercises 4 and 5 of chapter III of [6]), one can derive a formula for the kernel $K_t(q,q')$ (known in physics as the "propagator") where

$$(U(t)\psi)(q) = \int_{\mathbf{R}} K_t(q,q')\psi(q')dq'$$

One finds

$$K_t(q,q') = \frac{1}{\sqrt{2\pi \sin t}} \exp\left(-\frac{1}{2}\begin{pmatrix} q & q' \end{pmatrix}\begin{pmatrix} \frac{\cos t}{\sin t} & -\frac{1}{\sin t} \\ -\frac{1}{\sin t} & \frac{\cos t}{\sin t}\end{pmatrix}\begin{pmatrix} q \\ q'\end{pmatrix}\right) \tag{4.4}$$

This expression requires interpretation as a distribution defined as a boundary value of a holomorphic function, replacing t by $t - i\epsilon$ and taking the limit as positive ϵ vanishes.

One can show that

$$\lim_{\epsilon \to 0^+} U(\frac{\pi}{2} - i\epsilon) = e^{i\frac{\pi}{4}}\mathcal{F}$$

This is the oscillator representation operator for an element of the symplectic group corresponding to a $\frac{\pi}{2}$ rotation in the q,p plane, interchanging the role of q and p. As expected from the Stone-von Neumann theorem, one gets the Fourier transform, up to a phase factor. The calculation of the propagator fixes the phase factor. In some sense, rotations by arbitrary values of t will give "fractional Fourier transforms."

Rotation by π in the q,p plane is given by

$$i\mathcal{F}^2$$

The $\mathcal{F}^2$ is as expected since $\mathcal{F}^2$ acts on functions by

$$\psi(q) \to \mathcal{F}^2\psi(q) = \psi(-q)$$

corresponding to a rotation by π taking q to $-q$. Rotation by 2π is given by $-\mathcal{F}^4 = -\mathbf{1}$ rather than the $\mathbf{1}$ expected if $U(t)$ is to be a true (rather than up to ± 1) representation of $SO(2) \subset SL(2,\mathbf{R})$. This is a precise analog of what happens when we take the spinor Lie algebra representation of $SO(3)$ and exponentiate: we find that rotating around an axis by 2π gives a factor of -1. The representation is only a projective (up to sign) representation of $SO(3)$. To get a true representation, one needs the double cover $Spin(3) = SU(2)$. Here again we have a representation up to sign and need a double cover of $Sp(2,\mathbf{R})$. This will be the metaplectic group $Mp(2,\mathbf{R})$, which is not a matrix group.

4.2.3 The Bargmann-Fock model for the oscillator representation

The best way to calculate the phase factors in the exponentiated version of the oscillator representation is not to use the Schrödinger version of the representation and the complicated formula 4.4 for the propagator, but to instead use the

Bargmann-Fock version. Here the representation is on the space of polynomials $\mathbf{C}[w]$ (with the Bargmann-Fock inner product) and the operators

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{d}{dw}, \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = w$$

provide a representation of the complexified Heisenberg Lie algebra (which is the standard one on the real Lie algebra).

As in the Schrödinger case, one can extend this representation to the oscillator representation of $\mathfrak{sp}(2n, \mathbf{R})$ by taking quadratic combinations of the Heisenberg Lie algebra operators. In particular, using

$$\frac{1}{2}(Q^2 + P^2) = \frac{1}{2}(a^\dagger a + aa^\dagger) = a^\dagger a + \frac{1}{2}$$

one has (writing elements of $\mathfrak{sl}(2, \mathbf{R})$ both as quadratic polynomials and as matrices)

$$\pi'_{BF}(\frac{1}{2}(q^2 + p^2)) = \pi'_{BF}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = -i(a^\dagger a + \frac{1}{2}) = -i(w\frac{d}{dw} + \frac{1}{2})$$

This operator can easily be exponentiated:

$$e^{\theta \pi'_{BF}(\frac{1}{2}(q^2+p^2))}$$

acts on $\mathbf{C}[w]$ by multiplying the monomial w^n by $e^{-i\theta(n+\frac{1}{2})}$. This gives the minus sign previously discussed for $\theta = 2\pi$.

In this representation the other two basis elements of $\mathfrak{sl}(2, \mathbf{R})$ are

$$\pi'_{BF}(-qp) = \pi'_{BF}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = -\frac{1}{2}((a^\dagger)^2 - a^2)$$

$$\pi'_{BF}(q^2 - p^2) = \pi'_{BF}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = -\frac{i}{2}((a^\dagger)^2 + a^2)$$

Note that these operators do not change the parity of monomials they act on, and you can get from any monomial of a given parity to any other other of the same parity by applying these operators repeatedly. So, the oscillator representations we have constructed here is the sum of two irreducibles (all polynomials of even degree, and all polynomials of odd degree).

4.2.4 The symplectic group and automorphisms of the Heisenberg Lie group

Since the definition of the Heisenberg Lie algebra and Lie group only depend on the antisymmetric bilinear form Ω on $M = \mathbf{R}^{2n}$, the group $Sp(2n, \mathbf{R})$ of linear maps preserving Ω acts on this Lie algebra and group as automorphisms. Using $(v, z) \in V \oplus \mathbf{R}$ as coordinates on H_{2n+1}, the action of $g \in Sp(2n, \mathbf{R})$ on the Heisenberg group is

$$\Phi_g(v, z) = (gv, z)$$

Using this automorphism, one can construct the semi-direct product

$$H_{2n+1} \rtimes Sp(2n, \mathbf{R})$$

which is sometimes called the "Jacobi group."

We also can use these automorphisms to act on the set of representations of H_{2n+1}, taking

$$\pi \to \pi_g$$

where

$$\pi_g(v, z) = \pi(\Phi_g(v, z))$$

If π is irreducible, π_g will also be irreducible, and by the Stone-von Neumann theorem there will be unitary operators U_g such that

$$\pi_g = U_g \pi_S U_g^{-1}$$

By Schur's lemma, these operators will be unique up to a phase factor. They will then provide a representation of $Sp(2n, \mathbf{R})$ up to a phase factor (a projective representation)

$$U_{g_1} U_{g_2} = e^{i\theta(g_1, g_2)} U_{g_1 g_2}$$

By changing the U_g by a phase factor

$$U_g \to V(g) = e^{i\phi(g)} U(g)$$

one can try and remove the projective factor from the multiplication law. As we have seen explicitly in the case $n = 1$, this can only be done up to sign, a problem much like that which occurs in the case of the spin representation of the rotation group. As in the case of the rotation group, one can get a true representation by going to a double cover of $Sp(2n, \mathbf{R})$, which we'll denote $Mp(2n, \mathbf{R})$ and call the "metaplectic group." Two differences from the rotation group case are:

- In the rotation group case $\pi_1(SO(n)) = \mathbf{Z}_2$ and the double cover $Spin(n)$ is the universal cover. In the symplectic case $\pi_1(Sp(2n, \mathbf{R})) = \mathbf{Z}$ and the metaplectic double cover is just one of many possible covering groups.
- $Spin(n)$ can be identified with a group of finite-dimensional matrices. This is not true for $Mp(2n, \mathbf{R})$, a group which has no finite-dimensional faithful representations. It provides a very unusual example of where thinking of Lie theory just in terms of matrix groups is inadequate.

We will refer to the representation of $Mp(2n, \mathbf{R})$ as the "oscillator representation (it goes my many other names, including Weil representation, Segal-Shale-Weil representation, etc.). The representation will be on the same space $\mathcal{H}$ as the Schrödinger representation, extending the action of the Heisenberg Lie group, so we will often denote it by the same symbol π_S and also call the representation of the Heisenberg group by the same name. We will also describe this representation as being "essentially unique", meaning that all versions of it are the same up to unitary transformations, possible rescaling, and differences in the definition of $\mathcal{H}$ related by dense inclusions.

4.3 Choice of polarization

4.3.1 Real polarizations and the Schrödinger representation

From the discussion in section 4.1.6, the real symplectic vector space M can be written as

$$M = L \oplus L^*$$

where L is an n-dimensional vector space with basis X_j and L^* is the dual vector space with basis elements Y_j dual to the X_j (i.e. $Y_j(X_k) = \delta_{jk}$). Note that for any vectors $x, x' \in L \subset M$ one has $\Omega(x, x') = 0$. A subspace with this property is called "isotropic". The maximal dimension of a subspace of M on which Ω is zero is n, and such isotropic subspaces are called "Lagrangian". L^* is also Lagrangian.

Since the definitions of the Heisenberg Lie algebra and Lie group depend only on the symplectic form Ω, and by Stone-von Neumann there is only one irreducible representation, one might expect that the definition of this irreducible representation should depend just on Ω. It turns out though that all constructions of this representation depend upon a choice of additional structure. We have seen that the construction of the Schrödinger representation depends on a choice of n position coordinates q_j, correspondng to the basis elements X_j of the Lie algebra, which span a Lagrangian subspace of $\mathbf{R}^{2n}$. The Fourier transform takes this construction to a different one, depending on n momentum coordinates p_j, corresponding to the basis elements Y_j of the Lie algebra, which span a complementary Lagrangian subspace of $\mathbf{R}^{2n}$.

More generally, one can construct a version of the Schrödinger representation for any choice of Lagrangian subspace $\ell \subset \mathbf{R}^{2n}$. By the Stone-von Neumann theorem, for each ℓ there will be an operator U_ℓ giving a unitary equivalence with the construction for the standard Schrödinger choice of $\ell = L$ spanned by the X_j. For $\ell = L^*$ spanned by the Y_j, U_ℓ will be the Fourier transform, but for more general ℓ its construction is rather non-trivial. A choice of a Lagrangian ℓ and thus a decomposition $M = \ell \oplus \ell^*$ is called a "real polarization" of M.

Exercise. *Show that the choices of Lagrangian subspace ℓ are parametrized by the space $U(n)/O(n)$.*

For the case $n = 1$, $U(1)/O(1) = \mathbf{RP}^1$, which is a circle, so real polarizations l are parametrized by an angle θ. The operators U_ℓ are the operators $U(\theta)$ of equation 4.3, going once around $\mathbf{RP}^1$ as θ goes from 0 to π.

4.3.2 Complex polarizations

The Bargmann-Fock construction involves a different sort of polarization, called a "complex polarization." Here one complexifies M and asks for Lagrangian subspaces W and $\overline{W}$ such that

$$M \otimes_{\mathbf{R}} \mathbf{C} = W \oplus \overline{W}$$

where W and $\overline{W}$ are interchanged by the conjugation map on $\mathbf{C}$.

Such a decomposition is equivalent to the choice of a compatible complex structure on M, where

Definition (Complex structure)**.** *A complex structure on a real vector space M is a (real)-linear map*

$$J : M \to M$$

satisfying $J^2 = -\mathbf{1}$.

and

Definition (Compatible complex structure)**.** *A complex structure on M is compatible with a symplectic form Ω on M when*

$$\Omega(Jv_1, Jv_2) = \Omega(v_1, v_2)$$

Such J only exist if the dimension of M is even and one can think of them as ways of making M a complex vector space (so identifying $\mathbf{R}^{2n} = \mathbf{C}^n$), with multiplication by i given by J. J has no eigenvectors in M, but it does have complex eigenvalues $\pm i$, giving a decomposition

$$M \otimes \mathbf{C} = M_J^+ \oplus M_J^-$$

into $\pm i$ eigenspaces for J. This will be a polarization of M when J is compatible with Ω since then M_J^+ and M_J^- are Lagrangian subspaces. To see this, note that for $w_1, w_2 \in V_J^+$

$$\Omega(w_1, w_2) = \Omega(Jw_1, Jw_2) = \Omega(iw_1, iw_2) = -\Omega(w_1, w_2)$$

so must be zero.

Given both a symplectic form Ω and a compatible complex structure J on M, M becomes not just a complex vector space, but a complex vector space with Hermitian inner product, defined by

$$\langle v_1, v_2 \rangle_J = \Omega(v_1, Jv_2) + i\Omega(v_1, v_2)$$

One can easily check that this is Hermitian, but it is not necessarily positive. To get a positive Hermitian structure one needs to impose an additional condition on J, that, for non-zero $v \in M$ one has

$$\Omega(v, Jv) > 0$$

The possible choices of general complex structure J are parametrized by

$$GL(2n, \mathbf{R})/GL(n, \mathbf{C})$$

The compatibility condition implies that $J \in Sp(2n, \mathbf{R})$.

Exercise. *Show that the space of possible positive complex structures compatible with Ω is $Sp(2n, \mathbf{R})/U(n)$. This is called the Siegel upper half space.*

The $n = 1$ case

For the case $n = 1$, the geometry of the space $SL(2, \mathbf{R})/U(1)$ is best understood in terms of the geometry of $\mathbf{CP}^1$, the space of complex lines in $\mathbf{C}^2$. This is also the best way to understand the holomorphic line bundles on $SL(2, \mathbf{R})/U(1)$ and how representations of $SL(2, \mathbf{R})$ can be constructed geometrically.

$SL(2, \mathbf{C})$ acts linearly on $\mathbf{C}^2$ and transitively on the the space $\mathbf{CP}^1$. The space $\mathbf{CP}^1$ is a complex manifold, the Riemann version of the sphere S^2, and the action of $SL(2, \mathbf{C})$ is holomorphic and thus an action by conformal transformations. One can choose the coordinate of the line in $\mathbf{C}^2$ generated by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

to be $z = z_1/z_2$. This gives a good coordinate system away from one point, that of the line generated by $z_1 = 1, z_2 = 0$.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbf{C})$$

acts on this coordinate by the fractional linear transformation

$$z \to \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot z = \frac{\alpha z + \beta}{\gamma z + \delta}$$

The subgroup $SL(2, \mathbf{R})$ of real matrices acts in the coordinate z preserving the sign of Im z and so does not act transitively. There are three orbits of the action: the upper and lower open half planes, and the real line. On $\mathbf{CP}^1$, the three orbits are two open hemispheres and the equator separating them. The correspondence of the three orbits in the z coordinate with the three orbits on $\mathbf{CP}^1$ is that the point where z is not a good coordinate is on the equator orbit, and approached as one goes off to infinity in any direction in the z-plane.

Picking the point $z = i$ in the upper half plane, the subgroup of elements of $SL(2, \mathbf{R})$ of elements stabilizing the point is the an $SO(2) = U(1)$ subgroup given by elements of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

We can identify the upper half plane (which we'll denote $\mathfrak{H}$) with $SL(2, \mathbf{R})/U(1)$.

The Cayley transform

$$z \to z' = \frac{z - i}{z + i}$$

takes the upper half plane to the unit disk. Conjugating an element of $SL(2, \mathbf{R})$ by this transformation gives a matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$$

where α, β are complex numbers satisfying $|\alpha|^2 - |\beta|^2 = 1$. Such matrices give the subgroup $SU(1,1)$ of $SL(2, \mathbf{C})$ preserving a $(1,1)$ signature Hermitian form. It is isomorphic to $SL(2, \mathbf{R})$ by the conjugation map. At each point in the open unit disk, $SU(1,1)$ acts with stabilizer a $U(1)$ subgroup. The Cayley transform takes $z = i$ to $z' = 0$, which is stabilized by elements of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

The subgroup of such elements acts by rotation of the unit disk about its center.

In this $n = 1$ case, changing complex polarization corresponds to changing the linear combinations of Q and P that define annihilation and creation operators. One gets an analog of the Bargmann-Fock construction for any $\tau \in \mathbf{C}$ with positive imaginary part by changing

$$a = \frac{1}{\sqrt{2}}(Q - iP) \to a_\tau = \frac{1}{\sqrt{2}} \frac{|\tau|}{\sqrt{\text{Im } \tau}}(Q - \frac{1}{\tau}P)$$

$$a^\dagger = \frac{1}{\sqrt{2}}(Q + iP) \to a_\tau^\dagger = \frac{1}{\sqrt{2}} \frac{|\tau|}{\sqrt{\text{Im } \tau}}(Q - \frac{1}{\overline{\tau}}P)$$

a_τ and $a_\tau^\dagger$ are adjoint operators satisfying the commutation relation

$$[a_\tau, a_\tau^\dagger] = 1$$

and the representation is constructed by starting with a distinguished vector annihilated by a_τ and generating the rest of the representation by applying powers of $a_\tau^\dagger$.

The unitary transformation to the Schrödinger representation will then take the distinguished vector to a solution of

$$a_\tau \psi(q) = \frac{1}{\sqrt{2}} \frac{|\tau|}{\sqrt{\text{Im } \tau}}(Q - \frac{1}{\tau}P)\psi(q) = \frac{1}{\sqrt{2}} \frac{|\tau|}{\sqrt{\text{Im } \tau}}(q + \frac{i}{\tau}\frac{d}{dq})\psi(q) = 0$$

Solutions will be proportional to

$$\psi(q) = e^{\frac{i}{2}\tau q^2}$$

and normalizable for $\text{Im } \tau > 0$.

To visualize the entire space of possible choices of polarization that give constructions of the oscillator representation for $n = 1$, one should think of the unit disk, with interior points corresponding to complex polarizations and the Bargmann-Fock construction for different τ given above. As one approaches the boundary, the distinguished vectors annihilated by a_τ become non-normalizable and leave the space $L^2(\mathbf{R})$ (they will still be distributions in $\mathcal{S}'(\mathbf{R})$.

For more details and to see how this picture generalizes to $n \geq 1$, see Graeme Segal's notes on *Symplectic manifolds and quantization* [14].

4.4 Representations and holomorphic line bundles

While the oscillator representation is essentially unique, any construction of the representation requires specification of an additional structure. For complex polarizations, this additional structure Is a complex subspace

$$W \subset M \otimes \mathbf{C}$$

There are corresponding operators on the oscillator representation, the annihilation operators, and a distinguished vector

$$|0\rangle_\tau \subset \mathcal{H}$$

annhilated by these operators. Here the notation reflects that in the physical intepretation in which annihilation and creation operators annihilate and create quanta, this is the state with zero quanta. The subscript τ in general labels points in the Siegel upper half space. In the $n = 1$ case, τ is a complex number with positive imaginary part, and in the Schrödinger representation one has explicitly

$$|0\rangle_\tau = e^{\frac{i}{2}\tau q^2}$$

More precisely, what the choice of τ picks out is the one-dimensional complex line in $\mathcal{H}$ generated by $|0\rangle_\tau \subset \mathcal{H}$. The space of these complex lines gives a complex line bundle $\mathcal{L}$ over $Sp(2n, \mathbf{R})/U(n)$. This is not quite an $Sp(2n, \mathbf{R})$ equivariant line bundle (i.e. with an action of $Sp(2n, \mathbf{R})$ on $\mathcal{L}$ that projects to the action by left multiplication on the base $Sp(2n, \mathbf{R})$), since the only the double cover $Mp(2n, \mathbf{R})$ of $Sp(2n, \mathbf{R})$ acts on $\mathcal{H}$. $\mathcal{L}$ is a $Mp(2n, \mathbf{R})$ equivariant bundle over the Siegel upper half space, described as $Mp(2n, \mathbf{R})/\widetilde{U(n)}$ where $\widetilde{U(n)}$ is a double-cover of $U(n)$.

