The spinor and oscillator representation analogy

Mathematics GR6434, Spring 2023

The oscillator representation of a symplectic group that we have been discussing is closely analogous to the spinor representation of the orthogonal group. Here we'll make this analogy very explicit. This parallelism is well-known in physics, where the "canonical formalism" in quantum mechanics comes in both a "bosonic" version, with canonical commutation relations, and a "fermionic" version, with canonical anti-commutation relations. Much of this material is worked out in great detail in [1].

1 Classical theory, Lie groups and Lie algebras

Q: Symmetric non-degenerate bilinear form on $V = \mathbf{R}^n$	S: Antisymmetric non-degenerate bilinear form on $V = \mathbf{R}^{2d}$
Lie group $SO(n)$ preserving Q, with Lie algebra $\mathfrak{so}(n)$.	Lie group $Sp(2d, \mathbf{R})$ preserving S, with Lie algebra $\mathfrak{sp}(2d)$
$\pi_1(SO(2n)) = \mathbf{Z}_2$.	$\pi_1(Sp(2n), \mathbf{R}) = \mathbf{Z}$.
$Spin(n)$, double cover of $SO(n)$.	$Mp(2d, \mathbf{R})$, double cover of $Sp(2d, \mathbf{R})$.
$\Lambda^*(V^*)$: anti-symmetric algebra on V^*. Polynomials in "anti-commuting variables ξ_j, $j = 1, 2, \cdots n$. For physicists these are "fermionic" variables.	$S^*(V^*)$: symmetric algebra on V^*. Polynomial functions on V. Generated by a basis $q_j, p_k,\ j, k = 1, 2, \cdots d$ of V^*. For physicists these are "bosonic" variables.
Poisson bracket $\{\cdot, \cdot\}_+$. Lie bracket for Lie superalgebra of "anti-commuting functions on V, determined by Q.	Poisson bracket $\{\cdot, \cdot\}$. Lie bracket for Lie algebra of functions on V, determined by S.
Lie superalgebra of anticommuting polynomials on V of degree $0, 1, 2$. Semi-direct product of a Lie superalgebra (degree 0 and 1) and the orthogonal Lie algebra $\mathfrak{so}(n, \mathbf{R})$ (degree 2).	Lie algebra of polynomials on V of degree $0, 1, 2$. Semi-direct product of the Heisenberg Lie algebra $\mathfrak{h}_{2d+1}$ (degree 0 and 1) and the symplectic Lie algebra $\mathfrak{sp}(2d, \mathbf{R})$ (degree 2).
Pseudo-classical mechanics.	Classical mechanics.

2 Quantum theory and representations

Spin representation S (unitary) on a complex vector space of dimension $2^{\frac{n}{2}}$ for n even.	Oscillator representation (unitary) on $\mathcal{H}$, an infinite-dimensional Hilbert space.

Clifford algebra Cliff$(n, \mathbf{C})$. For n even this is the algebra $End(S)$, isomorphic to the matrix algebra $M(2^{\frac{n}{2}}, \mathbf{C})$.	Weyl algebra $U(\mathfrak{h}_{2d+1})/(Z-1)$. This algebra is infinite-dimensional over $\mathbf{C}$.
The group $SO(2n)$ acts by automorphisms on Cliff$(n, \mathbf{C})$.	The group $Sp(2n, \mathbf{R})$ acts by automorphism on the Weyl algebra.
For n even, Cliff$(n, \mathbf{C})$ has a single irreducible module, the spin module S. This is the spin representation as a Lie algebra representation of $\mathfrak{so}(2n)$. Integrating to the group, one gets a projective(up to $\pm$) representation of $SO(n)$, a true representation of the double cover $Spin(n)$.	Stone-von Neumann theorem: the Weyl algebra has a single irreducible module $\mathcal{H}$ that integrates to a representation of the Heisenberg group on $\mathcal{H}$. Integrating to the group, one gets a projective(up to $\pm$) representation of $Sp(2d, \mathbf{R})$, a true representation of the double cover $Mp(2d, \mathbf{R})$.
For n even, The spin representation has two irreducible components, the half-spinors S^+, S^-, each of dimension $2^{\frac{n}{2}-1}$	The oscillator representation has two irreducible components (an "even" and an "odd" component).
Generators γ_j of the Clifford algebra. On the spinor module S, identifying the Clifford algebra with a matrix algebra, these are the physicist's Dirac γ-matrices.	Generators Q_j, P_k of the Weyl algebra.
In even dimension, the Lie algebra representation operators for the spin representation are given by quadratic combinations of γ-matrices.	The Lie algebra representation operators for the oscillator representation are given by quadratic combinations of the Q_j, P_k operators.

