Groups and Representations II: Problem Set 2

Due Tuesday, February 14

Note: some of these problems involve material that is a prerequisite for this course, covered for instance last semester (Cartan subalgebra, roots, representations of $SU(2)$). In the next part of the course we'll start seriously using these, so if you haven't seen them before, having to work with them for these problems should help get you started.

Problem 1: In class I worked out the isomorphism between the Lie algebra of quadratic polynomials in variables q, p with bracket the Poisson bracket, and the matrix Lie algebra $\mathfrak{sp}(2, \mathbf{R}) = \mathfrak{sl}(2, \mathbf{R})$. Do the same for the case of n degrees of freedom (i.e. variables $q_1, \cdots q_n, p_1, \cdots p_n$, matrix group $Sp(2n, \mathbf{R})$ and matrix Lie algebra $\mathfrak{sp}(2n, \mathbf{R})$).

Problem 2: Find a Cartan sub-algebra and positive/negative roots for the Lie algebra $\mathfrak{sl}(2, \mathbf{R})$. More specifically, solve problems 6,7,8 in the notes/assignment I've stolen from a class of Shlomo Sternberg's, see
https://www.math.columbia.edu/~woit/LieGroups-2023/sternberg-math128-assignment.pdf

Problem 3: In class we made a standard choice of complex structure J, which picks out a $U(1) \subset SL(2, \mathbf{R})$, and then computed the action of the Lie algebra of $U(1)$ on the Bargmann-Fock version of the oscillator representation (finding the $\frac{1}{2}$ term that shows the representation has a sign problem). Can you do the same thing for arbitrary n, where a standard choice of J will determine a $U(n) \subset Sp(2n, \mathbf{R})$ and one wants to find the corresponding Lie algebra representation operators?

Problem 4: Show that, for J chosen as in problem 3, the representation space of the oscillator representation can be identified with the symmetric algebra $S^*(\mathbf{C}^n)$. Show that the action of $U(n)$ on this space is the standard action on the symmetric tensor product, up to a projective factor that you should identify. For $n = 2$, what is the decomposition into irreducibles of the oscillator representation as a representation of $SU(2)$?

Problem 5: Repeat the calculations of problems 3 and 4, but now for the spinor representation, using the analogy discussed in class. Again there will be a standard J, now a subgroup $U(n) \subset SO(2n)$ and the spinor representation space can be identified with the algebra of antisymmetric tensors $\Lambda^*(\mathbf{C}^n)$.