### BACKGROUND ON REPRESENTATIONS OF LIE GROUPS AND LIE ALGEBRAS

#### Mathematics GR6434, Spring 2023

I'll review here some basic facts about representations. Some of this will have been covered last semester, or perhaps in an undergraduate course on representations of finite groups. For more details, some suggested sources are [1], [2] and [3].

### **1** Some generalities

For groups in general:

**Definition 1** (Group Representation). A representation  $(\pi, V)$  of a group G on a vector space V is a homomorphism

$$\pi: G \to GL(V)$$

where GL(V) is the group of invertible linear transformations of V.

and for Lie algebras

**Definition 2** (Lie algebra Representation). A representation of a Lie algebra  $\mathfrak{g}$  is a module for the algebra  $U(\mathfrak{g})$ . This module is given by a choice of vector space V, and a homomorphism  $\pi : U(\mathfrak{g}) \to End(V)$ .

In the group case, one can alternatively define a representation as a module for the group algebra  $\mathbb{C}G$  (this is an algebra of functions, with product given by convolution), we may say more about this when we discuss the Peter-Weyl theorem.

We'll sometime refer to a representation  $(\pi, V)$  as V, with the action of G or  $\mathfrak{g}$  implicit, sometimes refer to it just using the homomorphism  $\pi$ .

Note that these definitions make sense for vector spaces (respectively modules) over an arbitrary field k. We'll restrict our attention to the simplest case,  $k = \mathbf{C}$ . The groups and Lie algebras themselves are also defined over a field, which for us will be **R** or **C**. More generally, for the groups we'll consider one can keep track of whether the complex representations we study are "real" or "quaternionic". If a *G*-equivariant anti-linear map  $J: V \to V$  exists satisfying  $J^2 = 1$ , one can think of J as a complex conjugation and restriction to its +1 eigenspace gives a representation on a real vector space. If a *G*-equivariant anti-linear map  $J: V \to V$  exists satisfying  $J^2 = -1$ , one can use this J to give V the structure of a vector space over the quaternions.

For applications of representation theory in number theory, one may want to consider representations over the *l*-adic numbers  $\mathbf{Q}_l$  for *l* a prime.

To study representations, one would like to decompose them into simpler ones,

**Definition 3** (Irreducible Representation). A representations is irreducible if it has no non-trivial sub-representations (i.e. no non-trivial subspace of V is invariant under the group action).

A weaker property is

**Definition 4** (Indecomposable Representation). A representation is indecomposable if it is not the direct sum of two non-trivial proper subrepresentations.

We'll see that for semi-simple Lie groups irreducibility and indecomposability coincide, but for solvable Lie groups one can have representations that are indecomposable, but not irreducible. For solvable Lie groups one has

**Theorem 1** (Lie-Kolchin theorem). If  $(\pi, V)$  is a representation of a solvable Lie group G of dimension n, there is a complete flag

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V$$

of invariant subspaces  $V_i$  of dimension *i*.

This implies that any such representation can be put in the form of a subgroup of the upper-triangular matrices pf GL(V), and any solvable Lie group has a one-dimensional irreducible representation  $(V_1)$ . One may however not be able to find a *G*-invariant complement to  $V_1$ , for example in the case of all upper-triangular matrices in GL(V) acting on *V*.

For many purposes it is a good idea to think of the set of representations of G not just as a set, but as a category Rep(G). We can take the objects of this category to be the equivalence classes of representations,

**Definition 5** (Equivalence of Representations). Two representations  $\pi_1$  and  $\pi_2$  on a vector space V are said to be equivalent (or isomorphic) if they are related by conjugation, *i.e.* 

$$\pi_2(g) = h(\pi_1(g))h^{-1}$$

for  $h \in GL(V)$ .

The morphisms in the category Rep(G) are not arbitrary linear maps between the representation space, but G-equivariant maps known as *intertwining operators*.

