The Heisenberg group and its representations

Mathematics GR6434, Spring 2023

Quantum mechanics as we know it was born in 1925 in a series of conceptual breakthroughs which began with Heisenberg's creation of a theory involving non-commuting quantities, soon reformulated (by Max Born) in terms of position and momentum operators Q and P satisfying the commutation relations

$$[Q, P] = i\hbar \mathbf{1}$$

(now known as the Heisenberg commutation relations). We are for now considering just one degree of freedom. $\hbar$ is a constant that depends on units used to measure position and momentum. We will choose units such that $\hbar = 1$. The mathematician Hermann Weyl soon recognized these relations as those of a unitary representation of a Lie algebra now known as the Heisenberg Lie algebra, and described the corresponding Heisenberg group.

Late in 1925, Schrödinger formulated a seemingly different version of quantum mechanics, in terms of wave-functions satisfying a differential equation. What Schrödinger had found was a construction of a representation of the Heisenberg Lie algebra on the vector space of functions $\psi(q)$ of a position variable q, with Q the multiplication by q operator and P the differential operator

$$P = -i\frac{d}{dq}$$

Ultimately the Stone-von Neumann theorem showed that there was essentially only one irreducible representation of the Heisenberg group, so the two formulations of quantum mechanics were two aspects of the same thing.

We'll begin with the Lie algebra corresponding to the Heisenberg commutation relations, then find the group with this Lie algebra and show that Schrödinger's wave-functions give an irreducible unitary representation of the Lie algebra and group. It turns out that any irreducible unitary representation of the Heisenberg group is essentially equivalent to this one (Stone-von Neumann theorem), but the family of different ways of constructing these representations carries an intricate structure.

1 The Heisenberg Lie algebra and Lie group

The Heisenberg Lie algebra $\mathfrak{h}$ will be the three-dimensional Lie algebra with a basis X, Y, Z and Lie bracket relations

$$[X, Z] = [Y, Z] = 0, \quad [X, Y] = Z$$

This Lie algebra can be identified with the Lie algebra of three by three strictly upper-triangular matrices by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad , Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A unitary representation (which we'll call π') will be given by three skew-adjoint operator $\pi'(X), \pi'(Y), \pi'(Z)$ satisfying

$$[\pi'(X), \pi'(Y)] = \pi'(Z), \quad [\pi'(X), \pi'(Z)] = 0, \quad [\pi'(Y), \pi'(Z)] = 0$$

These become the Heisenberg commutation relations if we identify

$$\pi'(X) = -iQ, \quad \pi'(Y) = -iP, \quad \pi'(Z) = -i\mathbf{1}$$

Note that factors of i are appearing here just because physicists want to work with self-adjoint operators, but for unitary representations the Lie algebra representation operators are skew-adjoint.

In terms of matrices, exponentiating elements of $\mathfrak{h}$ as in

$$exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

gives the elements of the Heisenberg group H. This is the group of upper triangular matrices with 1s on the diagonal. Using x, y, z as ("exponential") coordinates on the group, H is the space $\mathbf{R}^3$ with multiplication law

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$$

For computations with the Heisenberg group it is often convenient to use the Baker-Campbell-Hausdorf formula, which simplifies greatly in this case since all Lie brackets except $[X, Y] = Z$ vanish. As a result, for $A, B \in \mathfrak{h}$ one has

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

This group is a central extension

$$0 \to (\mathbf{R}, +) \to H \to (\mathbf{R}^2, +) \to 0$$

of the additive group of $\mathbf{R}^2$ by the additive group of $\mathbf{R}$.