In the case $n = 1$, recall that $SL(2, \mathbf{C})$ acts on $\mathbf{CP}^1$, with a subgroup $SU(2)$ acting transitively, identifying $SU(2)/U(1) = \mathbf{CP}^1$. Two ways to form $SU(2)$ equivariant line bundles over $\mathbf{CP}^1$ are

- Consider the product

  $$SU(2) \times \mathbf{C}$$

  and quotient by the action

  $$(g, w) \to (gh_\theta, e^{ik\theta}w)$$

  of $U(1)$, where

  $$h_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

  This will give a line bundle we'll call L^k, with sections

  $$\Gamma(L^k) = \{\phi : SU(2) \to \mathbf{C}, \quad \phi(gh_\theta) = e^{ik\theta}\phi(g)\}$$

- Since a point in $\mathbf{C}P^1$ is a complex line, one tautologically gets a line bundle (the fiber above a point is the point).

One can show that the tautological line bundle is isomorphic with L^{-1}, which is the dual line bundle to $L = L^1$. The bundle of holomorphic one-forms $\mathbf{C}P^1$ is the line bundle L^2, so in some sense L is the bundle of 1/2-forms (the spinors in this two-dimensional geometry).

The L^k for all $k \in \mathbf{Z}$ can be thought of as tensor powers of L or L^{-1}. It turns out that these are holomorphic line bundles and one can consider their holomorphic sections

$$\Gamma_{hol}(L^k)$$

(in the algebraic geometer's notation, this is $H^0(\mathbf{CP}^1, \mathcal{O}(k))$). The action of $SU(2)$ takes holomorphic sections to holomorphic sections and one finds (this is a simple example of the Borel-Weil theorem) that

$$\Gamma_{hol}(L^k) = \begin{cases} V^k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

where V^k is the irreducible representation of $SU(2)$ of dimension $k+1$ (in physicist's language, the spin $\frac{k}{2}$ representation).

For the subgroup $SU(1,1) \subset SL(2,\mathbf{C})$, the story is quite different, since the action of $SU(1,1)$ on $\mathbf{CP}^1$ is not transitive. Instead there are three orbits: two hemispheres and the equator between them. On a hemisphere D, one can use the same definition of the line bundle L^k as a quotient given above (replacing $SU(2)$ by $U(1)$) and get an irreducible representation of $SU(1,1)$ on $\Gamma_{hol}(L^k)$ but this space of sections is now infinite dimensional. This representation will be the discrete series representation D_k^+.

The even irreducible component of the oscillator representation can be realized as holomorphic sections of the line bundle $\mathcal{L}$, and one can show that $\mathcal{L} \otimes \mathcal{L} = L$. $\mathcal{L}$ is a square root of L, and a fourth-root of the holomorphic one-forms.

On the subset $D \subset \mathbf{CP}^1$ the line bundle L is the trivial bundle $D \times \mathbf{C}$, so one can choose coordinates on D and work with the first description of L given above, in which sections are holomorphic functions on D. L is an equivariant bundle under the action of $SL(2,\mathbf{R})$ and one wants to choose coordinates that transform simply under $SL(2,\mathbf{R})$.

To add some day: explicit description in coordinates

Chapter 5

Canonical quantization: fermions

In this chapter we'll discuss a precise analog of the canonical quantization formalism, in which one replaces the antisymmetric bilinear form by a symmetric one, and finds the spinor representation as an analog of the oscillator representation.

5.1 Anticommuting Variables and Pseudo-classical Mechanics

In the last chapter we studied canonical quantization for bosons starting with a classical phase space $P = \mathbf{R}^{2n}$ and a Poisson bracket determined by a non-degenerate antisymmetric bilinear form. We took as functions on phase space the polynomial functions, isomorphic to the symmetric tensor product $S^*(P^*)$. In this section we'll begin the study of canonical quantization for fermions with pseudo-classical mechanics, an analog of Hamiltonian mechanics based instead on a non-degenerate symmetric bilinear form.

5.1.1 The Grassmann algebra of polynomials on anticommuting generators

Instead of looking at polynomial functions on $P = \mathbf{R}^{2n}$, given by symmetric expressions in coordinates q_j, p_j, and identified with elements of the symmetric tensor algebra $S^*(P^*)$, one can consider a vector space $V = \mathbf{R}^m$, not necessarily of even dimension, and look at the algebra $\Lambda^*(V^*)$ of anti-symmetric tensor products on V^*. Taking a basis ξ_j of elements of V^*, one can identify $\Lambda^*(V^*)$ with what physicists call the "Grassmann algebra" (or sometimes the "exterior algebra"), thinking of this as polynomials in anti-commuting variables ξ_j:

Definition (Grassmann algebra)**.** *The algebra over the real numbers generated by $\xi_j, j = 1, \dots, n$, satisfying the relations*

$$\xi_j\xi_k + \xi_k\xi_j = 0$$

is called the Grassmann algebra.

Unlike the polynomial algebra, this algebra is finite dimensional over $\mathbf{R}$, with basis

$$1, \quad \xi_j, \quad \xi_j\xi_k, \quad \xi_j\xi_k\xi_l, \quad \cdots, \quad \xi_1\xi_2\cdots\xi_m$$

for indices $j < k < l < \cdots$ taking values $1, 2, \dots, m$.

Remarkably, an analog of calculus can be defined on such functions. For the case $n = 1$, an arbitrary function is

$$F(\xi) = c_0 + c_1\xi$$

and one can take its derivative to be

$$\frac{\partial}{\partial \xi}F = c_1$$

For larger values of n, an arbitrary function can be written as

$$F(\xi_1, \xi_2, \dots, \xi_n) = F_A + \xi_j F_B$$

where F_A, F_B are functions that do not depend on the chosen ξ_j (one gets F_B by using the anticommutation relations to move ξ_j all the way to the left). Then one can define

$$\frac{\partial}{\partial \xi_j}F = F_B$$

This derivative operator has many of the same properties as the conventional derivative, although there are unconventional signs one must keep track of. An unusual property of this derivative that is easy to see is that one has

$$\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_j} = 0$$

Taking the derivative of a product one finds this version of the Leibniz rule for monomials F and G

$$\frac{\partial}{\partial \xi_j}(FG) = \left(\frac{\partial}{\partial \xi_j}F\right)G + (-1)^{|F|}F\left(\frac{\partial}{\partial \xi_j}G\right)$$

where $|F|$ is the degree of the monomial F.

A notion of integration (often called the "Berezin integral") with many of the usual properties of an integral can also be defined. It has the peculiar feature of being the same operation as differentiation, defined in the $n = 1$ case by

$$\int (c_0 + c_1\xi)d\xi = c_1$$

and for larger n by

$$\int F(\xi_1,\xi_2,\cdots,\xi_n)d\xi_1 d\xi_2\cdots d\xi_n = \frac{\partial}{\partial\xi_n}\frac{\partial}{\partial\xi_{n-1}}\cdots\frac{\partial}{\partial\xi_1}F = c_n$$

where c_n is the coefficient of the basis element $\xi_1\xi_2\cdots\xi_n$ in the expression of F in terms of basis elements.

This notion of integration is a linear operator on functions, and it satisfies an analog of integration by parts, since if

$$F = \frac{\partial}{\partial\xi_j}G$$

then

$$\int F d\xi_j = \frac{\partial}{\partial\xi_j}F = \frac{\partial}{\partial\xi_j}\frac{\partial}{\partial\xi_j}G = 0$$

using the fact that repeated derivatives give zero.

5.1.2 Pseudo-classical mechanics and the fermionic Poisson bracket

Given an inner product (non-degenerate symmetric bilinear form) on $V = \mathbf{R}^m$, one can (Gram-Schmidt orthonormalization) choose an orthonromal basis with signature r, s (r elements have norm-squared $+1$, s have norm-squared -1 and $r+s=m$. Taking $\xi_j \in V^*$ to be the coordinates with respect to this basis, one can define a fermionic version of the Poisson bracket on elements of V^* by

$$\{\xi_j,\xi_k\}_+ = \pm\delta_{jk}$$

with a plus sign for $j=k=1,\cdots,r$ and a minus sign for $j=k=r+1,\cdots,m$, This is just the inner product on V^* corresponding to our choice of inner product on V. One can extend the definition of this inner product to all elements of $\Lambda^*(V^*)$ by imposing a symmetry propert and Leibniz rule (derivation property) that has signs consistent with the anticommutativity of the generators. For monomials F_1, F_2, F_3, define

- $$\{F_1F_2,F_3\}_+ = F_1\{F_2,F_3\}_+ + (-1)^{|F_2||F_3|}\{F_1,F_3\}_+F_2$$
  where $|F_2|$ and $|F_3|$ are the degrees of F_2 and F_3.

- $$\{F_1,F_2\}_+ = -(-1)^{|F_1||F_2|}\{F_2,F_1\}_+$$

These two properties can be used to compute the fermionic Poisson bracket for arbitrary functions in terms of the relations for generators.

Taking the case of a positive-definite inner product for simplicity, one can calculate explicitly the fermionic Poisson brackets for linear and quadratic combinations of the generators. One finds

$$\{\xi_j\xi_k,\xi_l\}_+ = \xi_j\{\xi_k,\xi_l\}_+ - \{\xi_j,\xi_l\}_+\xi_k = \delta_{kl}\xi_j - \delta_{jl}\xi_k \tag{5.1}$$

and

$$\begin{aligned}\{\xi_j\xi_k, \xi_l\xi_m\}_+ =&\{\xi_j\xi_k, \xi_l\}_+\xi_m + \xi_l\{\xi_j\xi_k, \xi_m\}_+\\ =&\delta_{kl}\xi_j\xi_m - \delta_{jl}\xi_k\xi_m + \delta_{km}\xi_l\xi_j - \delta_{jm}\xi_l\xi_k\end{aligned} \tag{5.2}$$

The second of these equations shows that the quadratic combinations of the generators ξ_j satisfy the relations of the Lie algebra of the group of rotations in n dimensions ($\mathfrak{so}(n) = \mathfrak{spin}(n)$). The first shows that the $\xi_k\xi_l$ acts on the ξ_j as infinitesimal rotations in the kl plane.

While the Poisson bracket defines a Lie algebra on $S^*(P^*)$, the fermionic Poisson bracket on $\Lambda^*(V^*)$ provides an example of something called a Lie superalgebra. These can be defined for vector spaces with some usual and some fermionic coordinates:

Definition (Lie superalgebra)**.** *A Lie superalgebra structure on a real or complex vector space V is given by a Lie superbracket $[\cdot,\cdot]_\pm$. This is a bilinear map on V which on generators X, Y, Z (which may be usual or fermionic ones) satisfies*

$$[X, Y]_\pm = -(-1)^{|X||Y|}[Y, X]_\pm$$

and a super-Jacobi identity

$$[X, [Y, Z]_\pm]_\pm = [[X, Y]_\pm, Z]_\pm + (-1)^{|X||Y|}[Y, [X, Z]_\pm]_\pm$$

where $|X|$ takes value 0 *for a usual generator,* 1 *for a fermionic generator.*

Analogously to the bosonic case, the polynomials of order less than or equal to two provide a sub-Lie superalgebra of dimension $1 + n + \frac{1}{2}(n^2 - n)$ (since there is one constant, n linear terms ξ_j and $\frac{1}{2}(n^2 - n)$ quadratic terms $\xi_j\xi_k$). On functions of order two this Lie superalgebra is a Lie algebra, $\mathfrak{so}(n)$. We will see in chapter 5.3 that the definition of a representation can be generalized to Lie superalgebras, and quantization will give a distinguished representation of this Lie superalgebra, in a manner quite parallel to that of the Schrödinger or Bargmann-Fock constructions of a representation in the bosonic case.

5.1.3 Examples of pseudo-classical mechanics

In pseudo-classical mechanics, the dynamics will be determined by choosing a Hamiltonian h in $\Lambda^*(V^*)$. Observables will be other functions $F \in \Lambda^*(V^*)$, and they will satisfy the analog of Hamilton's equations

$$\frac{d}{dt}F = \{F, h\}_+$$

We'll consider two of the simplest possible examples.

The pseudo-classical spin degree of freedom

Using pseudo-classical mechanics, a "classical" analog can be found for something that is quintessentially quantum: the degree of freedom described by the spin $\frac{1}{2}$ system of section 3.2. Taking $V = \mathbf{R}^3$ with the standard inner product as fermionic phase space, one has three generators $\xi_1, \xi_2, \xi_3 \in V^*$ satisfying the relations

$$\{\xi_j, \xi_k\}_+ = \delta_{jk}$$

and an 8 dimensional space of functions with basis

$$1,\ \xi_1,\ \xi_2,\ \xi_3,\ \xi_1\xi_2,\ \xi_1\xi_3,\ \xi_2\xi_3,\ \xi_1\xi_2\xi_3$$

For the Hamiltonian function to be non-trivial and of even degree, it will have to be a linear combination

$$h = B_{12}\xi_1\xi_2 + B_{13}\xi_1\xi_3 + B_{23}\xi_2\xi_3$$

for some constants B_{12}, B_{13}, B_{23}. This can be written

$$h = \frac{1}{2}\sum_{j,k=1}^{3} L_{jk}\xi_j\xi_k$$

where L_{jk} are the entries of the matrix

$$L = \begin{pmatrix} 0 & B_{12} & B_{13} \\ -B_{12} & 0 & B_{23} \\ -B_{13} & -B_{23} & 0 \end{pmatrix}$$

The equations of motion on generators will be

$$\frac{d}{dt}\xi_j(t) = \{\xi_j, h\}_+ = -\{h, \xi_j\}_+$$

with solution

$$\xi_j(t) = e^{tL}\xi_j(0)$$

This will be a time-dependent rotation of the ξ_j in the plane perpendicular to

$$\mathbf{B} = (B_{23}, -B_{13}, B_{12})$$

at a constant speed proportional to $|\mathbf{B}|$.

The pseudo-classical fermionic oscillator

To get a fermionic analog of the classical harmonic oscillator, for the case of d oscillators, take $V = \mathbf{R}^{2d}$ and Hamiltonian

$$h = \frac{1}{2}\sum_{j=1}^{d}(\xi_{2j}\xi_{2j-1} - \xi_{2j-1}\xi_{2j}) = \sum_{j=1}^{d}\xi_{2j}\xi_{2j-1}$$

From 5.1 and 5.2, quadratic products $\xi_j\xi_k$ act on the generators by infinitesimal rotations in the jk plane, and satisfy the commutation relations of $\mathfrak{so}(2d)$.

As in the bosonic case, we can make the standard choice of complex structure $J = J_0$ on $\mathbf{R}^{2d}$ and get a decomposition

$$V^* \otimes \mathbf{C} = \mathbf{R}^{2d} \otimes \mathbf{C} = \mathbf{C}^d \oplus \mathbf{C}^d$$

into eigenspaces of J of eigenvalue $\pm i$. This is done by defining

$$\theta_j = \frac{1}{\sqrt{2}}(\xi_{2j-1} - i\xi_{2j}), \quad \overline{\theta}_j = \frac{1}{\sqrt{2}}(\xi_{2j-1} + i\xi_{2j})$$

for $j = 1, \ldots, d$. These satisfy the fermionic Poisson bracket relations

$$\{\theta_j, \theta_k\}_+ = \{\overline{\theta}_j, \overline{\theta}_k\}_+ = 0, \quad \{\overline{\theta}_j, \theta_k\}_+ = \delta_{jk}$$

(where we have extended the inner product $\{\cdot,\cdot\}_+$ to $V^* \otimes \mathbf{C}$ by complex linearity).

In terms of the θ_j, the Hamiltonian is

$$h = -\frac{i}{2}\sum_{j=1}^{d}(\theta_j\overline{\theta}_j - \overline{\theta}_j\theta_j) = -i\sum_{j=1}^{d}\theta_j\overline{\theta}_j$$

Using the derivation property of $\{\cdot,\cdot\}_+$ one finds

$$\{h, \theta_j\}_+ = -i\sum_{k=1}^{d}(\theta_k\{\overline{\theta}_k, \theta_j\}_+ - \{\theta_k, \theta_j\}_+\overline{\theta}_k) = -i\theta_j$$

and, similarly,

$$\{h, \overline{\theta}_j\}_+ = i\overline{\theta}_j$$

so one sees that h is the generator of $U(1) \subset U(d)$ phase rotations on the variables θ_j. The equations of motion are

$$\frac{d}{dt}\theta_j = \{\theta_j, h\}_+ = i\theta_j, \quad \frac{d}{dt}\overline{\theta}_j = \{\overline{\theta}_j, h\}_+ = -i\overline{\theta}_j$$

with solutions

$$\theta_j(t) = e^{it}\theta_j(0), \quad \overline{\theta}_j(t) = e^{-it}\overline{\theta}_j(0)$$

5.2 Clifford Algebras

In the bosonic case, quantization was a homomorphism of Lie algebras takng coordinates q_j, p_j with Poisson bracket $\{q_j, p_k\} = \delta_{jk}$ to operators $-iQ_j, -iP_j$ with commutator

$$[-iQ_j, -iP_k] = -i\delta_{jk}\mathbf{1}$$

In the fermionic case we will have a homomorphism of Lie superalgebras, taking ξ_j with fermionic Poisson bracket $\{\xi_j, \xi_k\} = \pm\delta_{jk}$ to operators γ_j satisfying anticommutation relations

These operators are the generators of a Clifford algebra, which we'll now turn to.