**Definition 6** (Intertwining Operators). Given two representations  $(\pi_1, V_1)$  and  $(\pi_2, V_2)$  of G, the space of intertwining operators is the space  $Hom_G(V_1, V_2)$  of linear maps  $\phi: V_1 \to V_2$  satisfying

$$\phi \circ \pi_1(g) = \pi_2(g) \circ \phi$$

A basic fact about intertwining operators is

**Theorem 2** (Schur's Lemma). Given two finite-dimensional complex irreducible representations  $V_1, V_2$  of a Lie group G, the intertwiners satisfy

- $Hom_G(V_1, V_2) = \{0\}$  (the zero map) if  $V_1$  is not isomorphic to  $V_2$ .
- $Hom_G(V_1, V_2) = \mathbf{C}$  if  $V_1$  is isomorphic to  $V_2$ .

and one has the same fact for Lie algebra representations. Over a general field k, one gets that  $Hom_G(V_1, V_1)$  ( $V_1$  irreducible) can be a division algebra over k, not just k. In the case of infinite dimensional complex representations, if these are unitary Schur's lemma remains true, with the proof using the spectral theorem for self-adjoint operators. For Lie algebras, Schur's lemma remains true for arbitrary irreducible  $U(\mathfrak{g})$  modules, even when infinite-dimensional[?].

Note that equivalence classes of representations form a ring

**Definition 7** (Representation Ring). The representation ring R(G) is the ring of equivalence classes [V] of representations of G, with sum and product given by

$$[V_1] + [V_2] = [V_1 \oplus V_2], \ [V_1] \cdot [V_2] = [V_1 \otimes V_2]$$

For Lie groups, one way to distinguish non-isomorphic representations is by their character:

**Definition 8** (Character of a (finite-dimensional) representation). The character  $\chi_V$  of a representation  $(\pi, V)$  of G is the function on G

$$\chi_V = Tr(\pi(g))$$

It gives a ring homomorphism from R(G) to the ring of conjugation-invariant functions on G.

For infinite-dimensional representations, one can often make sense of the character as a distribution, defining the Harish-Chandra character on a function f by

$$\Theta_{\pi}(f) = Tr(\int_{G} f(g)\pi(g)dg)$$

For a Lie algebra  $\mathfrak{g}$ , one can define a different sort of invariant of a representation:

**Definition 9.** If the center  $Z(\mathfrak{g}) \subset U(\mathfrak{g})$  acts by scalars on a representation  $(\pi, V)$ , the representation is said to have infinitesimal character, and this is given by a homomorphism

$$\chi_{\pi}: Z(\mathfrak{g}) \to C$$

with  $\chi_{\pi}(z)$  the scalar by which z acts.

The infinitesimal character can be used to study Lie algebra representations that are not representations of the corresponding Lie group. Examples are the Verma modules we will begin studying next time. One source of Lie algebra representations that are not Lie group representations is functions on an open subset of a Lie group G that is invariant under  $\mathfrak{g}$ , but not under G. For representations that have both a distributional character and an infinitesimal character, the two are related by

$$\Theta_{\pi}(zf) = \chi_{\pi}(z)\Theta_{\pi}(f)$$

i.e. the characters is an eigendistribution for  $Z(\mathfrak{g})$ , with eigenvalues given by the infinitesimal character.

# 2 Representations of Finite Groups, Generalities

In this course we will stick to the case of complex representations, i.e. representations on complex vector spaces.

**Definition 10** (Representation). A representation  $(\pi, V)$  of a group G on a vector space V is a homomorphism

$$\pi: G \to GL(V)$$

where GL(V) is the group of invertible linear transformations of V.

For now we will just be considering vector spaces V of finite dimension. Note that we will sometimes refer to a representation by just specifying  $\pi$  or V.

If V is an n-dimensional complex vector space and we choose a basis, a representation is given explicitly by, for each  $g \in G$ , an n by n invertible complex matrix, with entries  $[\pi(g)]_{ij}$ , satisfying

$$\sum_{j} [\pi(g_1)]_{ij} [\pi(g_2)]_{jk} = [\pi(g_3)]_{ik}$$

when  $g_1 g_2 = g_3$ .

Given any representation, one can look for subrepresentations. These are smaller representations contained in the representation, i.e. proper subspaces of V left invariant by the action of the group. An *irreducible representation* will be a representation with no proper subrepresentations, if there is a proper subrepresentation, the representation is called *reducible*. The main goal of representation theory will be to understand these irreducible representations, as well as how to decompose an arbitrary representation in terms of them. A group G is said have the property of *complete reduciblity* if any representation can be decomposed into a direct sum of irreducible representations.