A slightly different version of the Heisenberg group (which we'll call H_{red}) that is sometimes used takes a quotient by $\mathbf{Z}$ and replaces the central $\mathbf{R}$ with a central $U(1)$, so is a central extension

$$0 \to U(1) \to H_{red} \to (\mathbf{R}^2, +) \to 0$$

Elements are labeled by (x, y, u) where x and y are in $\mathbf{R}$ and $u \in U(1)$, and the group law is

$$(x, y, u)(x', y', u') = (x + x', y + y', uu' e^{i\frac{1}{2}(xy' - x'y)})$$

2 The Schrödinger representation

The Schrödinger representation π_S will be a representation on a vector space $\mathcal{H}$ of complex valued functions $\psi(q)$ on $\mathbf{R}$, with derivative the Lie algebra representation

$$\pi'_S(X) = -iQ = -iq, \quad \pi'_S(Y) = -iP = -\frac{d}{dq}, \quad \pi'_S(Z) = -i1$$

Exponentiating these operators gives unitary operators that generate π_S

$$\pi_S(x) = e^{-ixq}, \quad \pi_S(y) = e^{-y\frac{d}{dq}}, \quad \pi_S(Z) = e^{-iz}\mathbf{1}$$

Note that $\pi_S(y)$ acts on the representation space by translation

$$\pi_S(y)\psi(q) = \psi(q-y)$$

Definition (Schrödinger representation)**.** *The Schrödinger representation of the Heisenberg group H is given by*

$$\pi_S(x,y,z)\psi(q) = e^{-iz}e^{i\frac{1}{2}xy}e^{-ixq}\psi(q-y)$$

for $(x,y,z) \in H$.

One can easily check that this is a representation, since it satisfies the homomorphism property

$$\pi_S(x,y,z)\pi_S(x',y',z') = \pi_S(x+x', y+y', z+z'+\frac{1}{2}(xy'-x'y))$$

Taking as representation space $\mathcal{H} = L^2(\mathbf{R})$, for the Lie algebra representation π'_S there will be domain (functions on which operators not defined) and range (operators take something in $L^2(\mathbf{R})$ to something not in $L^2(\mathbf{R})$) problems. As an alternative, one can take $\mathcal{H} = \mathcal{S}(\mathbf{R})$ so that the representation operators are well-defined (but then the dual space is something different, the tempered distributions $S'(\mathbf{R})$). For the group representation the operators π_S are well defined on $\mathcal{H} = L^2(\mathbf{R})$. Giving up on a well-defined inner-product and unitarity, one can take $\mathcal{H} = \mathcal{S}'(\mathbf{R})$. This is a general phenomenon for infinite-dimensional representations of non-compact Lie groups that we will see again later in other examples: one has inequivalent representations on a sequence of inclusions of representation spaces

$$\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$$

but there is a sense in which these are all "the same", since the inclusions are dense.

3 The Stone-von Neumann theorem

The remarkable fact about representations of the Heisenberg group is that there is essentially only one representation (once one has specified the constant by which Z acts, but non-zero choices are related by a rescaling). More specifically, any irreducible representation of H will be unitarily equivalent to the Schrödinger representation. One has the following theorem

Theorem (Stone-von Neumann). *For any irreducible unitary representation π of H (with action of the center $\pi(0,0,z) = e^{-iz}$) on a Hilbert space $\mathcal{H}$, there is a unitary operator $U : \mathcal{H} \to L^2(\mathbf{R})$ such that*

$$U\pi U^{-1} = \pi_S$$

We will not give a proof here, since the analysis is somewhat involved, but what follows should make clear some problems that any proof needs to overcome and motivate the strategy for an actual proof.

Recall (see for example the notes on simple quantum mechanical examples), that one can define the adjoint pair of operators

$$a = \frac{1}{\sqrt{2}}(Q + iP) = \frac{1}{\sqrt{2}}(q + \frac{d}{dq}), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})$$

and for the harmonic oscillator Hamiltonian the lowest energy eigenspace is the one-dimensional space of solutions in $L^2(\mathbf{Q})$ of

$$a\psi_0(q) = 0$$

These are all proportional to

$$\psi_0 = e^{-\frac{1}{2}q^2}$$

The rest of the state space can be generated by repeatedly applying the operator $a^\dagger$ to ψ_0. One of the exercises will be to use this basis to prove that the Schrödinger representation is irreducible.

