The symplectic group and the oscillator representation

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The irreducible representation of the Heisenberg group we have been studying provides a projective representation of the symplectic group. This has various names, of which we'll choose Roger Howe's "oscillator representation" (also popular is the "Weil representation"). For more details of, a good source is [1].

1 The symplectic group and automorphisms of the Heisenberg Lie group

Since the definition of the Heisenberg Lie algebra and Lie group only depend on the antisymmetric bilinear form S on $V = \mathbf{R}^{2n}$, the group $Sp(2n, \mathbf{R})$ of linear maps preserving S acts on this Lie algebra and group as automorphisms. Using $(v, z) \in V \oplus \mathbf{R}$ as coordinates on H_{2n+1}, the action of $g \in Sp(2n, \mathbf{R})$ on the Heisenberg group is

$$\Phi_g(v, z) = (gv, z)$$

Using this automorphism, one can construct the semi-direct product

$$H_{2n+1} \rtimes Sp(2n, \mathbf{R})$$

which is sometimes called the "Jacobi group."

We also can use these automorphisms to act on the set of representations of H_{2n+1}, taking

$$\pi \to \pi_g$$

where

$$\pi_g(v, z) = \pi(\Phi_g(v, z))$$

If π is irreducible, π_g will also be irreducible, and by the Stone-von Neumann theorem there will be unitary operators U_g such that

$$\pi_g = U_g \pi_S U_g^{-1}$$

By Schur's lemma, these operators will be unique up to a phase factor. The will then provide a representation of $Sp(2n, \mathbf{R})$ up to a phase factor (a projective representation)

$$U_{g_1} U_{g_2} = e^{i\theta(g_1, g_2)} U_{g_1 g_2}$$

By changing the U_g by a phase factor

$$U_g \to V(g) = e^{i\phi(g)} U(g)$$

one can try and remove the projective factor from the multiplication law. It turns out though that there will this can only be done up to sign, a problem much like that which occurs in the case of the spin representation of the rotation group. As in the case of the rotation group, one can get a true representation by going to a double cover of $Sp(2n, \mathbf{R})$, which we'll denote $Mp(2n, \mathbf{R})$ and call the "metaplectic group." Two differences from the rotation group case are:

- In the rotation group case $\pi_1(SO(n)) = \mathbf{Z}_2$ and the double cover $Spin(n)$ is universal cover. In the symplectic case $\pi_1(Sp(2n, \mathbf{R})) = \mathbf{Z}$ and the metaplectic double cover is just one of many possible covering groups.
- $Spin(n)$ can be identified with a group of finite-dimensional matrices. This is not true for $Mp(2n, \mathbf{R})$, a group which has no finite-dimensional faithful representations. It provides a very unusual example of where thinking of Lie theory just in terms of matrix groups is inadequate.

2 The Poisson bracket and the Lie algebras $\mathfrak{h}_{2n+1}$ and $\mathfrak{sp}(2n, \mathbf{R})$

As is usual in the subject of Lie groups and their representations, it is much easier to work with a Lie algebra and its representation then with the corresponding Lie group and its representation. This is especially true in the case we are now considering: the semi-direct product of the Heisenberg and metaplectic groups and the oscillator representation. One aspect of the problem is that the metaplectic group is not a matrix group and thus hard to describe explicitly. On the other hand, the Lie algebra of the metaplectic group is the same as the Lie algebra of the symplectic group and can be described by matrices.

It is however often much more convenient to work with a realization of the Lie algebra not as matrices, but as low degree polynomial functions. The Lie bracket is then the Poisson bracket on functions, which we'll now describe. We take $V = \mathbf{R}^{2n}$, with coordinates q_j, p_k for $j, k = 1, 2, \cdots, n$. Then

Definition 1 (Poisson bracket)**.** *The Poisson bracket of two functions f_1, f_2 on V is the function*

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f_1}{\partial q_j} \frac{\partial f_2}{\partial p_j} - \frac{\partial f_2}{\partial q_j} \frac{\partial f_1}{\partial p_j} \right)$$