**Theorem 3** (Complete Reducibility). Finite groups G have the property of complete reducibility.

*Proof.* Given any (positive-definite, Hermitian) inner product  $\langle \cdot, \cdot \rangle_0$  on V, one can form a *G*-invariant inner product by averaging over the *G* action

$$\langle v_1, v_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \pi(g) v_1, \pi(g) v_2 \rangle_0$$

If V is not already irreducible, one can pick a subrepresentation W. Then one can check that its orthogonal complement  $W^{\perp}$  will also be a subrepresentation and  $V = W \oplus W^{\perp}$ .

If W or  $W^{\perp}$  are not irreducible, one can decompose them into direct sums of subrepresentations, continuing this process until one has expressed V as a direct sum of irreducibles.

Note that this proof just relies on the possibility of constructing an invariant inner product. The same argument works for compact Lie groups. Another implication of the existence of such an invariant inner product is that these representations of finite groups are unitary: the representation  $\pi$  is a homomorphism of G into a subgroup  $U(n) \subset GL(V)$ , the subgroup preserving the inner product.

We will be using the convention

$$\langle z,w\rangle = \sum_i \overline{z_i} w_i$$

Note that this is the standard convention used by physicists, but many mathematicians use the opposite convention, conjugating the second variable.

One of our main goals is to classify all representations of G. In doing this, we don't want to distinguish between representations that differ just by a change of basis, i.e. by conjugation in GL(V):

**Definition 11** (Equivalence of Representations). Two representations  $\pi_1$  and  $\pi_2$  on a vector space V are said to be equivalent if they are related by conjugation, *i.e.* 

$$\pi_2(g) = h(\pi_1(g))h^{-1}$$

for  $h \in GL(V)$ .

The set of *n*-dimensional isomorphism classes of representations will be given by taking the quotient of the set  $Hom(G, GL(n, \mathbb{C}))$  by this conjugation action of  $GL(n, \mathbb{C})$ .

For many purposes it is a good idea to think of the set of representations of G not just as a set, but as a category. If we take the objects of this category to be the isomorphism classes of representations, the morphisms in the category are not arbitrary linear maps between the representation space, but G-equivariant maps known as *intertwining operators*.

**Definition 12** (Intertwining Operators). Given two representations  $(\pi_1, V_1)$ and  $(\pi_2, V_2)$  of G, the space of intertwining operators is the space  $Hom_G(V_1, V_2)$ of linear maps  $\phi: V_1 \to V_2$  satisfying

$$\phi \circ \pi_1(g) = \pi_2(g) \circ \phi$$

The structure of the category of representations of a finite group is rather simple since the intertwiners between irreducibles satisfy the following property

**Theorem 4.** Given two irreducible representations  $V_1, V_2$  of a finite group G, the intertwiners satisfy

•  $Hom_G(V_1, V_2) = \{0\}$  (the zero map) if  $V_1$  is not isomorphic to  $V_2$ .

•  $Hom_G(V_1, V_2) = \mathbf{C}$  if  $V_1$  is isomorphic to  $V_2$ .

The first part of the theorem follows from the observation that the kernel and image of an intertwining map are both invariant subspaces. They can't be proper subspaces since  $V_1$  and  $V_2$  are irreducible. So the only possibility is for the map to be either zero or an isomorphism. The second part of the theorem is known as Schur's lemma, and is equivalent to the following:

**Lemma 1** (Schur's Lemma). If V is an irreducible representation of a finite group G, then every linear map  $\phi : V \to V$  commuting with the action of all elements of g on V is a scalar.

*Proof.* One can easily see that the fact that  $\phi$  commutes with G implies that the eigenspace  $V_{\lambda}$  of V corresponding to an eigenvalue  $\lambda$  of  $\phi$  is invariant under G. By irreducibility  $V_{\lambda} = V$ , and  $\phi$  is just multiplication by the scalar  $\lambda$ .

Note that this argument crucially relies on the fact that we are dealing with complex representations. For real representations  $\phi$  may have no real eigenvalues. The story over a general field K is that  $Hom_G(V, V)$  can be seen to be an algebra over K, with multiplication given by composition of maps, and that it must be a finite dimensional division algebra. Schur's lemma corresponds to the fact that **C** is the only finite dimensional division algebra over the field **C**. Over **R** there are three finite dimensional division algebras (**R**, **C**, and **H**), and all of these occur as possible automorphisms of irreducible real representations of finite groups.