A possible approach to the Stone-von Neumann theorem would be to look at the operators

$$b = UaU^{-1}, \quad b^\dagger = Ua^\dagger U^{-1}$$

show that b has a one-dimensional kernel, and that the rest of the representation is given by repeated applications of $b^\dagger$. Unfortunately, this can't work, since the domain/range problems mean there is no guarantee that vectors in the range of $b^\dagger$ will be in its domain, so generating the representation by repeatedly applying $b^\dagger$ won't work. It turns out that the Stone-von Neumann theorem is not true for general Lie algebra representations of $\mathfrak{h}$, only works for Lie algebra representations that integrate to give a group representation.

To get a proof that does work, one needs to work not with Q, P or $a, a^\dagger$, but with their exponentiated versions. For details, see [3], chapter 14.

An important example of an irreducible representation unitarily equivalent to the Schrödinger representation is given by the Fourier transform $\mathcal{F}$ which takes

$$\psi(q) \to \widetilde{\psi}(p) = (\mathcal{F}\psi)(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ipq}\psi(q)dq$$

and is a unitary transformation on $L^2(\mathbf{R})$, with inverse $\widetilde{\mathcal{F}}$ given by Fourier inversion

$$\widetilde{\psi}(p) \rightarrow (\widetilde{\mathcal{F}}\widetilde{\psi})(q) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ipq}\widetilde{\psi}(p)dp$$

This is thus an example where the Stone-von Neumann theorem applies, with $U = \widetilde{\mathcal{F}}, \ \ U^{-1} = \mathcal{F}$.

4 The Bargmann-Fock representation

The Stone-von Neumann theorem also applies to very different constructions of representations on other versions of Hilbert space. In particular, it is clear from looking at the harmonic oscillator calculations that energy eigenstates can be identified with monomials in a complex variable , with a and $a^\dagger$ decreasing and increasing the degree. To find a construction of the Heisenberg group irreducible representation on $\mathbf{C}[w]$, one needs a Hilbert space structure, which we can define as follows:

Definition (Fock Space)**.** *Fock space $\mathcal{H}_F$ is the space of entire functions on $\mathbf{C}$, with finite norm in the inner product*

$$\langle f(w), g(w)\rangle = \frac{1}{\pi}\int_{\mathbf{C}} \overline{f(w)}g(w)e^{-|w|^2}$$

An orthonormal basis of $\mathcal{H}$ is given by apropriately normalized monomials. Since

$$\begin{aligned}\langle w^m, w^n\rangle &= \frac{1}{\pi}\int_{\mathbf{C}} \overline{w}^m w^n e^{-|w|^2}\\ &= \frac{1}{\pi}\int_0^\infty (\int_0^{2\pi} e^{i\theta(n-m)}d\theta)r^{n+m}e^{-r^2}rdr\\ &= n!\delta_{n,m}\end{aligned}$$

we see that the functions $\frac{w^n}{\sqrt{n!}}$ are orthornormal.

To get a representation of the (complexified) Heisenberg Lie algebra on this space, define

$$a = \frac{d}{dw}, \quad a^\dagger = w$$

As an exercise, you should show that these operators are each other's adjoints with respect to the inner product on Fock space. On the real Heisenberg Lie algebra, this representation exponentiates to a representation of the Heisenberg group. By the Stone-von Neumann theorem it is unitarily equivalent to the Schrödinger representation on $L^2(\mathbf{R})$.

To explicitly write the Bargmann-Fock representation of the Heisenberg Lie algebra, we can complexify and work with operators that depend on complex linear combinations of the real basis X, Y, Z. If we do this first in the Schrödinger representation we have

$$\pi'_S(iX) = Q, \quad \pi'_S(iY) = P, \quad \pi'_S(iZ) = \mathbf{1}$$

and so

$$\pi'_S(\frac{1}{\sqrt{2}}(iX + i(iY))) = a = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})$$

(with at similar formula for $a^\dagger$). To get Bargmann-Fock we want a π'_{BF} that takes the same linear combinations to $\frac{d}{dw}$ and w, acting on $\mathcal{H}_F$. Thus

$$\pi'_{BF}(\frac{1}{\sqrt{2}}(iX+i(iY)) = a = \frac{d}{dw},\ \ \pi'_{BF}(\frac{1}{\sqrt{2}}(iX-i(iY)) = a^\dagger = w,\ \ \pi'_{BF}(iZ) = \mathbf{1}$$