In the Hamiltonian form of classical mechanics, V will be the phase space and functions on V observables. There will be a distinguished function, the Hamiltonian h, which determines the dynamics, with time dependence of observables given by Hamilton's equation:

$$\frac{df}{dt} = \{f, h\}$$

The Poisson bracket can easily be seen to satisfy the following properties:

- Anti-symmetry:

$$\{f_1, f_2\} = -\{f_2, f_1\}$$

- Jacobi identity:

$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$$

- Leibniz rule (derivation property)

$$\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2 \{f_1, f_3\}$$

The first two properties imply that the Poisson bracket provides a Lie algebra structure on the space of functions on V. This is an infinite-dimensional Lie algebra. The corresponding infinite dimensional group is the subgroup of all diffeomorphisms of $\mathbf{R}^{2n}$ that preserve a symplectic form ("symplectomorphisms").

The Leibniz rule implies that, at least for polynomial functions, the Poisson bracket is determined by what it does on linear functions, where

$$\{q_j, q_k\} = \{p_j, p_k\} = 0, \quad \{q_j, p_k\} = \delta_{jk}$$

These are just the Lie bracket relations for the $2n+1$-dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$. Thinking of the q_j, p_k as basis elements of V^*, we have a Lie algebra structure on

$$\mathbf{R} \oplus V^*$$

the polynomial functions of degree less than or equal to one on V. As we have seen earlier, a more basis-independent point of view is that we have a symplectic form S on V^*, with q_j, p_k the basis of V^* that puts S in standard form.

The space of degree two monomials on V has a basis of the $2n^2+n$ elements $q_j q_k, p_j p_k$ for $j \leq k$ and all $q_j p_k$. The Poisson bracket of two of these is a linear combination of degree two monomials, so these provide a real Lie algebra of dimension $2n^2+n$. As an exercise, show that this is isomorphic to the Lie algebra $\mathfrak{sp}(2n, \mathbf{R})$. One way to see this is to look at the Poisson brackets between degree two and degree one monomials, which are the infinitesimal version of the action by the symplectic group on the Heisenberg Lie algebra as automorphisms.

Here we will work out explicitly what happens for $n=1$. The Heisenberg Lie algebra $\mathfrak{h}_3$ has basis $q, p, 1$ with only non-zero Lie bracket

$$\{q, p\} = 1$$

The symplectic Lie algebra $\mathfrak{sp}(2, \mathbf{R})$ has basis q^2, p^2, qp with non-zero Lie brackets

$$\{\frac{q^2}{2}, \frac{p^2}{2}\} = qp, \quad \{qp, p^2\} = 2p^2, \quad \{qp, q^2\} = -2q^2$$

This is isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbf{R})$ of 2 by 2 traceless real matrices, with bracket the commutator, where a conventional basis is

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The isomorphism is explicitly given by

$$\frac{q^2}{2} \leftrightarrow E, \quad -\frac{p^2}{2} \leftrightarrow F, \quad qp \leftrightarrow G$$

or by

$$-aqp + \frac{bq^2}{2} - \frac{cp^2}{2} \leftrightarrow \begin{pmatrix} a & b \\ c & -a \end{pmatrix} -$$

The semi-direct product of H_3 and $SL(2, \mathbf{R})$ puts the above two Lie algebras together, with the action of $SL(2, \mathbf{R})$ on H_3 by automorphisms reflected in the non-zero Lie brackets

$$\{qp, q\} = -q. \quad \{qp, p\} = p$$

$$\{\frac{p^2}{2}, q\} = -p, \quad \{\frac{q^2}{2}, p\} = q$$

From these relations one can see that

$$qp \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

generates a group $\mathbf{R}$ acting on the q direction in the qp plane by e^t, on the p direction by e^{-t}. The element

$$\frac{1}{2}(q^2 + p^2) \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generates an $SO(2)$ subgroup of rotations in the qp plane.