Schur's lemma has a wide variety of important corollaries, including:

#### **Corollary 1.** If G is abelian, all its irreducible representations are one-dimensional.

*Proof.* For G abelian, every  $g \in G$  and every representation  $(\pi, V)$  give elements  $\pi(g) \in Hom_G(V, V)$ , since the  $\pi(g)$  for different g commute. If V is irreducible, these  $\pi(g)$  must all be given by scalar multiplication. Then any subspace of V is an invariant subspace, implying the existence of subrepresentations and thus a contradiction if V is not one-dimensional.

Schur's lemma is also important in the following crucial theorem, that describes how an arbitrary representation decomposes into irreducibles. It tells us that to do this we need to understand two things:

- The irreducible representations  $(\pi_i, V_i)$ .
- The spaces of intertwining operators  $Hom_G(V_i, V)$ .

Sometimes the  $Hom_G(V_i, V)$  have no interesting structure other than their dimension as vector spaces, these dimensions are integers called the *multiplicities*  $n_i$  of the i'th irreducible in V. Often V will have some other structure, such as the action of another group H, commuting with G. In this case the  $Hom_G(V_i, V)$ will provide interesting representations of H. **Theorem 5** (Canonical Decomposition Theorem). If *i* is an index varying over a complete set of irreducibles  $(\pi_i, V_i)$  of a finite group *G*, and

$$\mu_i: Hom_G(V_i, V) \otimes V_i \to V$$

is the canonical G-map given by

$$\mu_i(f\otimes v_i)=f(v_i)$$

then

$$\mu = \oplus_i \mu_i : \oplus_i Hom_G(V_i, V) \otimes V_i \to V$$

is an isomorphism of G representations (on the left side, G acts trivially on the first factor, as the irreducible representation on the second).

*Proof.* In general, we know that V can be decomposed into a direct sum of irreducibles  $V_i$ , this will take the form

$$V = \oplus_i n_i V_i$$

where  $n_i V_i$  is direct sum of  $n_i$  copies of  $V_i$ . The domain of  $\mu$  is the same representation since

$$\oplus_i Hom_G(V_i, V) \otimes V_i = \oplus_i Hom_G(V_i, \oplus_j n_j V_j) \otimes V_i$$
(1)

$$= \oplus_i (\oplus_j n_j Hom_G(V_i, V_j)) \otimes V_i$$
(2)

$$= \oplus_i n_i Hom_G(V_i, V_i) \otimes V_i \tag{3}$$

$$= \oplus_i n_i (\mathbf{C} \otimes V_i) = \oplus_i n_i V_i \tag{4}$$

where we used Schur's lemma to get the third and fourth equalities. Again by Schur's lemma,  $\mu$  is an isomorphism when V is an irreducible  $V_i$ , and this remains true when  $V = n_i V_i$ . A final application of Schur's lemma shows that  $\mu$  must take  $n_i V_i$  to  $n_i V i$ .

Note that this theorem implies that the multiplicities are uniquely determined. If one knows that  $V = \bigoplus_i n_i V_i$  and  $\bigoplus_i m_i V_i$ , one must have  $n_i = m_i = dim Hom_G(V_i, V)$ .

Given two representations  $(\pi_1, V_1)$  and  $(\pi_2, V_2)$ , besides forming the direct sum representation  $V_1 \oplus V_2$ , one can also form tensor product representations  $V_1 \otimes V_2$ , as well as  $Hom(V_1, V_2)$ . The group acts on the tensor product in the obvious way by  $\pi_1 \otimes \pi_2$ . It acts on  $f \in Hom(V_1, V_2)$  by

$$(\pi(g)f)(v) = \pi_2(g)(f(\pi_1(g^{-1})v))$$

This is the action that makes the following diagram commute:

$$V_1 \xrightarrow{f} V_2$$

$$\pi_1(g) \downarrow \qquad \qquad \qquad \downarrow \pi_2(g)$$

$$V_1 \xrightarrow{\pi(g)f} V_2$$

An important special case of the construction  $Hom(V_1, V_2)$  is to take  $V = V_1, V_2 = \mathbf{C}$ , giving the dual representation  $(\pi_{V^*}, V^*) = Hom(V, \mathbf{C})$ , with  $\pi_{V^*}(g) = \pi_V(g^{-1})$ .

One can define an algebraic gadget that captures much of the structure of the set of irreducible representations, this is the *representation ring* 

**Definition 13.** Let R(G) be the free abelian group generated by the equivalence classes of irreducible representations of G. R(G) is a ring under the multiplication induced by taking the tensor product of representations.

