Groups and Representations II: Problem Set 5

Due Thursday, April 13

Problem 1:

- For the Heisenberg Lie group H_3 and Lie algebra $\mathfrak{h}_3$, find the co-adjoint orbits, and explicitly give the Kirillov symplectic two-form on each orbit.
- Same for the Lie group $SU(2)$ and its Lie algebra $\mathfrak{su}(2) = \mathbf{R}^3$.

Problem 2: For representations V, W of compact Lie groups $H \subset G$, prove the Frobenius reciprocity theorem

$$Hom_G(V, Ind_H^G W) = Hom_H(Res_H^G V, W)$$

by explicitly constructing inverse maps in both directions.

Problem 3: Consider the complex-valued functions on S^2 known as spherical harmonics (see, e.g. section 8.3 of `https://www.math.columbia.edu/~woit/QMbook/qmbook-latest.pdf`. Using Peter-Weyl, Frobenius reciprocity and $S^2 = SU(2)/U(1)$ of $SO(3)$ as well as the classification of irreducible $SO(3)$ or $SU(2)$ representations, should that the Y_m^l for $m = -l, , -l+1, \cdots, l-1, l$ provide decomposition of the L^2 functions on S^2 into orthogonal subspaces transforming as irreducible representations of dimension $2l+1$, for $l = 0, 1, 2, \cdots$.

Problem 4: Read the discussion of Borel-Weil given in chapters 11 and 14 of Graeme Segal's Lectures on Lie Groups, in the volume Lectures on Lie Groups and Lie Algebras, available online through the Columbia library at `http://www.columbia.edu/cgi-bin/cul/resolve?clio14116206`. Adapt the holmorphic induction constructions give there to the case of $SL(n, \mathbf{C})$, giving a detailed construction of the irreducible representations corresponding to $k\omega_1, \omega_2, \cdots, \omega_{n-1}$ where k is a positive integer, and ω_j is a fundamental weight.

Problem 5: For the case of $G = SU(3)$, for each i, identify explicity the set of integral weights such that

$$H^{0,i}(G/T, L_\lambda) \neq 0$$

Consider the highest weight of the adjoint representation of $SU(3)$. Letting the Weyl group act on this gives a set of six different weights λ_j Compute the cohomology

$$H^{0,i}(G/T, L_{\lambda_j})$$

for all choices of i, j (i.e. what is its dimension? what is it as an $SU(3)$ representation?).