5.2.1 Real Clifford algebras

We can define real Clifford algebras $\mathrm{Cliff}(r, s, \mathbf{R})$ for an inner product of arbitrary signature by

Definition (Real Clifford algebras, arbitrary signature)**.** *The real Clifford algebra in $m = r + s$ variables is the algebra $\mathrm{Cliff}(r, s, \mathbf{R})$ over the real numbers generated by $1, \gamma_j$ for $j = 1, 2, \ldots, m$ satisfying the relations*

$$[\gamma_j, \gamma_k]_+ = \pm 2\delta_{jk}1$$

where we choose the $+$ sign when $j = k = 1, \ldots, r$ and the $-$ sign when $j = k = r + 1, \ldots, m$.

In other words, different γ_j anticommute, but only the first r of them satisfy $\gamma_j^2 = 1$, with the other s of them satisfying $\gamma_j^2 = -1$.

Working out some of the low dimensional examples, one finds:

- $\mathrm{Cliff}(0, 1, \mathbf{R})$. This has generators 1 and γ_1, satisfying

  $$\gamma_1^2 = -1$$

  Taking real linear combinations of these two generators, the algebra one gets is just the algebra $\mathbf{C}$ of complex numbers, with γ_1 playing the role of $i = \sqrt{-1}$.

- $\mathrm{Cliff}(0, 2, \mathbf{R})$. This has generators $1, \gamma_1, \gamma_2$ and a basis

  $$1, \quad \gamma_1, \quad \gamma_2, \quad \gamma_1\gamma_2$$

  with

  $$\gamma_1^2 = -1, \quad \gamma_2^2 = -1, \quad (\gamma_1\gamma_2)^2 = \gamma_1\gamma_2\gamma_1\gamma_2 = -\gamma_1^2\gamma_2^2 = -1$$

  This four dimensional algebra over the real numbers can be identified with the algebra $\mathbf{H}$ of quaternions by taking

  $$\gamma_1 \leftrightarrow \mathbf{i}, \quad \gamma_2 \leftrightarrow \mathbf{j}, \quad \gamma_1\gamma_2 \leftrightarrow \mathbf{k}$$

- $\mathrm{Cliff}(1, 1, \mathbf{R})$. This is the algebra $M(2, \mathbf{R})$ of real 2 by 2 matrices, with one possible identification as follows

  $$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_1 \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1\gamma_2 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\mathrm{Cliff}(3, 0, \mathbf{R})$. This is the algebra $M(2, \mathbf{C})$ of complex 2 by 2 matrices, with one possible identification using Pauli matrices given by

  $$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\gamma_1 \leftrightarrow \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 \leftrightarrow \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_3 \leftrightarrow \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma_1\gamma_2 \leftrightarrow i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma_2\gamma_3 \leftrightarrow i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_1\gamma_3 \leftrightarrow -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma_1\gamma_2\gamma_3 \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

It turns out that $\mathrm{Cliff}(r, s, \mathbf{R})$ is always one or two copies of matrices of real, complex or quaternionic elements, of dimension a power of 2, but this requires a rather intricate algebraic argument that we will not enter into here. For the details of this and the resulting pattern of algebras one gets, see for instance [8]. One special case where the pattern is relatively simple is when one has $r = s$. Then $n = 2r$ is even dimensional and one finds

$$\mathrm{Cliff}(r, r, \mathbf{R}) = M(2^r, \mathbf{R})$$

5.2.2 Clifford algebras and geometry

The Clifford algebra was defined above in terms of generators and relations, but it also has a coordinate invariant definition, based on the choice of a non-degenerate symmetric bilinear form $(\cdot, \cdot)$, i.e., an inner product. It gives a powerful tool for the study of the orthogonal group of transformations that preserve the inner product.

To see the relation between Clifford algebras and geometry, consider first the positive definite case $\mathrm{Cliff}(m, \mathbf{R}) = \mathrm{Cliff}(m, 0, \mathbf{R})$ with the standard inner product. In a later chapter we'll discuss the geometry of Minkowski spacetime and special relativity, which uses the case $m = 4$ with signature $3, 1$. The generators of the Clifford algebra are well-known in that case as the Dirac γ-matrices.

To an arbitrary vector

$$\mathbf{v} = (v_1, v_2, \ldots, v_m) \in \mathbf{R}^m$$

one can associate the Clifford algebra element $\not{v} = \gamma(\mathbf{v})$ where γ is the map

$$\mathbf{v} \in \mathbf{R}^m \rightarrow \gamma(\mathbf{v}) = v_1\gamma_1 + v_2\gamma_2 + \cdots + v_n\gamma_m \in \mathrm{Cliff}(m, \mathbf{R}) \tag{5.3}$$

Using the Clifford algebra relations for the γ_j, given two vectors $\mathbf{v}$, $\mathbf{w}$ the product of their associated Clifford algebra elements satisfies

$$\begin{aligned} \not{v}\not{w} + \not{w}\not{v} &= [v_1\gamma_1 + v_2\gamma_2 + \cdots + v_n\gamma_n,\ w_1\gamma_1 + w_2\gamma_2 + \cdots + w_n\gamma_n]_+ \\ &= 2(v_1w_1 + v_2w_2 + \cdots + v_mw_m) \\ &= 2(\mathbf{v}, \mathbf{w}) \end{aligned} \tag{5.4}$$

Note that taking $\mathbf{v} = \mathbf{w}$ one has

$$\not{v}^2 = (\mathbf{v}, \mathbf{v}) = ||\mathbf{v}||^2$$

The Clifford algebra Cliff$(m, \mathbf{R})$ thus contains $\mathbf{R}^m$ as the subspace of linear combinations of the generators γ_j. It can be thought of as a sort of enhancement of the vector space $\mathbf{R}^m$ that encodes information about the inner product, and it will sometimes be written Cliff$(\mathbf{R}^m, (\cdot,\cdot))$. In this larger structure vectors can be multiplied as well as added, with the multiplication determined by the inner product (equation 5.4). Note that different people use different conventions, with

$$\not{\mathbf{v}}\not{\mathbf{w}} + \not{\mathbf{w}}\not{\mathbf{v}} = -2(\mathbf{v}, \mathbf{w})$$

another common choice. One also sees variants without the factor of 2.

We'll consider two different ways of seeing the relationship between the Clifford algebra Cliff$(n, \mathbf{R})$ and the group $O(m)$ of rotations in $\mathbf{R}^m$. The first is based upon the geometrical fact (known as the Cartan-Dieudonné theorem) that one can get any rotation by doing at most m orthogonal reflections in different hyperplanes. Orthogonal reflection in the hyperplane perpendicular to a vector $\mathbf{w}$ takes a vector $\mathbf{v}$ to the vector

$$\mathbf{v}' = \mathbf{v} - 2\frac{(\mathbf{v}, \mathbf{w})}{(\mathbf{w}, \mathbf{w})}\mathbf{w}$$

something that can easily be seen from the following picture

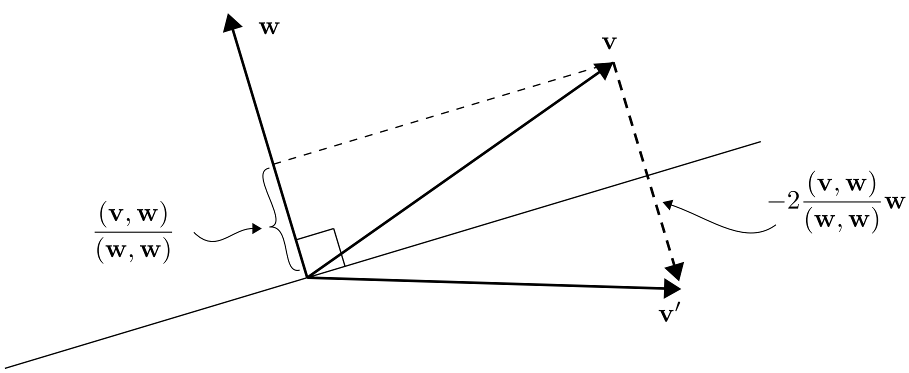


Figure 5.1: Orthogonal reflection in the hyperplane perpendicular to $\mathbf{w}$.

From now on we identify vectors $\mathbf{v}, \mathbf{v}', \mathbf{w}$ with the corresponding Clifford algebra elements by the map γ of equation 5.3. The linear transformation given by reflection in $\mathbf{w}$ is

$$\begin{aligned}\not{\mathbf{v}} \rightarrow \not{\mathbf{v}}' =& \not{\mathbf{v}} - 2\frac{(\mathbf{v}, \mathbf{w})}{(\mathbf{w}, \mathbf{w})}\not{\mathbf{w}} \\ =& \not{\mathbf{v}} - (\not{\mathbf{v}}\not{\mathbf{w}} + \not{\mathbf{w}}\not{\mathbf{v}})\frac{\not{\mathbf{w}}}{(\mathbf{w}, \mathbf{w})}\end{aligned}$$

Since

$$\not{\mathbf{w}}\frac{\not{\mathbf{w}}}{(\mathbf{w}, \mathbf{w})} = \frac{(\mathbf{w}, \mathbf{w})}{(\mathbf{w}, \mathbf{w})} = 1$$

we have (for non-zero vectors $\mathbf{w}$)

$$\not{\mathbf{w}}^{-1} = \frac{\not{\mathbf{w}}}{(\mathbf{w}, \mathbf{w})}$$

and the reflection transformation is just conjugation by $\not{\mathbf{w}}$ times a minus sign

$$\not{\mathbf{v}} \to \not{\mathbf{v}}' = \not{\mathbf{v}} - \not{\mathbf{v}} - \not{\mathbf{w}}\not{\mathbf{v}}\not{\mathbf{w}}^{-1} = -\not{\mathbf{w}}\not{\mathbf{v}}\not{\mathbf{w}}^{-1}$$

Identifying vectors with Clifford algebra elements, the orthogonal transformation that is the result of one reflection is given by a conjugation (with a minus sign). These reflections lie in the group $O(m)$, but not in the subgroup $SO(m)$, since they change orientation. The result of two reflections in hyperplanes orthogonal to $\mathbf{w}_1$ and $\mathbf{w}_2$ will be a conjugation by $\not{\mathbf{w}}_2\not{\mathbf{w}}_1$

$$\not{\mathbf{v}} \to \not{\mathbf{v}}' = -\not{\mathbf{w}}_2(-\not{\mathbf{w}}_1\not{\mathbf{v}}\not{\mathbf{w}}_1^{-1})\not{\mathbf{w}}_2^{-1} = (\not{\mathbf{w}}_2\not{\mathbf{w}}_1)\not{\mathbf{v}}(\not{\mathbf{w}}_2\not{\mathbf{w}}_1)^{-1}$$

This will be a rotation preserving the orientation, so of determinant one and in the group $SO(n)$.

This construction not only gives an efficient way of representing rotations (as conjugations in the Clifford algebra), but it also provides a construction of the group $Spin(n)$ in arbitrary dimension n. One can define:

Definition ($Spin(m)$)**.** *The group $Spin(m)$ is the group of invertible elements of the Clifford algebra $Cliff(m)$ of the form*

$$\not{\mathbf{w}}_1\not{\mathbf{w}}_2\cdots\not{\mathbf{w}}_k$$

where the vectors $\mathbf{w}_j$ for $j = 1, \cdots, k$ ($k \leq n$) are vectors in $\mathbf{R}^m$ satisfying $|\mathbf{w}_j|^2 = 1$ and k is even. Group multiplication is Clifford algebra multiplication.

The action of $Spin(m)$ on vectors $\mathbf{v} \in \mathbf{R}^n$ will be given by conjugation

$$\not{\mathbf{v}} \to (\not{\mathbf{w}}_1\not{\mathbf{w}}_2\cdots\not{\mathbf{w}}_k)\not{\mathbf{v}}(\not{\mathbf{w}}_1\not{\mathbf{w}}_2\cdots\not{\mathbf{w}}_k)^{-1} \tag{5.5}$$

and this will correspond to a rotation of the vector $\mathbf{v}$. One can see here the characteristic fact that there are two elements of the $Spin(m)$ group giving the same rotation in $SO(m)$ by noticing that changing the sign of the Clifford algebra element $\not{\mathbf{w}}_1\not{\mathbf{w}}_2\cdots\not{\mathbf{w}}_k$ does not change the conjugation action, where signs cancel.

For a second approach to understanding rotations in arbitrary dimension, one can use the fact that these are generated by taking products of rotations in the coordinate planes. A rotation by an angle θ in the jk coordinate plane ($j < k$) will be given by

$$\mathbf{v} \to e^{\theta\epsilon_{jk}}\mathbf{v}$$

where ϵ_{jk} is an m by m matrix with only two non-zero entries: jk entry -1 and kj entry $+1$. Restricting attention to the jk plane, $e^{\theta\epsilon_{jk}}$ acts as the standard rotation matrix in the plane

$$\begin{pmatrix} v_j \\ v_k \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_j \\ v_k \end{pmatrix}$$

In the $SO(3)$ case there are three of these matrices

$$\epsilon_{23} = l_1, \quad \epsilon_{13} = -l_2, \quad \epsilon_{12} = l_3$$

providing a basis of the Lie algebra $\mathfrak{so}(3)$. In m dimensions there will be $\frac{1}{2}(m^2 - m)$ of them, providing a basis of the Lie algebra $\mathfrak{so}(m)$.

In dimension m we can use elements of the Clifford algebra to get these same rotation transformations, but as conjugations in the Clifford algebra. To see how this works, consider the quadratic Clifford algebra element $\gamma_j\gamma_k$ for $j \neq k$ and notice that

$$(\gamma_j\gamma_k)^2 = \gamma_j\gamma_k\gamma_j\gamma_k = -\gamma_j\gamma_j\gamma_k\gamma_k = -1$$

so one has

$$\begin{aligned} e^{\frac{\theta}{2}\gamma_j\gamma_k} &= \left(1 - \frac{(\theta/2)^2}{2!} + \cdots\right) + \gamma_j\gamma_k\left(\theta/2 - \frac{(\theta/2)^3}{3!} + \cdots\right) \\ &= \cos\left(\frac{\theta}{2}\right) + \gamma_j\gamma_k \sin\left(\frac{\theta}{2}\right) \end{aligned}$$

Conjugating a vector $v_j\gamma_j + v_k\gamma_k$ in the jk plane by this, one can show that

$$e^{-\frac{\theta}{2}\gamma_j\gamma_k}(v_j\gamma_j + v_k\gamma_k)e^{\frac{\theta}{2}\gamma_j\gamma_k} = (v_j\cos\theta - v_k\sin\theta)\gamma_j + (v_j\sin\theta + v_k\cos\theta)\gamma_k$$

which is a rotation by θ in the jk plane. Such a conjugation will also leave invariant the γ_l for $l \neq j, k$. Thus one has

$$e^{-\frac{\theta}{2}\gamma_j\gamma_k}\gamma(\mathbf{v})e^{\frac{\theta}{2}\gamma_j\gamma_k} = \gamma(e^{\theta\epsilon_{jk}}\mathbf{v}) \tag{5.6}$$

and, taking the derivative at $\theta = 0$, the infinitesimal version

$$\left[-\frac{1}{2}\gamma_j\gamma_k, \gamma(\mathbf{v})\right] = \gamma(\epsilon_{jk}\mathbf{v}) \tag{5.7}$$

One gets a double cover of the group of rotations, with here the elements $e^{\frac{\theta}{2}\gamma_j\gamma_k}$ of the Clifford algebra giving a double cover of the group of rotations in the jk plane (as θ goes from 0 to 2π). General elements of the spin group can be constructed by multiplying these for different angles in different coordinate planes. The Lie algebra $\mathfrak{spin}(n)$ can be identified with the Lie algebra $\mathfrak{so}(n)$ by

$$\epsilon_{jk} \leftrightarrow -\frac{1}{2}\gamma_j\gamma_k$$

Yet another way to see this would be to compute the commutators of the $-\frac{1}{2}\gamma_j\gamma_k$ for different values of j, k and show that they satisfy the same commutation relations as the corresponding matrices ϵ_{jk}.

5.2.3 Complex Clifford algebras

If one allows complex coefficients in a real Clifford algebra Cliff$(r, s, \mathbf{R})$, then one gets a complex Clifford algebra:

Definition (Complex Clifford algebras)**.** *The complex Clifford algebra in m variables is the algebra* Cliff$(m, \mathbf{C})$ *over the complex numbers generated by* $1, \gamma_j$ *for* $j = 1, 2, \ldots, m$ *satisfying the relations*

$$[\gamma_j, \gamma_k]_+ = 2\delta_{jk}$$

When one complexifies, the signature of the inner product no longer matters, since one can multiply a generator by i to change the sign of its square. One can write this fact as

$$\text{Cliff}(r, s, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = \text{Cliff}(m, \mathbf{C})$$

In a situation like this of several different real algebras that complexify to the sam complex algebra, these real algebras are called "real forms" of the complex algebra.

While the structure of real Clifford algebras depends in a complicated way on r and s, the structure of the complex Clifford algebras is much simpler. We will not prove this here, but one has algebra isomorphisms:

- In the even dimensional case

$$\text{Cliff}(2d, \mathbf{C}) \leftrightarrow M(2^d, \mathbf{C})$$

- In the odd dimensional case

$$\text{Cliff}(2d + 1, \mathbf{C}) \leftrightarrow M(2^d, \mathbf{C}) \oplus M(2^d, \mathbf{C})$$

Two properties of Cliff$(n, \mathbf{C})$ are

- As a vector space over $\mathbf{C}$, a basis of Cliff$(m, \mathbf{C})$ is the set of elements

$$1, \quad \gamma_j, \quad \gamma_j\gamma_k, \quad \gamma_j\gamma_k\gamma_l, \ \ldots, \ \gamma_1\gamma_2\gamma_3 \cdots \gamma_{m-1}\gamma_m$$

  for indices $j, k, l, \cdots \in 1, 2, \ldots, n$, with $j < k < l < \cdots$. To show this, consider all products of the generators, and use the commutation relations for the γ_j to identify any such product with an element of this basis. The relation $\gamma_j^2 = 1$ shows that repeated occurrences of a γ_j can be removed. The relation $\gamma_j\gamma_k = -\gamma_k\gamma_j$ can then be used to put elements of the product in the order of a basis element as above.

- As a vector space over $\mathbf{C}$, Cliff$(m, \mathbf{C})$ has dimension 2^m. One way to see this is to consider the product

$$(1 + \gamma_1)(1 + \gamma_2) \cdots (1 + \gamma_m)$$

  which will have 2^m terms that are exactly those of the basis listed above.