We won't work this out here, but these operators can be exponentiated to get operators for a Heisenberg Lie group representation. By Stone-von Neumann, there will be a unitary operators

$$U : \mathcal{H}_F \to L^2(\mathbf{R}),\ \ U^{-1} : L^2(\mathbf{R}) \to \mathcal{H}_F$$

These operators are quite non-trivial and interesting in analysis, giving unitary isomorphisms between two very different kinds of function spaces. The explicit form for U^{-1} is often called the Bargmann transform and is given by

$$(U^{-1}\psi)(w) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}w^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}q^2} e^{\sqrt{2}wq}\psi(q)dq$$

The relation between the Schrödinger and Bargmann-Fock operators will be given by

$$U\frac{d}{dw}U^{-1} = \frac{1}{\sqrt{2}}(q + \frac{d}{dq}),\ \ UwU^{-1} = \frac{1}{\sqrt{2}}(q - \frac{d}{dq})$$

For more on the Bargmann-Fock representation and the Bargmann transform a good source is Chapter 1, Section 6 of [2].

5 The Weyl algebra

A closely related algebra to the Heisenberg Lie algebra is the Weyl algebra, which can be defined as the non-commutative algebra of polynomial coefficient differential operators for a complex variable w. The generators of the algebra are

- Multiplication by w.
- Differentiation by w: $\frac{d}{dw}$

These satisfy the same commutation relations as $a, a^\dagger$

$$[\frac{d}{dw}, w] = 1$$

since

$$\frac{d}{dw}(wf) - w\frac{df}{dw} = f$$

Recall that one can think of representations of a Lie algebra $\mathfrak{g}$ as modules for the associative algebra $U(\mathfrak{g})$. It is convenient here also to complexify, and for any Lie algebra we'll use the notation $U(\mathfrak{g})$ to refer to $U(\mathfrak{g})\otimes\mathbf{C} = U(\mathfrak{g}\otimes\mathbf{C})$. For the Heisenberg Lie algebra $\mathfrak{h}$, $U(\mathfrak{h})$ is given by all complex linear combinations of products of basis elements X, Y, Z, modulo the relations

$$[X,Z] = [Y,Z] = 0, \quad [X,Y] = Z$$

The center $Z(\mathfrak{h})$ of $U(\mathfrak{h})$ is the commutative algebra $\mathbf{C}[Z]$ of polynomials in Z. Note that we are following convention and using $Z(\mathfrak{h})$ to mean the center of $U(\mathfrak{h})$ not the center of the Lie algebra itself (which in this case is one dimensional, so in a confusing notation $\mathbf{C}Z$ not $\mathbf{C}[Z]$).

In any irreducible representation π' of a Lie algebra $\mathfrak{g}$, by Schur's lemma elements of the center $Z(\mathfrak{g})$ act by scalars. This gives a homomorphism

$$\chi_{\pi'} : Z(\mathfrak{g}) \to \mathbf{C}$$

called the infinitesimal character of the representation. In the case of $\mathfrak{g} = \mathfrak{h}$, since $Z(\mathfrak{h})$ is the polynomial functions on $\mathfrak{h}$, the infinitesimal character is evaluation of the polynomial at some $c \in \mathbf{C}$. This c is the scalar given by the action of $\pi'(Z)$ on the representation space. The Schrödinger representation as we have defined it is an irreducible representation with $c = -i$.

For general Lie algebra representations of the complexified Lie algebra $\mathfrak{h}\otimes\mathbf{C}$, for each $c \neq 0$ we have the irreducible representation unitarily equivalent to the Schrödinger representation (rescaled from $c = -i$. These will be unitary for c imaginary.