3 The Schrödinger model for the oscillator representation

We have seen that the Schrödinger representation is given as a representation of $\mathfrak{h}_3$ by the operators

$$\pi'_S(q) = -iQ = -iq, \quad \pi'_S(p) = -iP = -\frac{d}{dq}, \quad \pi'_S(1) = -i\mathbf{1}$$

Dirac's original definition of "quantization" asked for an extension of this representation from linear functions to all functions on phase space, i.e. a choice of operators that would take any polynomial in q and p to an operator, with Poisson bracket of functions going to commutator of operators, so a Lie algebra homomorphism. But going from functions of q and p to operators built out of Q and P, one runs into "operator-ordering" ambiguities since Q and P do not commute. It turns out that one can get a Lie algebra homomorphism for polynomials up to degree two, but this is impossible in higher degree (Groenewold-van Hove theorem).

What works in degree two is to extend the Schrödinger representation to a representation of $\mathfrak{sl}(2,\mathbf{R})$ (and of the semi-direct product with $\mathfrak{h}_3$) by taking

$$\pi'_S(q^2) = -iQ^2 = -iq^2, \quad \pi'_S(p^2) = -iP^2 = i\frac{d^2}{dq^2}$$

and making the choice (which gives a skew-adjoint operator)

$$\pi'_S(qp) = -i\frac{1}{2}(QP + PQ) = -i\frac{1}{2}(2QP - i\mathbf{1}) = -q\frac{d}{dq} - \frac{1}{2}\mathbf{1}$$

These operators will satisfy the commutation relations given by the Lie bracket of $\mathfrak{sl}(2,\mathbf{R})$. One would like to exponentiate them to get a representation of the Lie group $SL(2,\mathbf{R})$, which we will call the "oscillator representation" (it is also often known as the "Weil representation"). In the case of $\pi'_S(qp)$ the operator exponentiates to an operator on functions which rescalesin the q variable. But it is unclear how to exponentiate the second order differential operator

$$-iP^2 = i\frac{d^2}{dq^2}$$

If one takes Fourier transform to turn derivatives in q into multiplications operators, the problem just moves to the operator $-iQ^2$ which changes from a multiplication operator to a second-order differential operator.

The problem is best thought of as having to do with exponentiating the Lie algebra element

$$\frac{1}{2}(q^2 + p^2)$$

which generates the $SO(2) \subset SL(2,\mathbf{R})$ subgroup of rotations in the qp plane. So, for the oscillator representation, we need to explictly construct the operator

$$e^{\theta\pi'_S\left(\frac{1}{2}(q^2+p^2)\right)}$$

where

$$\pi'_S\left(\frac{1}{2}(q^2 + p^2)\right) = -i\frac{1}{2}(Q^2 + P^2) = -i\frac{1}{2}(q^2 - \frac{d^2}{dq^2})$$

Changing notation from θ to t, this is just the standard physics problem of solving the Schrödinger equation for the Hamiltonian $H = \frac{1}{2}(Q^2 + P^2)$ and so constructing the unitary operator

$$U(t) = e^{-it\frac{1}{2}(Q^2+P^2)}$$

With some effort (see for instance exercises 4 and 5 of chapter III of [2]), one can derive a formula for the kernel $K_t(q, q')$ (known in physics as the "propagator") where

$$(U(t)\psi)(q) = \int_{\mathbf{R}} K_t(q, q')\psi(q')dq'$$

One finds

$$K_t(q,q') = \frac{1}{\sqrt{2\pi \sin t}} \exp\left(-\frac{1}{2}\begin{pmatrix} q & q' \end{pmatrix}\begin{pmatrix} \frac{\cos t}{\sin t} & -\frac{1}{\sin t} \\ -\frac{1}{\sin t} & \frac{\cos t}{\sin t}\end{pmatrix}\begin{pmatrix} q \\ q'\end{pmatrix}\right)$$

This expression requires interpretation as a distribution defined as a boundary value of a holomorphic function, replacing t by $t - i\epsilon$ and taking the limit as positive ϵ vanishes.

One can show that

$$\lim_{\epsilon \to 0^+} U(\frac{\pi}{2} - i\epsilon) = e^{i\frac{\pi}{4}}\mathcal{F}$$

This corresponds to a $\frac{\pi}{2}$ rotation in the q,p plane, interchanging the role of q and p. By the Stone-von Neumann theorem, one expects this operator to be the Fourier transform, up to a phase factor. The calculation of the propagator fixes the phase factor. In some sense, rotations by arbitrary values of t will give "fractional Fourier transforms."