An element of R(G) is given by formal linear combinations  $\sum_i n_i[V_i]$ , where the  $n_i$  are integers (possibly zero or negative). These elements are also known as *virtual representations*. The representation ring comes with a natural inner product in which the irreducible representations form an orthonormal basis, using

$$\langle V_1, V_2 \rangle = \dim Hom_G(V_1, V_2)$$

The decomposition of any representation V into irreducibles can be computed from knowledge of these inner products. If  $V = \sum_{i} n_i V_i$ ,

$$n_i = \langle V_i, V \rangle = \dim Hom_G(V_i, V)$$

To understand the ring structure of R(G), we need to know how the product of irreducible decomposes into irreducibles, i.e. the structure constants  $n_{ij}^k$ defined by

$$V_i \otimes V_j = \sum_k n_{ij}^k V_k$$

Note that the  $n_{ij}^k$  can be computed in terms of the inner product

$$n_{ij}^k = \langle V_k, V_i \otimes V_j \rangle = \dim Hom_G(V_k, V_i \otimes V_k)$$

## 3 Character Theory

We would like to have some concrete way of easily distinguishing inequivalent representations, and computing the  $\dim Hom_G(V_i, V)$  that tell us how an arbitrary representation decomposed into irreducibles. This can be done by associating to a representation a function on G called its *character*. To motivate this definition, consider the following proposition:

**Lemma 2.** The operator  $e_1^V = \frac{1}{|G|} \sum_{g \in G} \pi(g) : V \to V$  is idempotent  $((e_1^V)^2 = e_1^V)$ , equal to the identity on  $V^G$  (the G-invariant component of V), and zero on the rest of V.

*Proof.* We'll show that the image of  $e_1^V$  is *G*-invariant:

For  $v \in V, g \in G$ 

$$\pi(g)e_1^V v = \frac{1}{|G|}\pi(g)\sum_{h\in G}\pi(h)v$$
(5)

$$= \frac{1}{|G|} \sum_{h \in G} \pi(g)\pi(h)v \tag{6}$$

$$= \frac{1}{|G|} \sum_{g'} \pi(g')v \tag{7}$$

$$= e_1^V v \tag{8}$$

and if G acts trivially on v,

$$e_1^V v = \frac{1}{|G|} \sum_{g \in G} (1)v = v$$

Using this, one way to compute the multiplicity n of the trivial representation in a representation V is to just take the trace of the operator  $e_V$ .

$$n = \dim V^G = \dim Hom_G(\mathbf{C}, V) = Tr(e_1^V)$$

Picking a basis of V, the trace is just the trace of the matrix  $e_1^V$ , but the trace of a matrix is independent of the basis, it is a conjugation invariant function on invertible matrices, satisfying

$$Tr(UMU^{-1}) = Tr(M)$$

This motivates to some extent the following definition, which associates to any representation a conjugation invariant function on the group, called the *char*-*acter* of the representation.

**Definition 14.** The character of a representation  $(\pi, V)$  is the complex function  $\chi_V : G \to \mathbf{C}$  given by

$$\chi(g) = Tr(\pi(g))$$

Note that our lemma above tells us that

dim 
$$V^G = Tr(e_1^V) = \frac{1}{|G|} \sum_{g \in G} Tr(\pi(g)) = \frac{1}{|G|} \sum_{g \in G} \chi(g)$$

Using the properties of the matrix trace, we can quickly see the following facts:

1. 
$$\chi_V(Id) = \dim V$$

2. 
$$\chi_V(hgh^{-1}) = \chi_V(g)$$

- 3. Equivalent representations have the same character.
- 4.  $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$
- 5.  $\chi_{V_1 \otimes V_2} = \chi_{V_1} \chi_{V_2}$
- 6.  $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$

These are all straightforward from properties of the matrix trace, with 6) following from the fact that we are working with unitary representations and thus unitary matrices. Using these properties of the trace we also have:

$$\chi_{Hom(V,W)} = \chi_{V^* \otimes W} = \overline{\chi_V} \chi_W$$

Finally, we can use these properties of the trace, together with our formula for the dimension of the invariant subspace of a representation to get an explicit formula for the multiplicity of an irreducible  $V_i$ , assuming that we know the character of the irreducible  $\chi_{V_i}$ 

$$n_i = \dim Hom_G(V_i, V) \tag{9}$$

$$= \dim (Hom(V_i, V))^G \tag{10}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{Hom(V_i,V)} \tag{11}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{V_i^* \otimes V} \tag{12}$$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{V_i}} \chi_V \tag{13}$$

The characters give us an explicit formula for the inner product on the representation ring

$$\langle V, W \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V} \chi_W$$

The map  $[V] \rightarrow \chi_V$  extends to an injective ring homomorphism

$$R(G) \to C(G)^G$$

from the representation ring to the ring of conjugation invariant functions (*class functions*) on G.

