An Introduction to the Volume Conjecture, III
Generalizations

Hitoshi Murakami

Tokyo Institute of Technology

9th June, 2009

Complexification

Conjecture (Volume Conjecture, R. Kashaev, J. Murakami+H.M.)

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K; \exp(2\pi\sqrt{-1}/N))|}{N} = \mathrm{Vol}(S^3 \setminus K).$$

Conjecture (Complexification of VC, J. Murakami, M. Okamoto, T. Takata, Y. Yokota,+H.M.)

$$2\pi \lim_{N\to\infty} \frac{\log J_N(K; \exp(2\pi\sqrt{-1}/N))}{N} = \mathrm{Vol}(S^3 \setminus K) + \sqrt{-1}\,\mathrm{CS}(S^3 \setminus K) \pmod{\pi^2\sqrt{-1}\mathbb{Z}}.$$

Here CS *is the* $SL(2;\mathbb{C})$ *Chern–Simons invariant.*

We may regard the left hand side as the definition of the Chern–Simons invariant for general knots.

Deform the parameter $2\pi\sqrt{-1}$

- In VC, the limit corresponds to the complete hyperbolic structure of $S^3 \setminus K$ (if it is hyperbolic).
- The complete structure can be deformed to incomplete ones.
- If we deform the parameter $2\pi\sqrt{-1}$, does the limit corresponds to an incomplete hyperbolic structure?
- Let us consider the limit

$$\lim_{N\to\infty} \frac{\log J_N\Big(K; \exp\big((u + 2\pi\sqrt{-1})/N\big)\Big)}{N}$$

When $u = 0$, we have the (complexified) Volume Conjecture.

Generalization for 

Theorem (Yokota+H.M.)

$\exists\mathcal{O}\subset\mathbb{C}$: *neighborhood of* 0. *If* $u\in\mathcal{O}\setminus\pi\sqrt{-1}\mathbb{Q}$, *the following limit exists*

$$\lim_{N\to\infty}\frac{\log J_N(\text{knot};\exp((u+2\pi\sqrt{-1})/N))}{N}$$

(knot in the formula above: )

Put

$$H(u):=(u+2\pi\sqrt{-1})\times\textit{(the limit above)}.$$

- $H(u)$ *is differentiable,*
- $v(u):=2\dfrac{d\,H(u)}{d\,u}-2\pi\sqrt{-1}$ *satisfies the following.*

  Vol( $_u$) $+\sqrt{-1}$ CS( $_u$)

  $$\equiv-\sqrt{-1}H(u)-\pi u+u\,v(u)\sqrt{-1}/4-\pi\kappa(\gamma_u)/2\pmod{\pi^2\sqrt{-1}\mathbb{Z}}.$$

Deformation of the hyperbolic structure

-  $_u$ is the closed hyperbolic three-manifold defined by u, that is, it is defined by the following representation of $\pi_1\left(S^3 \setminus$  $\right) \to SL(2;\mathbb{C})$:

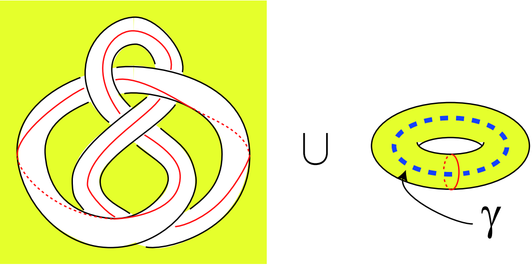
$$\begin{cases} \text{meridian} & \mapsto \begin{pmatrix} \exp(u/2) & * \\ 0 & \exp(-u/2) \end{pmatrix}, \\ \text{longitude} & \mapsto \begin{pmatrix} \exp(v(u)/2) & * \\ 0 & \exp(-v(u)/2) \end{pmatrix}. \end{cases}$$

  Here the meridian goes around  , and the longitude goes along  .

- When $u = 0$ this gives the holonomy representation, that is, each loop in $\pi_1\left(S^3 \setminus$  $\right)$ is identified with a deck transformation of the universal cover of $S^3 \setminus$  , which is $\mathrm{Isom}_+(\mathbb{H}^3) \cong PSL(2;\mathbb{C}) = SL(2;\mathbb{C})/\pm$.
- For $u \neq 0$, the hyperbolic structure is incomplete.

Dehn surgery

- If  $_u$ is incomplete, we can complete it by attaching either a point or a circle.
- γ_u is the attaching circle.
- If $pu + qv(u) = 2\pi\sqrt{-1}$, this is the (p, q)-Dehn surgery.



- $\kappa(\gamma_u) := \text{length}(\gamma_u) + \sqrt{-1}\,\text{torsion}(\gamma_u)$, where
  - length is its length,
  - torsion measures how the circle is twisted (mod 2π).