When $n = 2d$ is even, there is an alternate definition of the complex Clifford algebra in terms of fermionic versions of annihilation and creation operators. For the case $d = 1$, one can define

$$a_F = \frac{1}{2}(\gamma_1 + i\gamma_2), \quad a_F^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2)$$

The a_F and $a_F^\dagger$ will satisfy canonical anticommutation relations (CAR)

$$[a_F, a_F^\dagger]_+ = 1, \quad [a_F, a_F]_+ = [a_F^\dagger, a_F^\dagger]_+ = 0$$

called the "canonical anticommutation relations" (CAR).

The algebra generated over $\mathbf{C}$, by the $a_F, a_F^\dagger$ is four-dimensional, with basis.

$$1, \quad a_F, \quad a_F^\dagger, \quad a_F^\dagger a_F$$

It is isomorphic to the complex Clifford algebra $\mathrm{Cliff}(2, \mathbf{C})$ and can be identified with the algebra $M(2, \mathbf{C})$ of 2 by 2 complex matrices, using

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_F \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a_F^\dagger \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_F^\dagger a_F \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.8}$$

For arbitrary d, one can define

$$a_{Fj} = \frac{1}{2}(\gamma_{2j-1} + i\gamma_{2j}), \quad a_{Fj}^\dagger = \frac{1}{2}(\gamma_{2j-1} - i\gamma_{2j})$$

and get an alternate definition of the complex Clifford algebra:

Definition (Complex Clifford algebras, using annihilation and creation operators)**.** *The complex Clifford algebra* $\mathrm{Cliff}(2d, \mathbf{C})$ *is the algebra over* $\mathbf{C}$ *generated by* $1, a_{Fj}, a_{Fj}^\dagger$ *for* $j = 1, 2, \ldots, d$ *satisfying the CAR*

$$[a_{Fj}, a_{Fk}^\dagger]_+ = \delta_{jk}1, \quad [a_{Fj}, a_{Fk}]_+ = [a_{Fj}^\dagger, a_{Fk}^\dagger]_+ = 0$$

This shows that the complex Clifford algebra is a close analog of the Weyl algebra in the bosonic case, which could have been defined by

Definition (Complex Weyl algebras)**.** *The complex Weyl algebra is the algebra* $\mathrm{Weyl}(2n, \mathbf{C})$ *generated by the elements* $1, a_j, a_j^\dagger$, $j = 1, \ldots, n$ *satisfying the canonical commutation relations (CCR)*

$$[a_j, a_k^\dagger] = \delta_{jk}1, \quad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0$$

Unlike the Clifford algebra, as a vector space over $\mathbf{C}$, $\mathrm{Weyl}(2n, \mathbf{C})$ is infinite dimensional. Recall from the Bargmann-Fock construction that taking $a_j = \frac{\partial}{\partial w_j}, a_j^\dagger = w_j$ one can identify this algebra with the algebra of polynomial coefficient differential operators. We will see later that the complex Clifford algebra in this case can be identified with "differential operators in fermionic variables θ_j", analogous to what happens in the bosonic (Weyl algebra) case.

5.3 Fermionic Quantization and Spinors

In this chapter we'll begin by investigating the fermionic analog of the notion of quantization, which takes functions of anticommuting variables on a phase space with symmetric bilinear form $(\cdot,\cdot)$ and gives an algebra of operators with generators satisfying the relations of the corresponding Clifford algebra. We will then consider analogs of the constructions used in the bosonic case which there gave us the Schrödinger and Bargmann-Fock representations of the Weyl algebra on a space of states.

We know that for a fermionic oscillator with d degrees of freedom, the algebra of operators will be $\mathrm{Cliff}(2d,\mathbf{C})$, the algebra generated by annihilation and creation operators $a_{Fj}, a_{Fj}^\dagger$. These operators will act on $\mathcal{H}_F = \mathcal{F}_d^+$, a complex vector space of dimension 2^d, and this will provide a fermionic analog of the bosonic Γ'_{BF} acting on $\mathcal{F}_d$. Since the spin group consists of invertible elements of the Clifford algebra, it has a representation on $\mathcal{F}_d^+$. This is known as the "spinor representation", and it can be constructed by analogy with the construction of the metaplectic representation in the bosonic case. We'll also consider the analog in the fermionic case of the Schrödinger representation, which turns out to have a problem with unitarity, but finds a use in physics as "ghost" degrees of freedom.

5.3.1 Quantization of pseudo-classical systems

In the bosonic case, quantization was based on finding a representation of the Heisenberg Lie algebra of linear functions on phase space, or more explicitly, for basis elements q_j, p_j of this Lie algebra finding operators Q_j, P_j satisfying the Heisenberg commutation relations. In the fermionic case, the analog of the Heisenberg Lie algebra is not a Lie algebra, but a Lie superalgebra, with basis elements $1, \xi_j, j = 1, \ldots, n$ and a Lie superbracket given by the fermionic Poisson bracket, which on basis elements is

$$\{\xi_j, \xi_k\}_+ = \pm\delta_{jk}, \quad \{\xi_j, 1\}_+ = 0, \; \{1, 1\}_+ = 0$$

Quantization is given by finding a representation of this Lie superalgebra. The definition of a Lie algebra representation can be generalized to that of a Lie superalgebra representation by:

Definition (Representation of a Lie superalgebra)**.** *A representation of a Lie superalgebra is a homomorphism Φ preserving the superbracket*

$$[\Phi(X), \Phi(Y)]_\pm = \Phi([X, Y]_\pm)$$

This takes values in a Lie superalgebra of linear operators, with $|\Phi(X)| = |X|$ and $[\cdot,\cdot]_\pm$ the supercommutator

$$[\Phi(X), \Phi(Y)]_\pm = \Phi(X)\Phi(Y) - (-)^{|X||Y|}\Phi(Y)\Phi(X)$$

A representation of the pseudo-classical Lie superalgebra (and thus a quantization of the pseudo-classical system) will be given by finding a linear map Γ^+ that takes basis elements ξ_j to operators $\Gamma^+(\xi_j)$ satisfying the relations

$$[\Gamma^+(\xi_j), \Gamma^+(\xi_k)]_+ = \pm\delta_{jk}\Gamma^+(1), \quad [\Gamma^+(\xi_j), \Gamma^+(1)] = [\Gamma^+(1), \Gamma^+(1)] = 0$$

These relations can be satisfied by taking

$$\Gamma^+(\xi_j) = \frac{1}{\sqrt{2}}\gamma_j, \quad \Gamma^+(1) = \mathbf{1}$$

since then

$$[\Gamma^+(\xi_j), \Gamma^+(\xi_k)]_+ = \frac{1}{2}[\gamma_j, \gamma_k]_+ = \pm\delta_{jk}$$

are exactly the Clifford algebra relations. This can be extended to a representation of the functions of the ξ_j of order two or less by

Theorem. *A representation of the Lie superalgebra of anticommuting functions of coordinates ξ_j on $\mathbf{R}^n$ of order two or less is given by*

$$\Gamma^+(1) = \mathbf{1}, \quad \Gamma^+(\xi_j) = \frac{1}{\sqrt{2}}\gamma_j, \quad \Gamma^+(\xi_j\xi_k) = \frac{1}{2}\gamma_j\gamma_k$$

Proof. We have already seen that this is a representation for polynomials in ξ_j of degree zero and one. For simplicity just considering the case $s = 0$ (positive definite inner product), in degree two the fermionic Poisson bracket relations are given by equations 5.1 and 5.2. For 5.1, one can show that the products of Clifford algebra generators

$$\Gamma^+(\xi_j\xi_k) = \frac{1}{2}\gamma_j\gamma_k$$

satisfy

$$\left[\frac{1}{2}\gamma_j\gamma_k, \gamma_l\right] = \delta_{kl}\gamma_j - \delta_{jl}\gamma_k$$

by using the Clifford algebra relations, or by noting that this is the special case of equation 5.7 for $\mathbf{v} = \mathbf{e}_l$. That equation shows that commuting by $-\frac{1}{2}\gamma_j\gamma_k$ acts by the infinitesimal rotation ϵ_{jk} in the jk coordinate plane.

For 5.2, the Clifford algebra relations can again be used to show

$$\left[\frac{1}{2}\gamma_j\gamma_k, \frac{1}{2}\gamma_l\gamma_m\right] = \delta_{kl}\frac{1}{2}\gamma_j\gamma_m - \delta_{jl}\frac{1}{2}\gamma_k\gamma_m + \delta_{km}\frac{1}{2}\gamma_l\gamma_j - \delta_{jm}\frac{1}{2}\gamma_l\gamma_k$$

One could instead use the commutation relations for the $\mathfrak{so}(n)$ Lie algebra satisfied by the basis elements ϵ_{jk} corresponding to infinitesimal rotations. One must get identical commutation relations for the $-\frac{1}{2}\gamma_j\gamma_k$ and can show that these are the relations needed for commutators of $\Gamma^+(\xi_j\xi_k)$ and $\Gamma^+(\xi_l\xi_m)$. $\square$

Note that here we are not introducing the factors of i into the definition of quantization that in the bosonic case were necessary to get a unitary representation of the Lie group corresponding to the real Heisenberg Lie algebra $\mathfrak{h}_{2d+1}$. In the bosonic case we worked with all complex linear combinations of powers of the Q_j, P_j (the complex Weyl algebra Weyl$(2d, \mathbf{C})$), and thus had to identify the specific complex linear combinations of these that gave unitary representations of the Lie algebra $h_{2d+1} \rtimes \mathfrak{sp}(2d, \mathbf{R})$. Here we are not complexifying for now, but working with the real Clifford algebra Cliff$(r, s, \mathbf{R})$, and it is the irreducible representations of this algebra that provide an analog of the unique interesting irreducible representation of $\mathfrak{h}_{2d+1}$. In the Clifford algebra case, the may be on real vector spaces, with no analog of the unitarity property of the $\mathfrak{h}_{2d+1}$ representation.

In the bosonic case we found that $Sp(2d, \mathbf{R})$ acted on the bosonic dual phase space, preserving the antisymmetric bilinear form Ω that determined the Lie algebra $\mathfrak{h}_{2d+1}$, so it acted on this Lie algebra by automorphisms. We saw (see chapter 4.2.4) that intertwining operators there gave us a representation of the double cover of $Sp(2d, \mathbf{R})$ (the metaplectic representation), with the Lie algebra representation given by the quantization of quadratic functions of the q_j, p_j phase space coordinates. There is a closely analogous story in the fermionic case, where $SO(r, s, \mathbf{R})$ acts on the fermionic phase space V, preserving the symmetric bilinear form $(\cdot, \cdot)$ that determines the Clifford algebra relations. Here a representation of the spin group $Spin(r, s, \mathbf{R})$ double covering $SO(r, s, \mathbf{R})$ is constructed using intertwining operators, with the Lie algebra representation given by quadratic combinations of the quantizations of the fermionic coordinates ξ_j.

In order to have a full construction of a quantization of a pseudo-classical system, we need to construct the $\Gamma^+(\xi_j)$ as linear operators on a state space. As mentioned in section 5.2.1, it can be shown that the real Clifford algebras Cliff$(r, s, \mathbf{R})$ are isomorphic to either one or two copies of the matrix algebras $M(2^l, \mathbf{R}), M(2^l, \mathbf{C})$, or $M(2^l, \mathbf{H})$, with the power l depending on r, s. The irreducible representations of such a matrix algebra are just the column vectors of dimension 2^l, and there will be either one or two such irreducible representations for Cliff$(r, s, \mathbf{R})$ depending on the number of copies of the matrix algebra. This is the fermionic analog of the Stone-von Neumann uniqueness result in the bosonic case.

5.3.2 Two examples

Quantization of the pseudo-classical spin

As an example, one can consider the quantization of the pseudo-classical spin degree of freedom of section 5.1.3. In that case Γ^+ takes values in Cliff$(3, 0, \mathbf{R})$, for which an explicit identification with the algebra $M(2, \mathbf{C})$ of two by two complex matrices was given in section 5.2.1. One has

$$\Gamma^+(\xi_j) = \frac{1}{\sqrt{2}}\gamma_j = \frac{1}{\sqrt{2}}\sigma_j$$

and the Hamiltonian operator is

$$\begin{aligned}-iH = \Gamma^+(h) =& \Gamma^+(B_{12}\xi_1\xi_2 + B_{13}\xi_1\xi_3 + B_{23}\xi_2\xi_3)\\ =&\frac{1}{2}(B_{12}\sigma_1\sigma_2 + B_{13}\sigma_1\sigma_3 + B_{23}\sigma_2\sigma_3)\\ =&i\frac{1}{2}(B_1\sigma_1 + B_2\sigma_2 + B_3\sigma_3)\end{aligned}$$

Physically this describes a spin-$\frac{1}{2}$ degree of freedom in a magnetic field, with fixed position (imagine an infinitely heavy spin-$\frac{1}{2}$ paritcle).

The pseudo-classical equation of motion

$$\frac{d}{dt}\xi_j(t) = -\{h, \xi_j\}_+$$

after quantization becomes the Heisenberg picture equation of motion for the spin operators

$$\frac{d}{dt}\mathbf{S}_H(t) = -i[\mathbf{S}_H \cdot \mathbf{B}, \mathbf{S}_H]$$

for the case of Hamiltonian

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}$$

and magnetic moment operator

$$\boldsymbol{\mu} = \mathbf{S}$$

Here the state space is $\mathcal{H} = \mathbf{C}^2$, with an explicit choice of basis given by our chosen identification of Cliff$(3, 0, \mathbf{R})$ with two by two complex matrices. In the next sections we will consider the case of an even dimensional fermionic phase space, but there provide a basis-independent construction of the state space and the action of the Clifford algebra on it.

The Fermionic Oscillator

The simple change in the harmonic oscillator problem that takes one from bosons to fermions is the replacement of the bosonic annihilation and creation operators a and $a^\dagger$ by fermionic annihilation and creation operators called a_F and ${a_F}^\dagger$, and replacement of the commutator

$$[A, B] \equiv AB - BA$$

of operators by the anticommutator

$$[A, B]_+ \equiv AB + BA$$

The commutation relations are now (for $d = 1$, a single degree of freedom)

$$[a_F, a_F^\dagger]_+ = \mathbf{1}, \quad [a_F, a_F]_+ = 0, \quad [a_F^\dagger, a_F^\dagger]_+ = 0$$

with the last two relations implying that $a_F^2 = 0$ and $(a_F^\dagger)^2 = 0$

The fermionic number operator

$$N_F = a_F^\dagger a_F$$

now satisfies

$$N_F^2 = a_F^\dagger a_F a_F^\dagger a_F = a_F^\dagger(\mathbf{1} - a_F^\dagger a_F)a_F = N_F - {a_F^\dagger}^2 a_F^2 = N_F$$

(using the fact that $a_F^2 = {a_F^\dagger}^2 = 0$). So one has

$$N_F^2 - N_F = N_F(N_F - \mathbf{1}) = 0$$

which implies that the eigenvalues of N_F are just 0 and 1. We'll denote eigenvectors with such eigenvalues by $|0\rangle$ and $|1\rangle$. The simplest representation of the operators a_F and $a_F^\dagger$ on a complex vector space $\mathcal{H}_F$ will be on $\mathbf{C}^2$, and choosing the basis

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

the operators are represented as

$$a_F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a_F^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N_F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since

$$H = \frac{1}{2}(a_F^\dagger a_F + a_F a_F^\dagger)$$

is just $\frac{1}{2}$ the identity operator, to get a non-trivial quantum system, instead we make a sign change and set

$$H = \frac{1}{2}(a_F^\dagger a_F - a_F a_F^\dagger) = N_F - \frac{1}{2}\mathbf{1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

The energies of the energy eigenstates $|0\rangle$ and $|1\rangle$ will then be $\pm\frac{1}{2}$ since

$$H|0\rangle = -\frac{1}{2}|0\rangle, \quad H|1\rangle = \frac{1}{2}|1\rangle$$

Taking complex linear combinations of the operators

$$a_F, a_F^\dagger, N_F, \mathbf{1}$$

we get all linear transformations of $\mathcal{H}_F = \mathbf{C}^2$ (so this is an irreducible representation of the algebra of these operators). The relation to the Pauli matrices is

$$a_F^\dagger = \frac{1}{2}(\sigma_1 + i\sigma_2), \quad a_F = \frac{1}{2}(\sigma_1 - i\sigma_2), \quad H = \frac{1}{2}\sigma_3$$

For the case of d degrees of freedom one has

Definition (Canonical anticommutation relations)**.** *A set of* $2d$ *operators*

$$a_{Fj},\ a_{Fj}^\dagger,\quad j=1,\dots,d$$

is said to satisfy the canonical anticommutation relations (CAR) when one has

$$[a_{Fj},a_{Fk}^\dagger]_+=\delta_{jk}\mathbf{1},\quad [a_{Fj},a_{Fk}]_+=0,\quad [a_{Fj}^\dagger,a_{Fk}^\dagger]_+=0$$

In this case one may choose as the state space the tensor product of N copies of the single fermionic oscillator state space

$$\mathcal{H}_F=(\mathbf{C}^2)^{\otimes d}=\underbrace{\mathbf{C}^2\otimes\mathbf{C}^2\otimes\cdots\otimes\mathbf{C}^2}_{d\text{ times}}$$

The dimension of $\mathcal{H}_F$ will be 2^d. On this space an explicit construction of the operators a_{Fj} and $a_{Fj}^\dagger$ in terms of Pauli matrices is

$$a_{Fj}=\underbrace{\sigma_3\otimes\sigma_3\otimes\cdots\otimes\sigma_3}_{j-1\text{ times}}\otimes\begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}$$

$$a_{Fj}^\dagger=\underbrace{\sigma_3\otimes\sigma_3\otimes\cdots\otimes\sigma_3}_{j-1\text{ times}}\otimes\begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}$$

The factors of σ_3 are there as one possible way to ensure that

$$[a_{Fj},a_{Fk}]_+=[a_{Fj}^\dagger,a_{Fk}^\dagger]_+=[a_{Fj},a_{Fk}^\dagger]_+=0$$

are satisfied for $j\neq k$ since then one will get in the tensor product factors

$$[\sigma_3,\begin{pmatrix}0&0\\1&0\end{pmatrix}]_+=0\quad\text{or}\quad[\sigma_3,\begin{pmatrix}0&1\\0&0\end{pmatrix}]_+=0$$

While this sort of tensor product construction is useful for discussing the physics of multiple qubits, in general it is easier to not work with large tensor products, and the Fock space formalism we will describe in section 5.3.4 avoids this.