Z acts by a scalar we'll call c_π. Polynomials in $Z(\mathfrak{h})$ also act by a scalar, the evaluation of the polynomial at c_π. The Schrödinger representation as we have defined it is an irreducible representation with $c_{\pi_S} = -i$. Restricting attention to Lie algebra representations for which $\pi'(Z) = c\mathbf{1}$ for a chosen $c \in \mathbf{C}$, these will be modules for the quotient algebra

$$U(\mathfrak{h})/(Z - c)$$

By rescaling X and Y, for $c \neq 0$, we get the Weyl algebra, and so an irreducible Heisenberg algebra representation will be a module for the Weyl algebra. Among these modules is the standard one on polynomials on w, which corresponds to the one we have studying, which is integrable to a unitary Heisenberg group representation. But there are many different modules for the Weyl algebra, with the study of thes modules the beginning of the subject of D-modules in algebraic geometry (see for instance [1]).

6 The Heisenberg group and symplectic geometry

The three-dimensional Heisenberg group that we have been studying has a simple generalization that behaves in much the same way. For any n, define the

$2n+1$ dimensional Heisenberg Lie algebra to be the Lie algebra with basis X_j, Y_j, Z $(j = 1, 2, \cdots, n)$ and all Lie brackets zero except

$$[X_j, Y_k] = \delta_{jk} Z$$

One can easily easily get a corresponding Heisenberg Lie group generalizing the $n = 1$ case by exponentiating.

Instead of working with a basis like this, one can define this Lie group in a more coordinate-invariant way, starting with any symplectic form on $V = \mathbf{R}^{2n}$, where

Definition (Symplectic form)**.** *A symplectic form S on a vector space V is a non-degenerate anti-symmetric bilinear form*

$$(v_1, v_2) \in V \times V \to S(v_1, v_2) \in \mathbf{R}$$

on V.

This is the same definition as that of an inner product on V, with "symmetric" replaced by antisymmetric. For any even-dimensional real vector space S with a symplectic form S, one can define a Lie algebra structure on $V \oplus \mathbf{R}$ by taking the LIe bracket to be

$$[(v, z), (v', z')] = (0, S(v, v'))$$

where (v, z) are elements of $V \oplus \mathbf{R}$. One gets a corresponding Lie group by taking as group law on $V \oplus \mathbf{R}$

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2} S(v, v'))$$

In the inner product case, by Gram-Schmidt orthornomalization one can always find an orthonormal basis of V, with any other basis related to this one by an element of $GL(V)$. The subgroup of $GL(V)$ preseving the inner product and thus taking orthonormal bases to orthonormal bases is the orthogonal group $O(V)$. In the symplectic case, V has to be even-dimensional (to have a non-degenerate S), then one can always find (exercise?) a "symplectic basis": X_j and Y_j for $j = 1, 2, \cdots, n$ satisfying

$$S(X_j, X_k) = S(Y_j, Y_k) = 0, \quad S(X_j, Y_k) = \delta_{jk}$$

In this basis one recovers the earlier definition of the Heisenberg Lie algebra and Lie group of dimension $2n + 1$.

The subgroup of $GL(V)$ preserving S and taking symplectic bases to symplectic bases is by definition the symplectic group $Sp(V)$. Since V is even dimensional, this group will be a matrix group that can be denoted $Sp(2n, \mathbf{R})$. Note that this is different than the group often written as $Sp(n)$, the group of n by n quaternionic matrices preserving the standard hermitian form on $\mathbf{H}^n$. The groups $Sp(n)$ and $Sp(2n, \mathbf{R})$ are different real forms of the group $Sp(2n, \mathbf{C})$ of linear transformations preserving a non-degenerate anti-symmetric bilinear form on $\mathbf{C}^{2n}$.

7 Polarizations

From the discussion above, V can be written as

$$V = M \oplus M^*$$

where M is an n-dimensional vector space with basis X_j and M^* is the dual vector space with basis elements Y_j dual to the X_j (i.e. $Y_j(X_k) = \delta_{jk}$). Note that for any vectors $x, x' \in M \subset V$ one has $S(x, x') = 0$. A subspace with this property is called "isotropic". The maximal dimension of a subspace of V on which S is zero is n, and such isotropic subspaces are called "Lagrangian". M^* is also Lagrangian.