Precise expression of the limit

$J_N\Big($  $;q\Big) = \displaystyle\sum_{j=0}^{N-1} q^{jN} \prod_{k=1}^{j} \left(1 - q^{-N-k}\right)\left(1 - q^{-N+k}\right).$

Put

$$H(z,w) := \mathrm{Li}_2(z^{-1}w^{-1}) - \mathrm{Li}_2(zw^{-1}) + \log z \log w,$$

where

$$\mathrm{Li}_2(x) := -\int_0^x \frac{\log(1-t)}{t}\,dt.$$

If θ is near $2\pi\sqrt{-1} \in \mathbb{C}$ and not a rational multiple of $2\pi\sqrt{-1}$, then

$\theta \displaystyle\lim_{N\to\infty} \frac{\log J_N\big(}{}$  $\displaystyle\frac{;\exp(\theta/N)\big)}{N} = H\big(y, \exp(\theta)\big),$

where y satisfies

$$y + y^{-1} = \exp(\theta) + \exp(-\theta) - 1.$$

Approximation of the summand by dilogarithm

$q := \exp(\theta/N)$

$$
\begin{aligned}
&\log\left(\prod_{k=1}^{j}\left(1-q^{-N\pm k}\right)\right)\\
=&\sum_{k=1}^{j}\log\left(1-\exp(\pm k\theta/N-\theta)\right)\\
\underset{N\to\infty}{\sim}&N\int_{0}^{j/N}\log(1-\exp(\pm\theta s-\theta))\,ds\\
=&\frac{N}{\pm\theta}\int_{\exp(-\theta)}^{\exp(\pm j\theta/N-\theta)}\frac{\log(1-t)}{t}dt\\
=&\frac{N}{\pm\theta}\left(\mathrm{Li}_2\left(\exp(-\theta)\right)-\mathrm{Li}_2\left(\exp(\pm j\theta/N-\theta)\right)\right).
\end{aligned}
$$

Approximation of J_N by an integral

$J_N\Big($  $;\exp(\theta/N)\Big)$

$$\underset{N\to\infty}{\sim}\sum_{j=0}^{N-1}\exp(j\theta)\exp\left[\frac{N}{\theta}\Big(\mathrm{Li}_2\big(\exp(-j\theta/N-\theta)\big)-\mathrm{Li}_2\big(\exp(j\theta/N-\theta)\big)\Big)\right]$$

$$=\sum_{j=0}^{N-1}\exp\left[\frac{N}{\theta}H\big(\exp(j\theta/N),\exp(\theta)\big)\right]$$

$$\approx\int_C\exp\left[\frac{N}{\theta}H\big(x,\exp(\theta)\big)\right]\,dx$$

for a suitable contour C.

To find the 'maximum' of $\{H\big(x,\exp(\theta)\big)\}$, we will find a solution y to the equation $\dfrac{d\,H}{d\,x}\big(x,\exp(\theta)\big)=0$, which is

$$\frac{\log\left[\exp(\theta)+\exp(-\theta)-x-x^{-1}\right]}{x}=0.$$

Saddle point method

- Choose y so that

$$y + y^{-1} = \exp(\theta) + \exp(-\theta) - 1,$$

then

$J_N\Big($  $;\exp(\theta/N)\Big) \underset{N\to\infty}{\sim} \exp\left[\frac{N}{\theta}H\big(y,\exp(\theta)\big)\right]$

$\Rightarrow$

$\theta \lim_{N\to\infty} \frac{J_N\Big(}{}$  $\frac{;\exp(\theta/N)\Big)}{N} = H\big(y,\exp(\theta)\big).$

Putting $u := \theta - 2\pi\sqrt{-1}$, we have

$(u + 2\pi\sqrt{-1}) \lim_{N\to\infty} \frac{J_N\Big(}{}$  $\frac{;\exp((u+2\pi\sqrt{-1})/N)\Big)}{N} = H(u)$

with $H(u) := H\big(y,\exp(\theta)\big)$.
Note that this can be done rigorously.