The number operators will be

$$N_{Fj}=a_{Fj}^\dagger a_{Fj}$$

These will commute with each other, so can be simultaneously diagonalized, with eigenvalues $n_j=0,1$. One can take as a basis of $\mathcal{H}_F$ the 2^d states

$$|n_1,n_2,\cdots,n_d\rangle$$

which are the natural basis states for $(\mathbf{C}^2)^{\otimes d}$ given by d choices of either $|0\rangle$ or $|1\rangle$.

As an example, for the case $d=3$ the picture

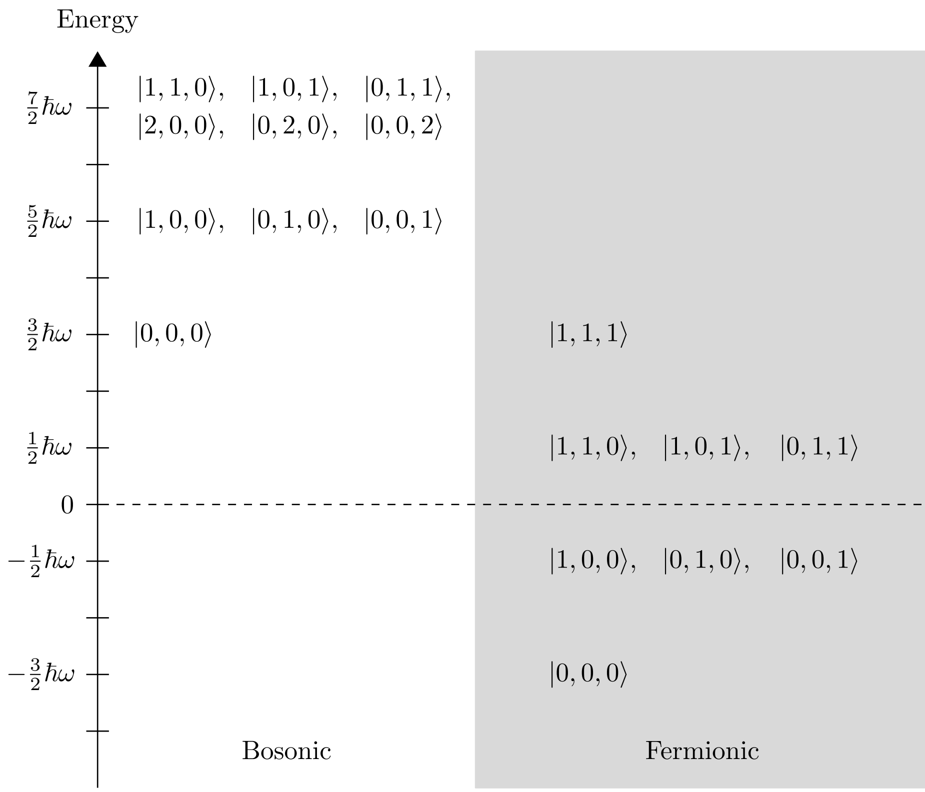


Figure 5.2: $N = 3$ oscillator energy eigenstates.

shows the pattern of states and their energy levels for the bosonic and fermionic cases. In the bosonic case the lowest energy state is at positive energy and there are an infinite number of states of ever increasing energy. In the fermionic case the lowest energy state is at negative energy, with the pattern of energy eigenvalues of the finite number of states symmetric about the zero energy level.

5.3.3 The Schrödinger representation for fermions: ghosts

We would like to construct representations of Cliff$(r, s, \mathbf{R})$ and thus fermionic state spaces by using analogous constructions to the Schrödinger and Bargmann-Fock ones in the bosonic case. The Schrödinger construction took the state space $\mathcal{H}$ to be a space of functions on a subspace of the classical phase space which had the property that the basis coordinate functions Poisson-commuted. Two examples of this are the position coordinates q_j, since $\{q_j, q_k\} = 0$, or the momentum coordinates p_j, since $\{p_j, p_k\} = 0$. Unfortunately, for symmetric bilinear forms $(\cdot,\cdot)$ of definite sign, such as the positive definite case Cliff$(m, \mathbf{R})$, the only subspace the bilinear form is zero on is the zero subspace.

To get an analog of the bosonic situation, one needs to take the case of signature (d, d). The fermionic phase space will then be $2d$ dimensional, with d dimensional subspaces on which $(\cdot,\cdot)$ and thus the fermionic Poisson bracket is zero. Quantization will give the Clifford algebra

$$\text{Cliff}(d, d, \mathbf{R}) = M(2^d, \mathbf{R})$$

which has just one irreducible representation, $\mathbf{R}^{2^d}$. This can be complexified to get a complex state space

$$\mathcal{H}_F = \mathbf{C}^{2^d}$$

This state space will come with a representation of $Spin(d, d, \mathbf{R})$ from exponentiating quadratic combinations of the generators of Cliff$(d, d, \mathbf{R})$. However, this is a non-compact group, and one can show that on general grounds it cannot have faithful unitary finite dimensional representations, so there must be a problem with unitarity.

To see what happens explicitly, consider the simplest case $d = 1$ of one degree of freedom. In the bosonic case the classical phase space is $\mathbf{R}^2$, and quantization gives operators Q, P which in the Schrödinger representation act on functions of q, with $Q = q$ and $P = -i\frac{\partial}{\partial q}$. In the fermionic case with signature $(1, 1)$, basis coordinate functions on phase space are ξ_1, ξ_2, with

$$\{\xi_1, \xi_1\}_+ = 1, \quad \{\xi_2, \xi_2\}_+ = -1, \quad \{\xi_1, \xi_2\}_+ = 0$$

Defining

$$\eta = \frac{1}{\sqrt{2}}(\xi_1 + \xi_2), \quad \pi = \frac{1}{\sqrt{2}}(\xi_1 - \xi_2)$$

one gets objects with fermionic Poisson bracket analogous to those of q and p

$$\{\eta, \eta\}_+ = \{\pi, \pi\}_+ = 0, \quad \{\eta, \pi\}_+ = 1$$

Quantizing, we get analogs of the Q, P operators

$$\widehat{\eta} = \Gamma^+(\eta) = \frac{1}{\sqrt{2}}(\Gamma^+(\xi_1) + \Gamma^+(\xi_2)), \quad \widehat{\pi} = \Gamma^+(\pi) = \frac{1}{\sqrt{2}}(\Gamma^+(\xi_1) - \Gamma^+(\xi_2))$$

which satisfy anticommutation relations

$$\widehat{\eta}^2 = \widehat{\pi}^2 = 0, \quad \widehat{\eta}\widehat{\pi} + \widehat{\pi}\widehat{\eta} = 1$$

and can be realized as operators on the space of functions of one fermionic variable η as

$$\widehat{\eta} = \text{multiplication by } \eta, \quad \widehat{\pi} = \frac{\partial}{\partial \eta}$$

This state space is two complex dimensional, with an arbitrary state

$$f(\eta) = c_1 1 + c_2 \eta$$

with c_j complex numbers. The inner product on this space is given by the fermionic integral

$$(f_1(\eta), f_2(\eta)) = \int f_1^*(\eta) f_2(\eta) d\eta$$

with

$$f^*(\xi) = \bar{c}_1 1 + \bar{c}_2 \eta$$

With respect to this inner product, one has

$$(1,1) = (\eta,\eta) = 0, \quad (1,\eta) = (\eta,1) = 1$$

This inner product is indefinite and can take on negative values, since

$$(1-\eta, 1-\eta) = -2$$

Having such negative-norm states ruins any standard interpretation of this as a physical system, since this negative number is supposed to the probability of finding the system in this state. Such quantum systems are called "ghosts", and do have applications in the description of various quantum systems, but only when a mechanism exists for the negative-norm states to cancel or otherwise be removed from the physical state space of the theory.

5.3.4 Spinors and the Bargmann-Fock construction

While the fermionic analog of the Schrödinger construction does not give a unitary representation of the spin group, it turns out that the fermionic analog of the Bargmann-Fock construction does, on the fermionic oscillator state space discussed in chapter 5.3.2. This will work for the case of a positive definite symmetric bilinear form $(\cdot,\cdot)$. Note though that this will only work for fermionic phase spaces $\mathbf{R}^m$ with m even, since a complex structure on the phase space is needed.

The corresponding pseudo-classical system will be the classical fermionic oscillator studied in section 5.1.3. Recall that this uses a choice of complex structure J on the fermionic phase space $\mathbf{R}^{2d}$, with the standard choice $J = J_0$ coming from the relations

$$\theta_j = \frac{1}{\sqrt{2}}(\xi_{2j-1} - i\xi_{2j}), \quad \overline{\theta}_j = \frac{1}{\sqrt{2}}(\xi_{2j-1} + i\xi_{2j}) \tag{5.9}$$

for $j = 1, \ldots, d$ between real and complex coordinates. Here $(\cdot,\cdot)$ is positive-definite, and the ξ_j are coordinates with respect to an orthonormal basis, so we have the standard relation $\{\xi_j, \xi_k\}_+ = \delta_{jk}$ and the $\theta_j, \overline{\theta}_j$ satisfy

$$\{\theta_j, \theta_k\}_+ = \{\overline{\theta}_j, \overline{\theta}_k\}_+ = 0, \quad \{\overline{\theta}_j, \theta_k\}_+ = \delta_{jk}$$

In the bosonic case, extending the Poisson bracket from M to $M \otimes \mathbf{C}$ by complex linearity gave an indefinite Hermitian form on $M \otimes \mathbf{C}$

$$\langle \cdot, \cdot \rangle = i\{\bar{\cdot}, \cdot\} = i\Omega(\bar{\cdot}, \cdot)$$

positive definite on M_J^+ for positive J. In the fermionic case we can extend the fermionic Poisson bracket from V to $V \otimes \mathbf{C}$ by complex linearity, getting a Hermitian form on $V \otimes \mathbf{C}$

$$\langle \cdot, \cdot \rangle = \{\bar{\cdot}, \cdot\}_+ = (\bar{\cdot}, \cdot)$$

This is positive definite on V_J^+ (and also on V_J^-) if the initial symmetric bilinear form was positive.

To quantize this system we need to find operators $\Gamma^+(\theta_j)$ and $\Gamma^+(\overline{\theta}_j)$ that satisfy

$$[\Gamma^+(\theta_j), \Gamma^+(\theta_k)]_+ = [\Gamma^+(\overline{\theta}_j), \Gamma^+(\overline{\theta}_k)]_+ = 0$$
$$[\Gamma^+(\overline{\theta}_j), \Gamma^+(\theta_k)]_+ = \delta_{jk}\mathbf{1}$$

but these are just the CAR satisfied by fermionic annihilation and creation operators. We can choose

$$\Gamma^+(\theta_j) = a_{F}{}_j^\dagger, \quad \Gamma^+(\overline{\theta}_j) = a_{F\,j}$$

and realize these operators as

$$a_{F\,j} = \frac{\partial}{\partial \theta_j}, \quad a_{F}{}_j^\dagger = \text{multiplication by } \theta_j$$

on the state space $\Lambda^* \mathbf{C}^d$ of polynomials in the anticommuting variables θ_j. This is a complex vector space of dimension 2^d, isomorphic with the state space $\mathcal{H}_F$ of the fermionic oscillator in d degrees of freedom, with the isomorphism given by

$$\begin{aligned}
1 &\leftrightarrow |0\rangle_F \\
\theta_j &\leftrightarrow a_{F}{}_j^\dagger |0\rangle_F \\
\theta_j\theta_k &\leftrightarrow a_{F}{}_j^\dagger a_{F}{}_k^\dagger |0\rangle_F \\
&\cdots \\
\theta_1 \ldots \theta_d &\leftrightarrow a_{F}{}_1^\dagger a_{F}{}_2^\dagger \cdots a_{F}{}_d^\dagger |0\rangle_F
\end{aligned}$$

where the indices $j, k, \ldots$ take values $1, 2, \ldots, d$ and satisfy $j < k < \cdots$.

If one defines a Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_F$ by taking these basis elements to be orthonormal, the operators $a_{F\,j}$ and $a_{F\,j}^\dagger$ will be adjoints with respect to this inner product. This same inner product can also be defined using fermionic integration by analogy with the Bargmann-Fock definition in the bosonic case as

$$\langle f_1(\theta_1, \cdots, \theta_d), f_2(\theta_1, \cdots, \theta_d) \rangle = \int e^{-\sum_{j=1}^d \theta_j \overline{\theta}_j} \overline{f_1} f_2 d\overline{\theta} d\theta_1 \cdots d\overline{\theta}_d d\theta_d \qquad (5.10)$$

where f_1 and f_2 are complex linear combinations of the powers of the anticommuting variables θ_j. For the details of the construction of this inner product, see chapter 7.2 of [15] or chapters 7.5 and 7.6 of [19]. We will denote this state space as $\mathcal{F}_d^+$ and refer to it as the fermionic Fock space. Since it is finite dimensional, there is no need for a completion as in the bosonic case.

The quantization using fermionic annihilation and creation operators given here provides an explicit realization of a representation of the Clifford algebra Cliff$(2d, \mathbf{R})$ on the complex vector space $\mathcal{F}_d^+$. The generators of the Clifford algebra are identified as operators on $\mathcal{F}_d^+$ by

$$\gamma_{2j-1} = \sqrt{2}\Gamma^+(\xi_{2j-1}) = \sqrt{2}\Gamma^+\left(\frac{1}{\sqrt{2}}(\theta_j + \overline{\theta}_j)\right) = a_{Fj} + a_{Fj}^\dagger$$

$$\gamma_{2j} = \sqrt{2}\Gamma^+(\xi_{2j}) = \sqrt{2}\Gamma^+\left(\frac{i}{\sqrt{2}}(\theta_j - \overline{\theta}_j)\right) = i(a_{Fj}^\dagger - a_{Fj})$$

Quantization of the pseudo-classical fermionic oscillator Hamiltonian h of section 5.1.3 gives

$$\Gamma^+(h) = \Gamma^+\left(-\frac{i}{2}\sum_{j=1}^{d}(\theta_j\overline{\theta}_j - \overline{\theta}_j\theta_j)\right) = -\frac{i}{2}\sum_{j=1}^{d}(a_{Fj}^\dagger a_{Fj} - a_{Fj}a_{Fj}^\dagger) = -iH \tag{5.11}$$

where H is the Hamiltonian operator for the fermionic oscillator used in section 5.3.2.

Taking quadratic combinations of the operators γ_j provides a representation of the Lie algebra $\mathfrak{so}(2d) = \mathfrak{spin}(2d)$. This representation exponentiates to a representation up to sign of the group $SO(2d)$, and a true representation of its double cover $Spin(2d)$. The representation that we have constructed here on the fermionic oscillator state space $\mathcal{F}_d^+$ is called the spinor representation of $Spin(2d)$, and we will sometimes denote $\mathcal{F}_d^+$ with this group action as S.

In the bosonic case, $\mathcal{H} = \mathcal{F}_d$ is an irreducible representation of the Heisenberg group, but as a representation of $Mp(2d, \mathbf{R})$, it has two irreducible components, corresponding to even and odd polynomials. The fermionic analog is that $\mathcal{F}_d^+$ is irreducible under the action of the Clifford algebra Cliff$(2d, \mathbf{C})$. One way to show this is to show that Cliff$(2d, \mathbf{C})$ is isomorphic to the matrix algebra $M(2^d, \mathbf{C})$ and its action on $\mathcal{H}_F = \mathbf{C}^{2^d}$ is isomorphic to the action of matrices on column vectors.

While $\mathcal{F}_d^+$ is irreducible as a representation of the Clifford algebra, it is the sum of two irreducible representations of $Spin(2d)$, the so-called "half-spinor" representations. $Spin(2d)$ is generated by quadratic combinations of the Clifford algebra generators, so these will preserve the subspaces

$$S_+ = \text{span}\{|0\rangle_F, \ \ a_{Fj}^\dagger a_{Fk}^\dagger |0\rangle_F, \cdots\} \subset S = \mathcal{F}_d^+$$

and

$$S_- = \text{span}\{a_{Fj}^\dagger |0\rangle_F, \ \ a_{Fj}^\dagger a_{Fk}^\dagger a_{Fl}^\dagger |0\rangle_F, \cdots\} \subset S = \mathcal{F}_d^+$$

corresponding to the action of an even or odd number of creation operators on $|0\rangle_F$. This is because quadratic combinations of the $a_{Fj}, a_{Fj}^\dagger$ preserve the parity of the number of creation operators used to get an element of S by action on $|0\rangle_F$.

5.3.5 Complex structures, $U(d) \subset SO(2d)$ and the spinor representation

The construction of the spinor representation given here has used a specific choice of the $\theta_j, \overline{\theta}_j$ (see equations 5.9) and the fermionic annihilation and creation operators. This corresponds to a standard choice of complex structure J_0, which appears in a manner closely parallel to that of the Bargmann-Fock case of section 4.3.2. The difference here is that, for the analogous construction of spinors, the complex structure J must be chosen so as to preserve not an antisymmetric bilinear form Ω, but the inner product, and one has

$$(J(\cdot), J(\cdot)) = (\cdot, \cdot)$$

We will here restrict to the case of $(\cdot, \cdot)$ positive definite, and unlike in the bosonic case, no additional positivity condition on J will then be required.

J splits the complexification of the real phase space $V = \mathbf{R}^{2d}$ with its coordinates ξ_j into a d dimensional complex vector space on which $J = +i$ and a conjugate complex vector space on which $J = -i$. As in the bosonic case one has

$$\mathcal{V} \otimes \mathbf{C} = V_J^+ \oplus V_J^-$$

and quantization of vectors in V_J^+ gives linear combinations of creation operators, while vectors in V_J^- are taken to linear combinations of annihilation operators. The choice of J is reflected in the existence of a distinguished direction $|0\rangle_F$ in the spinor space $S = \mathcal{F}_d^+$ which is determined (up to phase) by the condition that it is annihilated by all linear combinations of annihilation operators.