# 4 The Regular Representation

While groups act on all sorts of spaces, not just vector spaces, one reason why we restrict our attention in representation theory to group actions on vector spaces is that we can always "linearize" a group action on a space by looking at the induced action on functions on the space. If we are given a group action, acting on the left:

$$(g,m) \in G \times M \to gm \in M$$

and if C(M) is a space of functions on M, we get a representation  $(\pi, C(M))$  of G by defining

$$\pi(g)f(m) = f(g^{-1}m)$$

The inverse has to be there in order to make this a homomorphism. One way to see this is that given a map  $\phi: M_1 \to M_2$ , the induced "pullback" map on functions

$$\phi^* : f \in C(M_2) \to (\phi^* f)(m) = f(\phi(m)) \in C(M_1)$$

goes in the opposite direction. In our case:

$$(\pi(g_1)(\pi(g_2)f))(m) = (\pi(g_2)f)(g_1^{-1}m) = f(g_2^{-1}g_1^{-1}m) = f((g_1g_2)^{-1}m) = (\pi(g_1g_2)f)(m)$$

. One space that the group G always acts on is itself, and the representation one gets on functions is called the *regular* representation.

**Definition 15.** The (left) regular representation  $(\pi_L, \mathbf{C}(G))$  is the representation of G on complex valued functions on G given by

$$\pi_L(g)f(h) = f(g^{-1}h)$$

Note that we could also work with the action of G on itself given by right multiplication. In this case we get what is called the (right) regular representation, given by

$$\pi_R(g)f(h) = f(hg)$$

One can check that using the right action, this is the correct formula to get a homomorphism. Note that the right action commutes with the left action, and what we have is actually a representation  $(\pi, \mathbf{C}(G))$  of  $G \times G$ , with  $\pi(g_1, g_2)f(g) = f(g_1^{-1}gg_2)$ . For now, we will just just the left action, and consider this as a representation of G.

We would like to decompose the regular representation into irreducibles, and as we have seen, the way to do this is by using characters. First we'll compute the character of the regular representation:

**Claim 1.** The character  $\chi_L$  of the regular representation satisfies:

$$\chi_L(g) = \begin{cases} 0 & \text{if } g \neq e \\ |G| & \text{if } g = e \end{cases}$$

*Proof.* Whenever we have a group acting by permutations on a set X, the representation  $\pi$  on functions on that set will satisfy

$$\chi_{\pi}(g) = |X^g|$$

 $(|X^g|$  is the number of points in the set left fixed by the action of g). To see this, consider the representation as a matrix with respect to a basis  $\{\mathbf{e}_x\}$  consisting of functions that are 1 on x, 0 elsewhere. In this basis,  $\pi(g)$  has diagonal elements equal to 1 exactly corresponding to those x left fixed by g. Taking the trace just counts these. Applying this argument to the left action of G on itself, we get the proposition.

What we really want to know is, for each irreducible  $V_i$ , the multiplicity  $n_i = \dim Hom_G(V_i, \mathbf{C}(G))$ . Computing these using characters we get:

$$n_{i} = \langle V_{i}, \mathbf{C}(G) \rangle$$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{V_{i}}(g)} \chi_{\mathbf{C}(G)}(g)$$

$$= \frac{1}{|G|} \overline{\chi_{V_{i}}(e)} |G|$$

$$= \dim V_{i}$$

We see that every irreducible  $V_i$  occurs in the left regular representation, with multiplicity given by the dimension of the representation. So, as a Grepresentation (with G acting on the left), we have

$$\mathbf{C}(G) = \oplus_i (\dim V_i) V_i$$

The canonical decomposition theorem tells us that

$$\mathbf{C}(G) = \bigoplus_i Hom_G(V_i, \mathbf{C}(G)) \otimes V_i$$

so we have learned that

dim 
$$Hom_G(V_i, \mathbf{C}(G)) = \dim V_i$$

The space  $Hom_G(V_i, \mathbf{C}(G))$  is invariant under the action of G we have been using, but recall that there is another copy of G, acting from the right, and its action commutes with the left action of G. Under this right action of G, the  $V_i$ term in the tensor product is invariant, but the space of intertwining operators gives a nontrivial representation:

**Claim 2.** The right regular representation of G induces an action on  $Hom_G(V_i, \mathbf{C}(G))$ , equivalent to the representation of G on  $V_i^*$ .