Since the definition of the Heisenberg Lie algebra and Lie group depend only on the symplectic form S, and by Stone-von Neumann there is only one irreducible representation, this irreducible representation should depend just on S. It turns out though that all constructions of this representation depend upon a choice of additional structure. We have seen that the construction of the Schrödinger representation depends on a choice of n position coordinates q_j, correspondng to the basis elements X_j of the Lie algebra, which span a Lagrangian subspace of $\mathbf{R}^{2n}$. The Fourier transform takes this construction to a different one, depending on n momentum coordinates p_j, corresponding to the basis elements Y_j of the Lie algebra, which span a complementary Lagrangian subspace of $\mathbf{R}^{2n}$. More generally, one can construct a version of the Schrödinger representation for any choice of Lagrangian subspace $\ell \subset \mathbf{R}^{2n}$ (we will not show this here, may later give this construction). By the Stone-von Neumann theorem, for each ℓ there will be an operator U_ℓ giving a unitary equivalence with the construction for the standard choice of ℓ spanned by the X_j. For ℓ spanned by the Y_j, U_ℓ will be the Fourier transform, but for more general ℓ its construction is rather non-trivial.

As an exercise, you should show that the choices of Lagrangian subspace *ell* are parametrized by the space $U(n)/O(n)$. Choices of Lagrangian subspace $M \subset V$ give what is called a "real polarization" of V. The Bargmann-Fock construction involves a different sort of polarization, called a "complex polarization". Here one complexifies V and asks for Lagrangian subspaces W and $\overline{W}$ such that

$$V \otimes \mathbf{C} = W \oplus \overline{W}$$

where W and $\overline{W}$ are interchanged by the conjugation map on $\mathbf{C}$.

Such a decomposition is equivalent to the choice of a compatible complex structure on V, where

Definition (Complex structure)**.** *A complex structure on a real vector space V is a (real)-linear map*

$$J : V \to V$$

satisfying $J^2 = -\mathbf{1}$.

and

Definition (Compatible complex structure)**.** *A complex structure on V is compatible with a symplectic form S on V when*

$$S(Jv_1, Jv_2) = S(v_1, v_2)$$

Such J only exist if the dimension of V is even and one can think of them as ways of making V a complex vector space (so identifying $\mathbf{R}^{2n} = \mathbf{C}^n$), with multiplication by i given by J. J has no eigenvectors in V, but it does have complex eigenvalues $\pm i$ giving a decomposition

$$V \otimes \mathbf{C} = V_J^+ \oplus V_J^-$$

into $\pm i$ eigenspaces for J. This will be a polarization of V when J is compatible with S since then V_J^+ and V_J^- are Lagrangian subspaces. To see this, note that for $w_1, w_2 \in V_J^+$

$$S(w_1, w_2) = S(Jw_1, Jw_2) = S(iw_1, iw_2) = -S(w_1, w_2)$$

so must be zero.

Given both a symplectic form S and a compatible complex structure J on V, V becomes not just a complex vector space, but a complex vector space with Hermitian inner product, defined by

$$\langle v_1, v_2 \rangle_J = S(v_1, Jv_2) + iS(v_1, v_2)$$

One can easily check that this is Hermitian, but it is not necessarily positive. To get a positive Hermitian structure one needs to impose an addtional condtion on J, that, for non-zero $v \in V$ one has

$$S(v, Jv) > 0$$

The possible choices of general complex structure J are parametrized by $GL(2n, \mathbf{R})/GL(n, \mathbf{C})$. The compatibility condition implies that $J \in Sp(2n, \mathbf{R})$. As an exercise you will show that the space of possible positive complex structures compatible with S is $Sp(2, \mathbf{R})/U(n)$. Gor the case $n = 1$, this is $SL(2, \mathbf{R})/U(1)$, which can be identified either with the upper half plane or interior of the unit disk in $\mathbf{C}$.

Draw picture of $n = 1$ case.

References

[1] Coutinho, S. C., *A Primer of Algebraic D-modules,* Cambridge University Press, 1995.

[2] Folland, G., *Harmonic Analysis in Phase Space,* Princeton University Press, 1989.

[3] Hall, B., *Quantum Theory for Mathematicians,* Springer, 2013.