The choice of J also picks out a subgroup $U(d) \subset SO(2d)$ of those orthogonal transformations that commute with J. Just as in the bosonic case, two different representations of the Lie algebra $\mathfrak{u}(d)$ of $U(d)$ are used:

- The restriction to $\mathfrak{u}(d) \subset \mathfrak{so}(2d)$ of the spinor representation described above. This exponentiates to give a representation not of $U(d)$, but of a double cover of $U(d)$ that is a subgroup of $Spin(2d)$.

- By normal ordering operators, one shifts the spinor representation of $\mathfrak{u}(d)$ by a constant and gets a representation that exponentiates to a true representation of $U(d)$. This representation is reducible, with irreducible components the $\Lambda^k(\mathbf{C}^d)$ for $k = 0, 1, \ldots, d$.

In both cases the representation of $\mathfrak{u}(d)$ is constructed using quadratic combinations of annihilation and creation operators involving one annihilation operator and one creation operator, operators which annihilate $|0\rangle_F$. Non-zero pairs of two creation operators act non-trivially on $|0\rangle_F$, corresponding to the fact that elements of $SO(2d)$ not in the $U(d)$ subgroup take $|0\rangle_F$ to a different state in the spinor representation.

5.4 Spinor-oscillator analogy

The oscillator representation of a symplectic group that we have been discussing is closely analogous to the spinor representation of the orthogonal group. Here we'll make this analogy very explicit. This parallelism is well-known in physics, where the "canonical formalism" in quantum mechanics comes in both a "bosonic" version, with canonical commutation relations, and a "fermionic" version, with canonical anti-commutation relations. Much of this material is worked out in great detail in [18].

5.4.1 Classical theory, Lie groups and Lie algebras

Q: Symmetric non-degenerate bilinear form on $V = \mathbf{R}^m$	Ω: Antisymmetric non-degenerate bilinear form on $M = \mathbf{R}^{2n}$
Lie group $SO(d)$ preserving Q, with Lie algebra $\mathfrak{so}(m)$.	Lie group $Sp(2n,\mathbf{R})$ preserving Ω, with Lie algebra $\mathfrak{sp}(2n)$
$\pi_1(SO(m)) = \mathbf{Z}_2$.	$\pi_1(Sp(2n),\mathbf{R}) = \mathbf{Z}$.
$Spin(m)$, double cover of $SO(m)$.	$Mp(2n,\mathbf{R})$, double cover of $Sp(2n,\mathbf{R})$.
$\Lambda^*(V)$: anti-symmetric algebra on V. Polynomials in "anti-commuting variables $\xi_j,\ j = 1,2,\cdots m$. For physicists these are "fermionic" variables.	$S^*(M)$: symmetric algebra on M. Polynomial functions on M^*. Generated by a basis $q_j, p_k,\ j,k = 1,2,\cdots n$ of M. For physicists these are "bosonic" variables.
Poisson bracket $\{\cdot,\cdot\}_+$. Lie bracket for Lie super-algebra of "anti-commuting functions"on V^*, determined by Q. $$\{v_1,v_2\}_+ = Q(v_1,v_2)$$	Poisson bracket $\{\cdot,\cdot\}$. Lie bracket for Lie algebra of functions on M^*, determined by Ω.
Lie superalgebra of anticommuting polynomials on V^* of degree $0,1,2$. Semi-direct product of a Lie superalgebra (degree 0 and 1) and the orthogonal Lie algebra $\mathfrak{so}(m,\mathbf{R})$ (degree 2).	Lie algebra of polynomials on M^* of degree $0,1,2$. Semi-direct product of the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ (degree 0 and 1) and the symplectic Lie algebra $\mathfrak{sp}(2n,\mathbf{R})$ (degree 2).
Pseudo-classical mechanics.	Classical mechanics.

5.4.2 Quantum theory and representations

Spin representation S (unitary) on a complex vector space of dimension 2^d for $m = 2d$ even.	Oscillator representation (unitary) on $\mathcal{H}$, an infinite-dimensional Hilbert space.
Clifford algebra Cliff$(m,\mathbf{C})$. For $m =$	Weyl algebra $U(\mathfrak{h}_{2n+1})/(Z-1)$. This

$2d$ even this is the algebra $End(S)$, isomorphic to the matrix algebra $M(2^d, \mathbf{C})$.	algebra is infinite-dimensional over $\mathbf{C}$.
The group $SO(m)$ acts by automorphisms on Cliff$(m, \mathbf{C})$.	The group $Sp(2n, \mathbf{R})$ acts by automorphism on the Weyl algebra.
For $m = 2d$ even, Cliff$(2d, \mathbf{C})$ has a unique irreducible module, the spin module S. This is the spin representation as a Lie algebra representation of $\mathfrak{so}(2d)$. Integrating to the group, one gets a projective (up to $\pm$) representation of $SO(2d)$, a true representation of the double cover $Spin(2d)$.	Stone-von Neumann theorem: the Weyl algebra has an essentially unique irreducible module $\mathcal{H}$ that integrates to a representation of the Heisenberg group on $\mathcal{H}$. Integrating to the group, one gets a projective (up to $\pm$) representation of $Sp(2n, \mathbf{R})$, a true representation of the double cover $Mp(2n, \mathbf{R})$.
For $m = 2d$ even, the spin representation has two irreducible components, the half-spinors S^+, S^-, each of dimension 2^{d-1}	The oscillator representation has two irreducible components (an "even" and an "odd" component).
Generators γ_j of the Clifford algebra. On the spinor module S, identifying the Clifford algebra with a matrix algebra, these are the physicist's Dirac γ-matrices.	Generators Q_j, P_k of the Weyl algebra.
In even dimension, the Lie algebra representation operators for the spin representation are given by quadratic combinations of γ-matrices.	The Lie algebra representation operators for the oscillator representation are given by quadratic combinations of the Q_j, P_k operators.
Spin 1/2 degree of freedom in m dimensions.	Harmonic oscillator with n degrees of freedom.

5.4.3 Real and complex polarizations

When Q has signature (d, d), choosing a real polarization $V = L \oplus L^*$ ($Q = 0$ on L and on L^*), one can realize the spinor module as anticommuting functions on L. This will be an irreducible representation of the real form $SO(d, d)$, non-unitary.	Choosing a real polarization $M = L \oplus L^*$ one can realize (the Schrödinger representation) the Q_j, P_j operators respectively as multiplication and differentiation operators on $L^2(L)$. This representation will be unitary, both as a representation of the Heisenberg group and the metaplectic group.
For $m = 2d$ even, an orthogonal complex structure on V is a linear map J satisfying $J^2 = -\mathbf{1}$ and preserving the bilinear form Q. This picks out a	A symplectic complex structure on M is a linear map J satisfying $J^2 = -\mathbf{1}$ and preserving the bilinear form Ω. This picks out a $U(n) \subset Sp(2n, \mathbf{R})$. Such

$U(d) \subset SO(2d)$ and the space of such complex structures is the compact space $SO(2d)/U(d)$.	J satisfying the positivity conditions $S(\cdot, J\cdot)$ positive are parametrized by the non-compact space $Sp(2n, \mathbf{R})/U(n)$.
For $m = 2d$ even, such a J gives a complex polarization $V \otimes \mathbf{C} = W_J^+ \oplus W_J^-$ ($\pm i$ eigenspaces of J).	Such a J gives a complex polarization $M \otimes \mathbf{C} = W_J^+ \oplus W_J^-$ ($\pm i$ eigenspaces of J).
For $m = 2d$ even, taking complex linear combinations of the γ_j in W_J^+ one can form adjoint operators $a_j, a_j^\dagger$ on the spinor module, satisfying the canonical anti-commutation relations $$[a_j, a_k^\dagger]_+ = \delta_{jk}\mathbf{1}$$	Taking complex linear combinations of the Q_j, P_j in W_J^+ one can form adjoint operators $a_j, a_j^\dagger$ on the oscillator representation, satisfying the canonical commutation relations $$[a_j, a_k^\dagger] = \delta_{jk}\mathbf{1}$$
For each J there is a unique (up to scalar) vector in S annihilated by all the a operators. These are fibers of the line bundle $\Lambda^d(W_J^+)^{-\frac{1}{2}}$.	For each J there is a unique (up to scalar) vector (vacuum vector) in $\mathcal{H}$ annihilated by all the a operators. These are fibers of the line bundle $\Lambda^n(W_J^+)^{\frac{1}{2}}$.
Applying $a^\dagger$ operators $$S = \Lambda^*(W_J^+) \otimes (\Lambda^d(W_J^+)^{-\frac{1}{2}}$$	Applying $a^\dagger$ operators $$\mathcal{H} = S^*(W_J^+) \otimes (\Lambda^n(W_J^+)^{\frac{1}{2}}$$
Half-spinors are holomorphic sections of the line bundle $\Lambda^n(W_J^+)^{-\frac{1}{2}}$ over $SO(2d)/U(d)$ $$S^+ = \Gamma_{hol}(\Lambda^n(W_J^+)^{-\frac{1}{2}})$$	The even component of the oscillator representation is holomorphic sections of the line bundle $\Lambda^n(W_J^+)^{\frac{1}{2}}$ over $Sp(2n, \mathbf{R})/U(n)$. $$\mathcal{H}_{even} = \Gamma_{hol}(\Lambda^n(W_J^+)^{-\frac{1}{2}})$$

5.5 For further reading

Some more detail about spin groups and the relationship between geometry and Clifford algebras can be found in [8], and an exhaustive reference is [12]. A good source for more details about Clifford algebras and spinors is chapter 12 of the representation theory textbook [16]. For the details of what happens for all Cliff$(r, s, \mathbf{R})$, another good source is chapter 1 of [8].

For more about pseudo-classical mechanics and quantization, see [15] chapter 7 or the very readable original reference [1]. The Clifford algebra and fermionic quantization are discussed in chapter 20.3 of [5]. The fermionic quantization map, Clifford algebras, and the spinor representation are discussed in detail in [10]. For another discussion of the spinor representation from a similar point of view to the one here, see chapter 12 of [16]. Chapter 12 of [13] contains an extensive discussion of the role of different complex structures in the construction of the spinor representation.

Chapter 6

Free particles in three space dimensions

Generalizing the discussion of the quantum free particle in section 3.1.1 to the physical case of three spatial dimensions, states will be functions (or distributions) on $\mathbf{R}^3$ and the Hamiltonian operator will be

$$H = \frac{1}{2m}(P_1^2 + P_2^2 + P_3^2) = -\frac{\hbar^2}{2m}\nabla^2$$

Fourier transforming to momentum space, the energy eigenvalue equation becomes

$$\frac{1}{2m}(P_1^2 + P_2^2 + P_3^2)\widetilde{\psi}(\mathbf{p}) = \frac{1}{2m}|\mathbf{p}|^2\widetilde{\psi}(\mathbf{p}) = E\widetilde{\psi}(\mathbf{p})$$

States of energy E ($E \geq 0$) will be distributions on momentum space supported on the sphere of radius $\sqrt{2mE}$. Restricting to those distributions of the form

$$\widetilde{\psi}(\mathbf{p}) = \widetilde{\psi}_E(\phi, \theta)\delta(\rho - \sqrt{2mE})$$

in spherical coordinates, the space $\mathcal{H}_E$ of energy eigenstates of energy E can be identified with the space of functions $\widetilde{\psi}_E(\phi, \theta)$ on the sphere of radius $\sqrt{2mE}$. This space has a Hermitian inner product given by integration over the sphere and one can take $\mathcal{H}_E$ to be the square-integrable functions.

6.1 Irreducible representations of the Euclidean group E_3

There is a unitary representation on $\mathcal{H}_E$ of the Euclidean group $E_3 = \mathbf{R}^3 \rtimes SO(3)$, the semi-direct product of translations and rotations. This is the representation induced from the action of E_3 on $\mathbf{R}^3$. Recall that if a group G acts on a space X, it acts on functions on X by

$$f(x) \to f(g^{-1} \cdot x)$$

In this case the action of $(\mathbf{a}, R) \in E_3$ (translation by $\mathbf{a}$ and rotation by R) on momentum space wave functions will be by

$$\widetilde{\psi}(\mathbf{p}) \rightarrow e^{-i\mathbf{a}\cdot\mathbf{p}}\widetilde{\psi}(R^{-1}\mathbf{p})$$

One can show that this representation is irreducible and that one gets distinct representations for each $R > 0$.

Note that what we are doing here is just taking the Schrödinger representation in the case $n = 3$ and restricting it to a representation of E_3, which breaks up into a continuous sum over the $\mathcal{H}_E$ for $E > 0$. Here the $\mathbf{R}^3$ is the subgroup of the Heisenberg group that acts by translations in $\mathbf{q}$. The rotations $SO(3)$ are the subgroup of $Sp(6, \mathbf{R})$ given by rotating both $\mathbf{q}$ and $\mathbf{p}$ (preserving their dot product). On this subgroup the metaplectic group is a trivial double cover, so one can use $Sp(6, \mathbf{R})$ instead of the double cover $Mp(6, \mathbf{R})$.

Another way to get this decomposition into irreducibles is to consider the Casimir operators of the Lie algebra of $E(3)$. These are operators quadratic in the generators that commute with the generators. There are two of them:

$$|\mathbf{P}|^2 = P_1^2 + P_2^2 + P_3^2$$

and

$$\mathbf{J} \cdot \mathbf{P} = J_1P_1 + J_2P_2 + J_3P_3$$

Here J_j is the operator that generates the action of rotations about the j axis. By Schur's lemma, on an irreducible representation these operators will act by scalars, and these scalars will characterize the irreducible representation. On our representation on $\mathcal{H}_E$, the first of these will act by $2mE$, the second can be shown to act by 0.

The general theory of representations of groups $N \rtimes K$, where N is commutative, says that irreducible representations of such a group will correspond to pairs consisting of

- Orbits of K on the character group $\widehat{N}$ of the abelian group N. Here the definition of the semi-direct product gives the action of K on N by automorphisms, which induces an action on the characters (homomorphisms $\chi : N \rightarrow \mathbf{C}$).

- Irreducible representations of the stabilizer group $K_\chi \subset K$, the subgroup of K that leaves a character in the orbit invariant. Physicists often call this the "little group".

In the case we are looking at, $N = \mathbf{R}^3$ and $K = SO(3)$. The complete list of irreducible representations is labeled by the pairs:

- The trivial orbit $0 \in \widehat{N}$ with stabilizer $K_0 = SO(3)$, which has irreducible representations labeled by $n = 0, 1, 2, \ldots$. These are just the irreducible representations of E_3 on which $\mathbf{R}^3$ acts trivially.

- Non-trivial orbits are spheres in the momentum space $\mathbf{R}^3$ (which is the character group of the position space $\mathbf{R}^3$). We'll label these by $E = \frac{|\mathbf{p}|^2}{2m}$. The stabilizer group of a vector $\mathbf{p}$ on the sphere is the $SO(2) \subset SO(3)$ subgroup of rotations about $\mathbf{p}$. Irreducible representations of this $SO(2)$ are labeled by integers h.

Of the representations in the second class, our representation on $\mathcal{H}_E$ is the one labeled by $(E, 0)$. To get others, we need to find representations with non-zero eigenvalues of the second Casimir $\mathbf{J} \cdot \mathbf{P}$. On wave-functions of momentum $\mathbf{p}$, this operator is the operator that generates rotations about $\mathbf{p}$ and will have eigenvalues $h|\mathbf{p}|$, where $h \in \mathbf{Z}$ is called the "helicity". One way to get such representations is to take wavefunctions in $\mathcal{H} \otimes \mathbf{C}^{|h|+1}$, with the Hamiltonian acting trivially on the second factor, but the rotations acting on it by the spin$|h|$ representation. This will break up into irreducibles of different helicities. In the next section we'll se how this works in the case $|h| = \frac{1}{2}$, where one needs to take the spin double cover of $SO(3)$ (and the corresponding double cover of $E(3)$).

6.2 The Dirac operator and spin $\frac{1}{2}$

In chapters 4 and 5 we saw how to quantize a classical system and get the Schrödinger representation describing a spin-less free particle, as well as how to quantize a pseudo-classical system and get a spin-$\frac{1}{2}$ degree of freedom. In this section we'll see that one can put these two together, getting a description of a free spin $\frac{1}{2}$ particle, with a Hamiltonian that now has a square root, providing a non-relativistic version of the Dirac equation.

One can take as phase space the conventional classical phase space with 3 degrees of freedom and coordinates q_j, p_j, together with the three-dimensional pseudo-classical phase space with coordinates ξ_j. The non-zero Poisson super-brackets are

$$\{q_j, p_k\}_{\pm} = \delta_{jk}, \quad \{\xi_j, \xi_k\}_{\pm} = \delta_{jk}$$

The Hamiltonian function for the free particle now has a square root $p_1\xi_1 + p_2\xi_2 + p_3\xi_3$ in the sense that

$$h = \frac{1}{2m}\{\sum_{j=1}^{3} p_j\xi_j, \sum_{k=1}^{3} p_k\xi_k\}_{\pm} = \frac{1}{2m}\sum_{j,k=1}^{3} p_j\{\xi_j, \xi_k\}_{\pm}p_k = \frac{1}{2m}\sum_{j=1}^{3} p_j^2$$

This is a simple example of what is called a "supersymmetry": by extending the usual Lie algebra to a Lie superalgebra, we are able to find a generator which in some sense is a "square root" of the generator of time translation.

Quantization takes

$$p_1\xi_1 + p_2\xi_2 + p_3\xi_3 \rightarrow \frac{1}{\sqrt{2}}\boldsymbol{\sigma} \cdot \mathbf{P}$$

and the Hamiltonian operator can be written

$$H = \frac{1}{2m}[\frac{1}{\sqrt{2}}\boldsymbol{\sigma} \cdot \mathbf{P}, \frac{1}{\sqrt{2}}\boldsymbol{\sigma} \cdot \mathbf{P}]_+ = \frac{1}{2m}(\boldsymbol{\sigma} \cdot \mathbf{P})^2 = \frac{1}{2m}(P_1^2 + P_2^2 + P_3^2)$$

One can define the Dirac operator in this context as

$$\partial\!\!\!/ = \sigma_1 \frac{\partial}{\partial q_1} + \sigma_2 \frac{\partial}{\partial q_2} + \sigma_3 \frac{\partial}{\partial q_3} = \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}$$

The Schrödinger equation now is called the "Pauli-Schrödinger" equation (or "Pauli equation") and given by

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\partial\!\!\!/^2\psi$$

where ψ is a two-component wave function.