 $\mathit{Proof.}\,$  The details of the proof will be left as an exercise, but here is an outline: If

$$T \in Hom_G(V_i, \mathbf{C}(G))$$

show that

$$(\pi(g)T)(v) = \pi_R(Tv)$$

gives a well-defined action of G on  $Hom_G(V_i, \mathbf{C}(G))$ , and that this action is isomorphic with the action fo G on  $V_i^*$ , with the intertwining isomorphism given by

$$\lambda: T \in Hom_G(V_i, \mathbf{C}(G)) \to \lambda T \in V_i^*$$

where  $\lambda T$  is defined by

$$\lambda T(v) = (T(v))(e)$$

(where e is the identity of G).

This is a special case of something called the Frobenius reciprocity theorem, which we will review when we come to it later in the course.

We have shown that, knowing the irreducible representations of G, the space  $\mathbf{C}(G)$  decomposes under the combined  $G \times G$  action as

$$\mathbf{C}(G) = \oplus_i (V_i^* \otimes V_i)$$

with one copy of G acting on the  $V_i$ , the other on the  $V_i^*$ . This is the decomposition of  $\mathbf{C}(G)$  into irreducibles as a representation of  $G \times G$ . In an exercise, you will show that irreducible representations of a product group  $G \times H$  are given by tensor products  $V \otimes W$ , where V is an irreducible representation of G, W is an irreducible representation of H.

This still does not tell us what the irreducible representations are. Note that  $V_i^* \otimes V_i = End(V_i)$ . Once we do know  $(\pi_{V_i}, V_i)$ , we can identify the corresponding subspace of  $\mathbf{C}(G)$  as the subspace spanned by the matrix elements in the representation  $V_i$ , i.e. all functions of the form

$$l(\pi_{V_i}(g)v), v \in V_i, l \in V_i^*$$

It is a standard result found in all the referenced texts on finite group representations that, choosing an orthonormal basis in  $V_i$  and dual basis in  $V_i^*$ , the elements of the matrices representing  $\pi_{V_i}$  with respect to this basis are orthogonal functions on G, using the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

## **5** An Example: $S_3$

Finally, let's work out a simple example, the group  $S_3$  of permutations of a set with three elements. This group has a total of six elements (the identity, three transpositions of order two, and two permutations of order three). The regular representation will be on  $\mathbf{C}^6$ , and it has to decompose into subrepresentations whose dimensions are squares. They can't all be one-dimensional, since the group is non-abelian, so the decomposition must go as 6 = 1 + 1 + 4, and there must be two irreducible representations of dimension 1, and one irreducible representation of dimension 2. More explicitly, these representations are

- The trivial representation  $(Id, \mathbf{C})$
- The sign representation  $(\pi_{sgn}, \mathbf{C})$ , given by  $\pi_{sgn}(g)v = sgn(g)v$ , where  $sgn(g) = \pm 1$  is the sign of the permutation.
- An irreducible representation on  $\mathbf{C}^2$  constructed as follows: Let  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  be a basis of  $\mathbf{C}^3$ , with G acting by taking  $\mathbf{e}_i$  to  $\mathbf{e}_{g(i)}$ . On coordinate functions  $z_i$  the action is given by  $\pi(g)z_i = z_{g^{-1}(i)}$ . This action leaves invariant the complex line proportional to (1, 1, 1), as well as the orthogonal subspace

$$V = \{(z - 1, z_2, z_3) \in \mathbf{C}^3 : z_1 + z_2 + z_3 = 0\}$$

This is our two-dimensional representation.

Finding the characters, checking orthogonality properties, etc. is left as an exercise.

## References

- [1] Serre, J.-P., Linear Representations of Finite Groups, Springer, 1977.
- [2] Teleman, C., Representation Theory, Notes from a course at Cambridge, Lent 2005, https://math.berkeley.edu/~teleman/math/RepThry.pdf
- [3] Simon, B., Representations of Finite and Compact Groups, American Mathematical Society, 1996.