Rotation by π is given by

$$i\mathcal{F}^2$$

. The $\mathcal{F}^2$ is as expected since $\mathcal{F}^2$ acts on functions by

$$\psi(q) \to \mathcal{F}^2\psi(q) = \psi(-q)$$

corresponding to a rotation by π taking q to $-q$. Rotation by 2π is given by $-\mathcal{F}^4 = -\mathbf{1}$ rather than the $\mathbf{1}$ expected if $U(t)$ is to be a true (rather than up to ± 1) representation of $SO(2) \subset SL(2,\mathbf{R})$. This is a precise analog of what happens when we take the spinor Lie algebra representation of $SO(3)$ and exponentiate: we find that rotating around an axis by 2π gives a factor of -1. The representation is only a projective (up to sign) representation of $SO(3)$. To get a true representation, one needs the double cover $Spin(3) = SU(2)$. Here again we have a representation up to sign and need a double cover of $Sp(2,\mathbf{R})$. This will be the metaplectic group $Mp(2,\mathbf{R})$, which is not a matrix group.

4 The Bargmann-Fock model for the oscillator representation

The best way to calculate the phase factors in the exponentialed version of the oscillator representation is not to use the Schrödinger version of the representation, but to instead use the Bargmann-Fock version. This is based upon choosing a compatible positive complex structure J, using it to get a complex polarization

$$V \otimes \mathbf{C} = W_J \oplus \overline{W_J}$$

and realizing the representation on the space of polynomial functions on W_J. We will just do this in the simplest case, $n = 1$ and J the standard choice for such a complex structure. In this case W_j the space of polynomials $\mathbf{C}[w]$ (with the Bargmann-Fock inner product) and the operators

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{d}{dw}, \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = w$$

provide a representation of the complexified Heisenberg Lie algebra (which is the standard one on the real Lie algebra).

As in the Schrödinger case, one can extend this representation to the oscillator representation of $\mathfrak{sp}(2n, \mathbf{R})$ by taking quadratic combinations of the Heisenberg Lie algebra opertors. In particular, using

$$\frac{1}{2}(Q^2 + P^2) = \frac{1}{2}(a^\dagger a + aa^\dagger) = a^\dagger a + \frac{1}{2}$$

one has (writing elements of $\mathfrak{sl}(2, \mathbf{R})$ both as quadratic polynomials and as matrices)

$$\pi'_{BF}(\frac{1}{2}(q^2 + p^2)) = \pi'_{BF}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = -i(a^\dagger a + \frac{1}{2}) = -i(w\frac{d}{dw} + \frac{1}{2})$$

This operator can easily be exponentiated:

$$e^{\theta \pi'_{BF}(\frac{1}{2}(q^2+p^2))}$$

act on $\mathbf{C}[w]$ by multiplying the monomial w^n by $e^{-i\theta(n+\frac{1}{2})}$. This gives the minus sign previously discusses for $\theta = 2\pi$.

In this represntation the other two basis elements of $\mathfrak{sl}(2, \mathbf{R})$ are

$$\pi'_{BF}(qp) = \pi'_{BF}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = -\frac{i}{2}((a^\dagger)^2 + a^2)$$

$$\pi'_{BF}(q^2 - p^2) = \pi'_{BF}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = -\frac{1}{2}((a^\dagger)^2 - a^2)$$

Note that these operators do not change the parity of monomials they act on, and you can get from any monomial of a given parity to any other other of the same parity by applying these operators repeatedly. So, the oscillator representations we have constructed here is the sum of two irreducibles (all polynomials of even degree, and all polynomials of odd degree).

Say more about dependence on J, and the existence of a lowest weight vector?

References

[1] Folland, G., *Harmonic Analysis in Phase Space,* Princeton University Press, 1989.

[2] Howe, R. and Tan, E. C., *Non-Abelian Harmonic Analysis,* Springer, 1992.