This is nothing but two decoupled copies of the usual Schrödinger equation. The total angular momentum operator that generates rotations is different, since on the state space $\mathcal{H} = L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$, rotations act on the $\mathbf{C}^2$ not trivially, but by the spin representation. The helicity operator will be

$$\frac{1}{|\mathbf{P}|}\mathbf{J} \cdot \mathbf{P} = \frac{1}{|\mathbf{P}|}(\mathbf{L} + \frac{1}{2}\boldsymbol{\sigma}) \cdot \mathbf{P}$$

The term involving the orbital angular momentum $\mathbf{L}$ will act trivially, as in the spin-less case, but the other term will act with eigenvalues $\pm\frac{1}{2}$. The space of solutions with a fixed energy will decompose into two different irreducible representations of (the double-cover of) $E(3)$, distinguished by the eigenvalue of the helicity operator.

The Pauli-Schrödinger equation becomes much more interesting when one couples the free particle to an electromagnetic field. We will discuss this in detail in a later chapter, but the main point is that derivatives get replaced by covariant derivatives of a gauge field A, so

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - ieA_0, \;\; \boldsymbol{\nabla} \to \boldsymbol{\nabla} - ie\mathbf{A} \tag{6.1}$$

The Pauli-Schrödinger equation becomes

$$i(\frac{\partial}{\partial t} - ieA_0)\psi = -\frac{1}{2m}(\boldsymbol{\sigma} \cdot (\boldsymbol{\nabla} - ie\mathbf{A}))^2\psi = -\frac{1}{2m}((\boldsymbol{\nabla} - ie\mathbf{A})^2 + e\boldsymbol{\sigma} \cdot \mathbf{B})\psi \tag{6.2}$$

with the magnetic field $\mathbf{B}$ corresponding to the vector potential A now giving a non-trivial coupling between the components of ψ.

Chapter 7

Non-relativistic quantum field theory

In the previous chapters we have discussed the quantum theory of a non-relativistic single free particle, as well as the oscillator and spinor representations. In this chapter we'll put these two subjects together, getting quantum field theories describing arbitrary numbers of identical free particles, using the oscillator representation for bosons, the spinor representation for fermions. This subject is sometimes known as "second quantization", with first quantization giving solutions of the Schrödinger equation, second quantization treating this space of solutions as a classical or pseudo-classical phase space. Quantizing this phase space as a bosonic or fermionic oscillator, using an infinite-dimensional version of the Bargmann-Fock quantization, quantum field theory appears as an oscillator or spinor representation for an infinite number of degrees of freedom.

The fact that we are quantizing an infinite-dimensional phase space gives the subject a very different flavor. The material discussed in chapters 4 and 5 assumes a finite number of degrees of freedom and it is not clear how this generalizes (for instance, what is the right infinite-dimensional version of the metaplectic or spin group and their oscillator or spinor representations?). A crucial part of the story in finite dimensions, the Stone-von Neumann theorem for bosons or the analogous uniqueness of the spinor module for the Clifford algebra, no longer holds. This non-uniqueness means that for quantum field theories finding the correct state space becomes a major problem, even before one comes to questions about operators acting on it.

For free quantum field theories, which can be decomposed into an infinite number of finite-dimensional oscillator problems that do not interact with each other, the mathematical issues can be addressed. But for interacting field theories, corresponding physically to arbitrary numbers of quantum particles that are not free particles but that interact with each other, the problems are serious enough that no non-trivial interacting relativistic quantum field theory in four spacetime dimensions has yet been rigorously constructed. For weakly coupled particles approximate computations in principle are possible, but these require very careful treatment since a straightforward computation will give infinite results. We will not get into the details of methods for making sense of such calculations ("renormalization theory") in general, but will outline what is known for the Standard Model quantum field theory in later chapters. In this chapter we'll begin by restricting our attention mostly to free field theories and do the non-relativistic case.

Note that changing at this point we will be changing notation for position variables from q to x. In quantum mechanics, position coordinates q get quantized and become an operator Q, whereas in quantum field theory, the position in space plays a different role, parametrizing field operators.

7.1 Oscillators and quantum fields

In quantum mechanics one can deal with multi-particle systems by taking tensor products of the state spaces and operators for the single particle system. If one does this, one finds that one needs to restrict to symmetric tensor products (bosons) or anti-symmetric tensor products (fermions), a procedure that lacks an obvious motivation. In quantum field theory, the use of oscillator methods automatically gives state spaces that are symmetric or antisymmetric tensor products.

We'll first consider the case of the quantum field theory describing non-relativistic quantum particles moving in one spatial dimension. Recall that solutions of the Schrödinger equation break up into complex one-dimensional spaces of solutions proportional to

$$\psi_p(x,t) = e^{ipx}e^{-i\frac{p^2}{2m}t}$$

describing a particle with momentum p. We can do an oscillator second quantization of this complex one-dimensional space by defining annihilation and creation operators $a_p, a_p^\dagger$ satisfying the commutation relations (here we'll do bosons, could instead use anti-commutators and do fermions)

$$[a_p, a_p^\dagger] = 1$$

The number operator $N_p = a_p^\dagger a_p$ will have eigenvalues $0, 1, 2, \ldots$, which will be interpreted as the number of quanta of momentum p. The Hamiltonian will be

$$H_p = \frac{p^2}{2m}(N_p + \frac{1}{2})$$

If there were only a finite number of possible momentum values, we could define a quantum theory of an indefinite number of indistinguishable particles with those possible momenta by using a finite number of operators $a_p, a_p^\dagger$ satisfying

$$[a_p, a_{p'}^\dagger] = \delta_{p,p'}$$

and a Hamiltonian

$$H = \sum_p H_p$$

The problem of course is that the space of possible momenta is not a finite set, but is $\mathbf{R}$. Physicists deal with this by introducing

- An "infrared cutoff" that makes the set of momenta discrete, also called "putting the system in a box." The idea is to restrict space to a finite interval $[-\frac{L}{2}, \frac{L}{2}]$ of size L and choose periodic boundary conditions (or, equivalently, take space to be a circle). The periodicity condition $e^{ipx} = e^{ip(x+L)}$ then implies that p takes only the discrete values

  $$p_j = j\frac{2\pi}{L}$$

  labeled by $j \in \mathbf{Z}$.

- An "ultraviolet cutoff" that makes the set of discrete momenta finite. This is the choice of some $\Lambda > 0$ and restriction of the set of momenta to the interval $[-\Lambda, \Lambda]$.

One then tries to recover the theory with continuous momenta by taking the limits $L \to \infty$ and $\Lambda \to \infty$. Before taking the limit, the theory will just be a harmonic oscillator with a large but finite number of degrees of freedom, and the Stone-von Neumann theorem will apply. Unfortunately one immediately runs into problems when trying to take the limit. In particular, each p will contribute a term $\frac{1}{2}$ to the energy of the vacuum state, so the vacuum will have infinite energy. This is just the first of various problems that will need to be addressed to get a well-defined limit.

Physicists often define the theory formally, in a notation assuming that the limits can be made sense of, with sums becoming integrals and Kronecker $\delta_{j,k}$ terms becoming Dirac delta-functions as p becomes a continuous variable in the limit. In this notation, one has operators $a(p), a(p)^\dagger$ for $p \in \mathbf{R}$ satisfying

$$[a(p), a(p')^\dagger] = \delta(p - p')$$

and a Hamiltonian operator

$$H = \int_{-\infty}^{\infty} \frac{p^2}{2m}(a(p)a(p)^\dagger + \frac{1}{2})dp$$

In this notation a quantum field will be the (inverse) Fourier transform of the a_p operator,

$$\widehat{\psi}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} a(p) dp$$

with adjoint

$$\widehat{\psi}^\dagger(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} a(p)^\dagger dp$$

and these quantum field operators will satisfy the commutations relations

$$[\widehat{\psi}(x), \widehat{\psi}(x')] = [\widehat{\psi}^\dagger(x), \widehat{\psi}^\dagger(x')] = 0, \quad [\widehat{\psi}(x), \widehat{\psi}^\dagger(x')] = \delta(x - x')$$

The physical interpretation of these operators is that $\widehat{\psi}^\dagger(x)$ creates an additional quantum, thus a particle, at position x at $t = 0$, while $\widehat{\psi}(x)$ removes a particle at position x. While the $a(p)$ and $a(p)^\dagger$ operators have a stable physical interpretation (a particle of momentum p at $t = 0$ will continue to be a particle of momentum p at all later times), this is not true at all if one creates a physical quantum particle at a specific position x' at time $t = 0$. The initial wave function $\delta(x - x')$ has constant Fourier transform, so contributions from arbitrarily large momenta p, which will each evolve separately as $e^{-i\frac{p^2}{2m}t}$. For arbitrarily short times after $t = 0$ there wil be non-zero probability that the particle is observerved arbitrarily far away.

We will make heavy use of the Fourier transform for functions and for distributions. For the spatial variables x the Fourier transform variable will be p and the Fourier and inverse Fourier transforms are

$$\widetilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x) dx, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \widetilde{f}(p) dp$$

For the time variable there will be an opposite choice of sign

$$\widetilde{f}(E) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iEt} f(t) dt, \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iEt} \widetilde{f}(E) dE$$

this convention is motivated by the relativistic case, where the Lorentz-invariant inner product of energy-momentum and time-space vectors has opposite signs for time and space. Under Fourier transformation differentiation becomes multiplication with

$$\widetilde{\frac{\partial f}{\partial t}} = -iE\widetilde{f}, \quad \widetilde{\frac{\partial f}{\partial x}} = ip\widetilde{f}$$

(use integration by parts, assume functions vanishing at $\pm\infty$).

7.2 Quantum fields as operator-valued distributions

In order to have a well-defined notion of a quantum field theory, one needs to take into account that one gets physically sensible results not for states with a definite position or momentum, but for states that are in $L^2(\mathbf{R})$. We can rigorously define quantum fields as operator-valued distributions, meaning $\widehat{\psi}(x)$ by itself is not an operator, but there will be a well-defined operator $\widehat{\psi}(f)$ for $f \in L^2(\mathbf{R})$, which we'll write as

$$\widehat{\psi}(f) = \int_{-\infty}^{\infty} \widehat{\psi}(x) f(x) dx$$

In this section we'll outline a rigorous construction of the state space of the non-relativistic quantum field theory, for more details see [2]. This is based on exactly the Fock space construction we explained in the finite-dimensional case for the bosonic and fermionic oscillators. There we used polynomials in generators z_j or θ_j, with creation operators given by multiplication, annihilation operators given by differentiation. Here instead of polynomials we'll use the symmetric and antisymmetric tensor algebras.

Note that in earlier chapters we were starting with a finite-dimensional real phase space M, and choosing an appropriate complex structure J, giving a decomposition $M \otimes \mathbf{C} = M_J^+ \oplus M_J^-$. The oscillator state space of the Bargmann-Fock construction would be $S^*(M_J^+)$ (bosonic) or $\Lambda^*(M_J^+)$ (fermionic). Here we want to construct oscillator state spaces starting not with a finite-dimensional real phase space M, but with an infinite-dimensional complex space $\mathcal{H}_1 = L^2(\mathbf{R})$, the space of initial data for a solution of the Schrödinger equation. We still need to complexify the phase space, but don't need to choose a J. Instead we have

$$\mathcal{H}_1 \otimes \mathbf{C} = \mathcal{H}_1 \oplus \overline{\mathcal{H}}_1$$

where $\overline{\mathcal{H}}_1$ is $\mathcal{H}_1$ with the conjugate action of complex scalars.

Given a Hilbert space $\mathcal{H}_1$ one also gets a Hilbert space structure on the n-fold tensor product $T^n(\mathcal{H}_1)$. This in turn gives a Hilbert space structure on the entire tensor algebra $T^*(\mathcal{H}_1)$, taking as norm-squared the infinite sum of the norm-squareds for each T^n. Note that another way to represent multi-particle wavefunctions is to use the fact that when one takes tensor products one has

$$L^2(\mathbf{R}) \otimes L^2(\mathbf{R}) = L^2(\mathbf{R}^2)$$

so one could identify $T^n(\mathcal{H}_1)$ with $L^2(\mathbf{R}^n)$ (although we will not be using this). On $T^n(\mathcal{H}_1)$ one has symmetrization and anti-symmetrization operators that project onto $S^n(\mathcal{H}_1)$ and $\Lambda^n(\mathcal{H}_1)$. These are given by

$$\Pi^+(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!}\sum_P (f_{P(1)} \otimes \cdots \otimes f_{P(n)})$$

and

$$\Pi^-(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!}\sum_P (sgn(P))(f_{P(1)} \otimes \cdots \otimes f_{P(n)})$$

where P are the elements of the permutation group S_n and $sgn(P)$ the sign of a permutation.

The action of the field operators will be given by

$$\widehat{\psi}^\dagger(f)\Pi^\pm(f_1 \otimes \cdots \otimes f_n) = \sqrt{n+1}\Pi^\pm(f \otimes f_1 \otimes \cdots \otimes f_n)$$

and

$$\widehat{\psi}(f)\Pi^\pm(f_1 \otimes \cdots \otimes f_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^n (f, f_j)\Pi^\pm(f_1 \otimes \cdots \otimes \widehat{f_j} \otimes \cdots \otimes f_n)$$

Here $(\cdot,\cdot)$ is the inner product on $\mathcal{H}_1$. and the hatted term in the tensor product is omitted. Note that the operator $\widehat{\psi}^\dagger(f)$ is complex linear in the complex function f, whereas $\widehat{\psi}(f)$ is complex anti-linear.

These definitions are set up to give us the usual CCR or CAR relations. In our earlier discussion we just wrote those down for the operators corresponding to basis element of the complex vector space M_J^+, but we could have extended these relations to arbitrary vectors in M_J^+, which would be what we have here. The relations we get are explicitly (for the bosonic case, for fermions use anti-commutators).

$$[\widehat{\psi}(f_1),\widehat{\psi}(f_2)] = [\widehat{\psi}^\dagger(f_1),\widehat{\psi}^\dagger(f_2)] = 0$$

$$[\widehat{\psi}(f_1),\widehat{\psi}^\dagger(f_2)] = (f_1, f_2)$$

More effort is needed to define quadratic products of operators such as the Hamiltonian, for details see [2]. Such products will not be defined only on pairs of elements of $\mathcal{H}_1$ that lie in a dense subspace given by functions f that are in the Schwarz space $\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R})$.

7.3 Dynamics of quantum fields

Until now we have been studying the dynamics of quantum systems in what physicist's call the "Schrödinger picture", with states in $\mathcal{H}$ evolving in time, while the interesting operators like Q and P are time-independent. There is an alternative way to proceed, the "Heisenberg picture", in which states are time independent, with time dependence instead in the operators. In quantum field theory the Heisenberg picture is much more convenient, since the set of states is very complicated (functions on an infinite-dimensional spaces), with the set of operators (combinations of field operators) much less so. In addition, in quantum field theory there is a distinguished state, the vacuum state, and one can study other states in terms of the operators that produce the state from the vacuum.

Going back to quantum mechanics, the time dependence of Schrödinger picture states is given by

$$\psi(t) = e^{-iHt}\psi(0)$$

and operator expectation values are give by inner products

$$\langle\psi'(t), O\psi(t)\rangle = \langle\psi'(0), e^{iHt}Oe^{-iHt}\psi(0)\rangle$$

In the Heisenberg picture one treats the state space as time-independent, for instance taking Schrödinger states at $t = 0$, but operators now become time-dependent, with the Heisenberg time-dependent operator $O(t)$ related to the Schrödinger time-independent operator O by

$$O(t) = e^{iHt}Oe^{-iHt}$$

Differentiating this equation, one finds that Heisenberg picture operators will satisfy the equation

$$\frac{d}{dt}O(t) = [O(t), -iH]$$

which is the quantization of Hamilton's equations of classical mechanics

$$\frac{d}{dt}f = \{f, h\}$$

For annihilation and creation operators with Hamiltonian $H = \omega a^\dagger a$ one has

$$\frac{d}{dt}a(t) = -i\omega a(t), \quad \frac{d}{dt}a^\dagger(t) = -i\omega a^\dagger(t)$$

so

$$a(t) = e^{-i\omega t}a, \quad a^\dagger(t) = e^{i\omega t}a^\dagger \tag{7.1}$$

where $a = a(0), a^\dagger - a^\dagger(0)$. In quantum field theory we will have momentum space annihilation and creation operators $a(t,p)$ and $a^\dagger(t,p)$ and position space field operators $\widehat{\psi}(t,x)$ and $\widehat{\psi}^\dagger(t,x)$. Note that all of these are operator-valued distributions, with actual operators given by $\widehat{\psi}(f)$ for f in the Schwartz space of functions of t and x.

For the quantum field theory of free non-relativistic particles, one can use the Heisenberg picture, and take as state space the Fock space $S^*(\mathcal{H}_1)$ or $\Lambda^*(\mathcal{H}_1)$ described above. Observables will be time-dependent fields, which are operator-valued distributions $\widehat{\psi}(t,x)$ satisfying the equation of motion

$$\frac{\partial\widehat{\psi}(t,x)}{\partial t} = i[H, \widehat{\psi}(t,x)]$$

The Hamilltonian is time-independent and for a free particle is given by

$$H = \int_{-\infty}^{\infty}\frac{p^2}{2m}a^\dagger(p)a(p)dp = \int_{-\infty}^{\infty}\widehat{\psi}^\dagger(x)\left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2}\right)\widehat{\psi}(x)dx$$

To describe interacting particles, the simplest sort of interaction is a single-particle interaction with an external potential $V(x)$. This is no longer so simply described using momentum space $a(p)$ and $a(p)^\dagger$ operators since one no longer has translation invariance and particle momentum is no longer conserved. There is a simple description using field operators, with the Hamiltonian now

$$\int_{-\infty}^{\infty}\widehat{\psi}^\dagger(x)\left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\widehat{\psi}(x)dx$$

The field is now an operator-valued distribution satisfying the usual linear Schrödinger equation

$$i\frac{\partial\widehat{\psi}(t,x)}{\partial t} = \left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\widehat{\psi}(t,x)$$

In momentum space the $a(t,p)$ are no longer decoupled, with the Hamiltonian now

$$H = \int_{-\infty}^{\infty} \frac{p^2}{2m} a^\dagger(p)a(p)dp + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{V}(q)a^\dagger(p+q)a(p)dpdq$$

The second term has a physical interpretation as causing a change in momentum of a particle by momentum q, with amplitude proportional to the q Fourier component of the potential. One can also easily write down the non-relativistic field theory Hamiltonian for charged particles coupled to a background electromagnetic vector potential by changing derivatives to covariant derivatives (see equation 6.1. Using a two component version of $\mathcal{H}_1$, one can also write down the field theory version of the Pauli-Schrödinger equation, describing spin $\frac{1}{2}$ particles coupled to an electromagnetic field.