3 Polarizations

For n even, choosing a real polarization $V = M \oplus M^*$ one can realize the spinor module as anticommuting functions on M. This will be an irreducible representation of the real form $SO(n, n)$, non-unitary.	Choosing a real polarization $V = M \oplus M^*$ one can realize (the Schrödinger representation) the Q_j, P_j operators respectively as multiplication and differentiation operators on $L^2(M)$. This representation will be unitary, both as a representation of the Heisenberg group and the metaplectic group.
For $n = 2d$ even, an orthogonal complex structure on V is a linear map J satisfying $J^2 = -\mathbf{1}$ and preserving the bilinear form Q. This picks out a $U(d) \subset SO(2d)$ and the space of such	A symplectic complex structure on V is a linear map J satisfying $J^2 = -\mathbf{1}$ and preserving the bilinear form S. This picks out a $U(n) \subset Sp(2n, \mathbf{R})$. Such J satisfying the positivity conditions

complex structures is the compact space $SO(2d)/U(n)$.

$S(\cdot, J\cdot)$ positive are parametrized by the non-compact space $Sp(2n, \mathbf{R})/U(n)$.

Such a J gives a complex polarization $V \otimes \mathbf{C} = W_J^+ \oplus W_J^-$ ($\pm i$ eigenspaces of J). This can be used to construct our representation as holomorphic functions (commuting or anticommuting) on W_J^+.

For n even, taking complex linear combinations of the γ_j one can form adjoint operators $a_j, a_j^\dagger$ on the spinor module, satisfying the canonical anti-commutation relations

$$[a, a^\dagger]_+ = \mathbf{1}$$

Taking complex linear combinations of the Q_j, P_k one can form adjoint operators $a_j, a_j^\dagger$ on the spinor module, satisfying the canonical commutation relations

$$[a, a^\dagger] = \mathbf{1}$$

Still to do, write spinors as functions on W_J.
Vacuum vectors, depend on J.
$a, a^\dagger$ as multiplication, differentiation.
Line bundle, relation to det bundle.
Representation at holomorphic sections of a line bundle.
More on the picture of space of polarizations. Non-positive polarizations?

Irreducible $C(n)$ module: $\Lambda^*(W_J)$	Irreducible H_n representation: $S^*(W_J)$
$S = \Lambda^*(W_J) \times (\Lambda^n(W_J))^{-\frac{1}{2}}$	$M = S^*$
Vacuum vector $\Omega_J \in S$	Vacuum vector $\Omega_J \in M$
Line bundle L, $L \otimes L = (\det)^{-1}$	Line bundle L, $L \otimes L = \det$
$S = \Gamma_{hol}(L)$	$M = \Gamma_{hol}(L)$
Particle with spin $\frac{1}{2}$ in $2n$ dimensions	Harmonic oscillator with n degrees of freedom

References

[1] Woit, P., *Quantum theory, groups and representations,* Springer, 2017.

[2] Segal, G., *Notes on quantum field theory, and on symplectic manifolds and quantization,* available at http://web.math.ucsb.edu/ drm/conferences/ITP99/segal/