The non-relativistic quantum field theory formalism can also describe particles that can interact not just with a potential but with each other. In the case where particles interact pairwise the Hamiltonian gets an added term

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{\psi}^\dagger(x)\widehat{\psi}^\dagger(y)v(x-y)\widehat{\psi}(x)\widehat{\psi}(y)dxdy$$

Here $v(x-y)$ is the interaction potential energy between the particles. As an example, for the Coulomb interaction this would be

$$v(x-y) = -\frac{e^2}{|x-y|}$$

By doing this, the Hamiltonian is no longer quadratic in the fields and the equation of motion of the operators is now non-linear. This introduces serious new difficulties, one of which is that of how to make any sense of a product of four operator valued distributions.

Note that the formalism we have discussed here applies equally well to bosonic and fermionc cases, with the only difference the use of symmetric versus anti-symmetric tensor product spaces and commutators versus anti-commutators.

7.4 Anti-particles

We have been discussing non-relativistic quantum field theory as a second quantization of the phase space $\mathcal{H}_1$ of solutions of the Schrödinger equation by oscillator methods. $\mathcal{H}_1$ is complex, so comes with a complex structure $J = i$ that one can use to do Bargmann-Fock quantization. One could however have chosen the opposite complex structure, $J = -i$, or equivalently choose to quantize the space $\overline{\mathcal{H}_1}$ of complex conjugates $\overline{\psi}$ of solutions to the Schrödinger equation, which are solutions to the equation

$$-i\frac{\partial}{\partial t}\overline{\psi} = \frac{1}{2m}P^2\overline{\psi}$$

We'll denote the annihilation and creation operators used to do this by $b(p), b^\dagger(p)$, satisfying the same (anti)-commutation relations as the $a(p), a^\dagger(p)$. We can define a new set of field operators, which for free particles will be given by

$$\widehat{\widehat{\psi}}(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}e^{-i\frac{p^2}{2m}t}b(p)^\dagger dp$$

This operator and its adjoint will have the physical interpretation of creating and annihilating an "anti-particle" of the particle corresponding to $\widehat{\psi}$. One can interpret such particles as moving backwards in time with negative energy (and negative the usual momentum). The usual particles we have been discussing are said to have "charge +1", which both describes their behavior under $U(1)$ phase transformations and their coupling to electromagnetism (to be discussed in chapter 16). Anti-particles will instead have charge -1. In the relativistic theory, there will unavoidably be both particles and anti-particles, but in the non-relativistic limit they decouple, so we can just consider particles and ignore the anti-particles.

7.5 The propagator

In the Heisenberg picture, one can use the existence of a distinguished vacuum state $|0\rangle \in \mathcal{H}$, with the other states in $\mathcal{H}$ given by applying linear combinations of sums of field operators of the form $\widehat{\psi}^\dagger(f)$. What we need to calculate are the distributions

$$\langle 0|\, O_1(t_1,x_1)O_2(t_2,x_2)\cdots O_n(t_n,x_n)\,|0\rangle$$

depending on n values of t and x, where the operator O_j can be $\widehat{\psi}$ or $\widehat{\psi}^\dagger$. These are known as "Wightman n-point functions", although they are distributions, not functions. For the case of the free particle, one only gets a non-zero result by pairing one of the $\widehat{\psi}$ and one of the $\widehat{\psi}^\dagger$, with the result factorizing into contributions from each possible pairing. We won't here explain the details of this, but just note that one just needs to compute the 2-point function

$$W(t,t',x,x') = \langle 0|\,\widehat{\psi}(t,x)\widehat{\psi}(t',x')\,|0\rangle$$

Furthermore, this will only depend on $t-t'$ and $x-x'$, since one can translate field operators in time and space using the Hamiltonian and momentum operators by

$$\widehat{\psi}(t+a,x) = e^{iaH}\widehat{\psi}(t,x)e^{-iaH}, \quad \widehat{\psi}(t,x+b) = e^{ibP}\widehat{\psi}(t,x)e^{-ibP}$$

and use

$$P\,|0\rangle = 0 = H\,|0\rangle$$

We thus just need to calculate

$$W(t,x) = \langle 0|\,\widehat{\psi}(t,x)\widehat{\psi}^\dagger(0,0)\,|0\rangle$$

The field operators are Fourier transforms of the momentum space operators $a(t,p), a^\dagger(t,p)$ which have time dependence given by 7.1. So

$$W(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ipx}e^{-i\frac{p^2}{2m}t}\langle 0|\, a(p)a^\dagger(p')\,|0\rangle\, dpdp'$$

Using

$$[a(p), a^\dagger(p')] = \delta(p-p')$$

this becomes

$$W(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}e^{-i\frac{p^2}{2m}t}dp$$

Note that this is just the inverse Fourier transform of

$$\widetilde{W}(t,p) = \frac{1}{\sqrt{2\pi}}e^{-i\frac{p^2}{2m}t}$$

One could also Fourier transform in t, with Fourier transform variable E, and find that

$$\widetilde{W}(E,p) = \delta(E - \frac{p^2}{2m}) \tag{7.2}$$

To work with this distribution, one can define it as a boundary value of a holomorphic function by looking at complex values of t, defining $z = \tau + it$. Then

$$W(z,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}e^{-\frac{p^2}{2m}z}dp$$

This integral is well-defined if $\tau > 0$ and then is holomorphic in the $\tau > 0$ half-plane. In this half-plane one can evaluate the integral by completing the square and getting a Gaussian integral, by shifting p by $i\sqrt{\frac{m}{z}}x$ with the result

$$W(z,x) = \sqrt{\frac{m}{2\pi z}}e^{-\frac{m}{2z}x^2}$$

The distribution $W(t,x)$ is then defined as

$$W(t,x) = \lim_{\tau\to 0^+}\sqrt{\frac{m}{2\pi(\tau+it)}}e^{-\frac{m}{2(\tau+it)}x^2}$$

Note that the choice of square root here is determined by the analytic continuation from the result on the $\tau > 0$ real z axis.

7.6 Euclidean quantum field theory

The calculation of the propagator in the last section makes it tempting to try and define the quantum field theory for complex values of time, getting operators

$$\psi(z,x) = e^{zH}\psi(0,x)e^{-zH}$$

that would be holomorphic in z for $\tau > 0$. This however cannot work for very general reasons. For any sensible theory with a stable lowest energy state, the Hamiltonian is supposed to have a spectrum bounded below that goes off to $+\infty$, and for $\tau > 0$ this will make e^{zH} ill-defined. What does make sense though are expectation values of operators of the form

$$\langle 0| O(z_1, x_1)O(z_2, x_2), \cdots O(z_n, x_n) |0\rangle$$

where successive z_j satisfy $\mathrm{Re}(z_j) > \mathrm{Re}(z_{j+1})$ (since, translating to $z = 0$, there will be factors $e^{-(z_j - z_{j+1})H}$ between the operators). An example of this is the propagator of the previous section, which only makes sense for $\tau > 0$.

In Euclidean quantum field theory (the origin of the name will become clear when we study the relativistic case), one defines the theory by defining n-point functions for real values of z_j (equivalently, for imaginary values of time). Analytic continuation to real values of time (imaginary values of z) will give Wightman distributions with properties needed to define a quantum field theory. These imaginary time n-point functions are called "Schwinger functions" and they are actual functions (not distributions) away from coinciding points. In particular, the Schwinger 2-point function for the non-relativistic free particle theory will be

$$S(\tau, x) = \sqrt{\frac{m}{2\pi\tau}} e^{-\frac{m}{2\tau}x^2}$$

and only defined for $\tau > 0$.

This Schwinger function is well-known in mathematics as the "heat kernel", and it is appearing because the Schrödinger equation for imaginary time

$$(\frac{\partial}{\partial\tau} - \frac{1}{2m}\frac{\partial^2}{\partial x^2})\psi(\tau, x) = 0$$

is just the heat equation (for the choice of constant $\frac{1}{2m}$). One normally looks for real-valued solutions, but complex-valued solutions will just be pairs of real-valued solutions. A standard problem one solves for the heat equation is to find $\psi(\tau, x)$ given initial data $\psi(0, x)$. This can be done only for $\tau > 0$. Unlike the Schrödinger equation case, initial data can be propagated in one direction only.

The heat kernel is the kernel of the transformation taking initial data to solutions at later times:

$$\psi(\tau, x) = \int_{-\infty}^{\infty} S(\tau, x - x')\psi(0, x')dx' \tag{7.3}$$

It has remarkable properties, in particular the way it smooths solutions, with initial data a distribution in $S'(\mathbf{R})$ propagating to a C^∞ function for arbitrarily small τ. Physically, it describes the diffusion of heat in a homogeneous material, and also models the way probability diffuses in a random walk. Note that the way probability appears is very different than in the Schrödinger equation case. In the random walk case the probability density at x is $\psi(x)$ and is real, whereas in the Schrödinger equation case ψ is complex and the probability density is $|\psi(x)|^2$.

7.7 Propagators and Green's functions

In the real time quantum field theory, field operators satisfy an equation of motion such as the Schrödinger equation, but in imaginary time there are not well-defined operators satisfying an equation of motion. Using the propagation equation 7.3 one can see that the Schwinger functions however do satisfy the imaginary time equation of motion for $\tau > 0$ (since the $\psi(\tau, x)$ do. They also must satisfy the distributional boundary condition $S(\tau, x - x') = \delta(x - x')$ and can be defined as 0 for $\tau < 0$. Shifting the x coordinate by x', the Schwinger function will satisfy

$$(\frac{\partial}{\partial\tau} - \frac{1}{2m}\frac{\partial^2}{\partial x^2})S(\tau, x) = \delta(\tau)\delta(x)$$

Fourier transforming in τ and p, this equation becomes

$$(-iE + \frac{p^2}{2m})\widetilde{S}(E, p) = \frac{1}{2\pi}$$

so

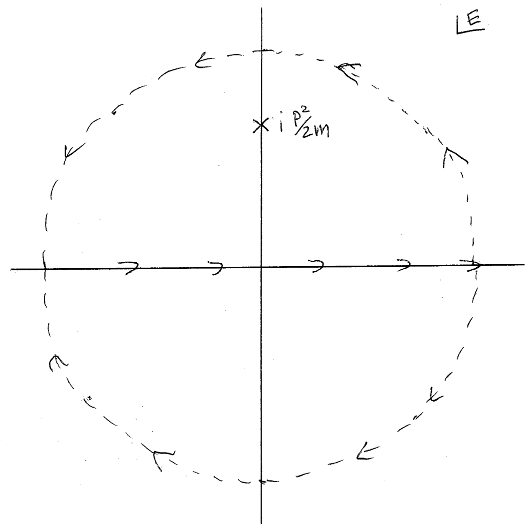
$$\widetilde{S}(E, p) = \frac{1}{2\pi}\left(\frac{1}{-iE + \frac{p^2}{2m}}\right) = \frac{1}{2\pi}\left(\frac{i}{E + i\frac{p^2}{2m}}\right)$$

The Schwinger function is thus the inverse of the heat equation differential operator, so an example of a "Green's function". Note that $\widetilde{S}(E, p)$ is holomorphic in the complex E-plane, except for a simple pole at $E = -i\frac{p^2}{2m}$ with residue one. Evaluating the integral

$$\widetilde{S}(\tau, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iE\tau}\frac{1}{2\pi}\left(\frac{i}{E + i\frac{p^2}{2m}}\right)dE$$

along the real E axis, one can close the contour in the upper half E plane for $\tau < 0$ and in the lower half E plane for $\tau > 0$. Using the residue formula one gets

$$\widetilde{S}(\tau, p) = \begin{cases} \frac{1}{\sqrt{2\pi}}e^{-\frac{p^2}{2m}\tau} & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$



In real time, it is the Wightman function $W(t,x)$ that gives the kernel for propagation in time t according to the Schrödinger equation. Defining (here $\theta(t)$ is the Heaviside function)

$$W_+(t,x) = \theta(t)W(t,x)$$

to get propagation just in the positive t direction, by the same calculation as above except for the Schrödinger rather than heat equation one gets

$$\widetilde{W}_+(E,p) = \frac{i}{2\pi}\frac{1}{E - \frac{p^2}{2m}}$$

Taking an inverse Fourier transform gives

$$\widetilde{W}_+(t,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iEt}\frac{i}{2\pi}\frac{1}{E - \frac{p^2}{2m}}dE$$

This integral is however ill-defined in the absence of any indication of how one should treat integration through the pole at $E = \frac{p^2}{2m}$.

It turns out that one can get a sensible result if one shifts the position of the pole by an infinitesimal negative imaginary amount $-\epsilon$ (or, equivalently, moves the contour into the upper half plane by a positive amount ϵ. Then $\widetilde{W}_+(t,p)$ is the limit as $\epsilon \to 0^+$ of

$$\frac{1}{\sqrt{2\pi}}\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-iEt}\frac{1}{E - \frac{p^2}{2m} + i\epsilon}dE = \begin{cases} 0 & t<0 \\ \frac{1}{\sqrt{2\pi}}e^{-i(\frac{p^2}{2m} - i\epsilon)t} & t>0 \end{cases}$$

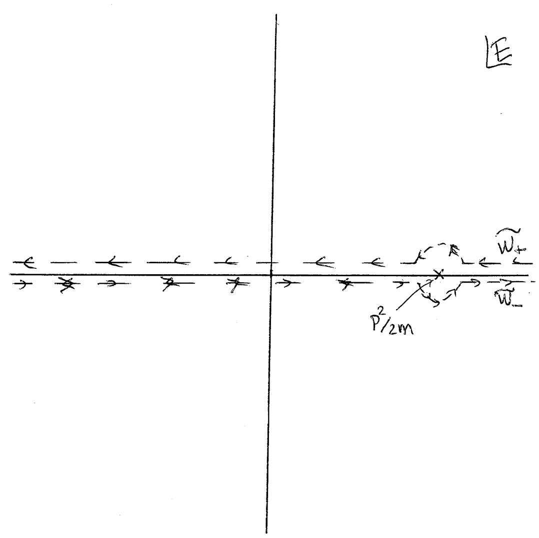
One gets this result from Cauchy's integral formula since, for $t < 0$ one can close the contour in the upper half-plane (which doesn't include the pole), and for $t > 0$ close in the lower half-plane, which does. The propagator is then

$$W_+(t,x) = \lim_{\epsilon \to 0^+} \theta(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} e^{-i(\frac{p^2}{2m} - i\epsilon)t} dp$$

To get progagation to negative t, one just needs to switch the Heaviside function and the sign of ϵ

$$W_-(t,x) = \lim_{\epsilon \to 0^-} \theta(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} e^{-i(\frac{p^2}{2m} - i\epsilon)t} dp$$

Adding together W_+ and W_- gives the earlier result that W is a delta-function, since the integrals in the sum cancel except one goes above the pole, the other below, with the integral around the pole the delta-function.



7.8 On and off mass-shell quantum fields

Non-relativistic quantum field theory has a state space $\mathcal{H} = S^*(\mathcal{H}_1)$ or $\Lambda^*(\mathcal{H}_1)$, where $\mathcal{H}_1$ is the single particle state space of solutions to the Schrödinger equation. The Schrödinger picture field operators $\widehat{\psi}^\dagger(f)$ of section 7.2 for $f \in \mathcal{H}_1$ act by increasing the number of particles by one, their adjoints $\widehat{\psi}(f)$ reduce the number by one. These satisfy commutation relations

$$[\widehat{\psi}(f_1), \widehat{\psi}^\dagger(f_2)] = (f_1, f_2)$$

and

$$\langle 0| \widehat{\psi}(f_1)\widehat{\psi}^\dagger(f_2) |0\rangle = (f_1, f_2)$$

More explicitly, the space $\mathcal{H}_1$ can be identified with initial data at $t = 0$ for a solution, which will be either $\psi(0, x) \in L^2(\mathbf{R})$ or the Fourier transfrom $\widetilde{\psi}(0, p) \in L^2(\mathbf{R})$. The Wightman function at $t = 0$ is the delta-function distribution $W(0, x) = \delta(x)$.

Heisenberg picture time dependent quantum fields are again operator-valued distributions, but now depend on t as well as x, so operators are given by $\widehat{\psi}(f), \widehat{\psi}^\dagger(f)$, where f is a function of two variables. Fourier transforming, $\widetilde{f}$ is a function of E and p. The Fourier transformed Wightman function $\widetilde{W}(E, p)$ is the distribution $\delta(E - \frac{p^2}{2m})$ and one has

$$\langle 0| \widehat{\psi}(f_1)\widehat{\psi}^\dagger(f_2) |0\rangle = (f_1, f_2)$$

where

$$(f_1, f_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\widetilde{f}_1}(E, p)\delta\left(E - \frac{p^2}{2m}\right)\widetilde{f}_2(E, p)dEdp$$

$\mathcal{H}_1$ now has a description as functions on the parabola $E = \frac{p^2}{2m}$.

While we have been writing down formulas just for one spatial dimension, the extension to the case of three dimensions is straightforward, with position and momentum x, p now 3-vectors $\mathbf{x}, \mathbf{p}$. The Wightman function in energy-momentum space will be

$$\widetilde{W}(E, \mathbf{p}) = \delta\left(E - \frac{|\mathbf{p}|^2}{2m}\right)$$

The locus $E = \frac{|\mathbf{p}|^2}{2m}$ is sometimes called the "mass-shell", for reasons that will become clear in the relativistic case. Heisenberg picture fields in general are described as "off mass-shell", with their component supported on the mass-shell decribed as "on mass-shell."

7.9 For further reading

For a more detailed rigorous version of the Fock space construction discussed here, see chapter 5 of [2]. For a physics text that covers clearly this material, see chapter 6 of [4].

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