Calculation of the volume using dilogarithm

- $\Delta(z)$, $\Delta(w)$: ideal hyperbolic tetrahedra parametrized by complex numbers z and w, respectively.
- $S^3 \setminus$  $= \Delta(z) \cup \Delta(w)$ if $z(z-1)w(w-1) = 1$. (This is just the glueing condition. The hyperbolic structure may not be complete. The completion condition is $w(1-z) = 1$.)
- Introduce parameters u and y so that

$$\exp u = w(1-z), \quad \text{(meridian)}$$
$$y + y^{-1} = \exp(u) + \exp(-u) - 1.$$

  Note that z, w and y are defined by u.
- Use the formula:

$$\operatorname{Vol}\bigl(\Delta(z)\bigr) = \operatorname{Im} \operatorname{Li}_2(z) + \log|z| \arg(1-z).$$

Calculation of the volume by H function

$\mathrm{Vol}(S^3 \setminus$  $)$

$$= \mathrm{Im}\, H(u) - \pi\, \mathrm{Re}\, u - \mathrm{Re}\, u\, \mathrm{Im} \log(z(1-z))$$

Since $\dfrac{d\,H(u)}{d\,u} = \log(z(z-1))$,

$\mathrm{Vol}(S^3 \setminus$  $) = \mathrm{Im}\, H(u) - \pi\, \mathrm{Re}\, u - \dfrac{1}{2}\mathrm{Re}\, u\, \mathrm{Im}\, v(u)$

putting $v(u) := 2\dfrac{d\,H(u)}{d\,u} - 2\pi\sqrt{-1}$.

Indeed, $\exp(v(u))$ corresponds to the longitude $z^2(1-z)^2$.

We will show:

$$\mathrm{length}\,\gamma_u = -\frac{1}{2\pi}\,\mathrm{Im}\left(u\overline{v(u)}\right).$$

Length of the geodesic γ_u (W. Neumann and D. Zagier)

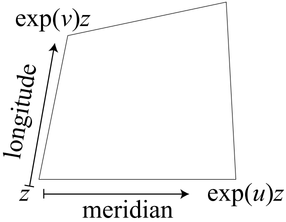
On $\partial\mathbb{H}^3 = S^2_\infty = \mathbb{C} \cup \{\infty\}$:

$$\mu := \text{meridian} \mapsto \left[z \mapsto \exp(u)z + c\exp(u/2)\right]$$
$$\lambda := \text{longitude} \mapsto \left[z \mapsto \exp(v)z + d\exp(v/2)\right].$$

- When $u = 0$, we have the complete structure.
  $\Rightarrow$ the corresponding representation is a parallel transport.
- When $u \neq 0$, we have an incomplete structure.
  $\Rightarrow$ Since the meridian and the longitude commute, their images have the same two fixed points; $\dfrac{c\exp(u/2)}{1-\exp(u)} = \dfrac{d\exp(v/2)}{1-\exp(v)}$ and ∞.
  Changing the coordinate, the fixed points are assumed to be O and ∞.
  $\Rightarrow$

$$\mu \mapsto \left[z \mapsto \exp(u)z\right]$$
$$\lambda \mapsto \left[z \mapsto \exp(v)z\right].$$

Calculation of the complex length



- Choose (p, q) so that $pu + qv = 2\pi\sqrt{-1}$ $(p, q \in \mathbb{R})$.
- Assume p and q are coprime integers.
- u defines an incomplete structure whose completion is the (p, q)-Dehn surgery.
- Choose (r, s) so that $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 1$.
- $\gamma_u = r\mu + s\lambda \in H_1(\partial(S^3 \setminus$  $))$.
  ($\because$ the meridian of the attached solid torus is identified with $p\mu + q\lambda$, and the meridian and γ_u make a basis of $H_1(\partial(S^3 \setminus$  $))$.)

Calculation of length and torsion

- γ_u corresponds to the multiplication by $\exp(ru+sv)$, and so
$$\exp(\text{length}+\sqrt{-1}\,\text{torsion}) = \exp(\pm(ru+sv)).$$
- In $\mathbb{H}^3$, this defines $\text{Im}(\pm(ru+sv))$-rotation, and an upward shift by $\exp\bigl(\text{Re}(\pm(ru+sv))\bigr)$ in coordinate, which has length $\text{Re}(\pm(ru+sv))$.
- $$\begin{cases} pu+qv & = 2\pi\sqrt{-1}, \\ ru+sv & = \pm(\text{length}+\sqrt{-1}\,\text{torsion}). \end{cases}$$
- $\text{length}\,\gamma_u = -\dfrac{1}{2\pi}\,\text{Im}\,(u\bar{v})$.

  (Here we choose the negative sign since $v = u \times \dfrac{|v|^2}{u\bar{v}}$ and the orientation of (u, v) should be positive on $\mathbb{C}$.)

Conclusion

$$\operatorname{length}\gamma_u = -\frac{1}{2\pi}\operatorname{Im}(u\overline{v}) = -\frac{1}{2\pi}\operatorname{Im}u\operatorname{Re}v + \frac{1}{2\pi}\operatorname{Re}u\operatorname{Im}v.$$

$\Rightarrow$

$$\mathrm{Vol}(S^3 \setminus$$

$$) = \operatorname{Im}H(u) - \pi\operatorname{Re}u - \frac{1}{2}\operatorname{Re}u\operatorname{Im}v(u)$$
$$= \operatorname{Re}(-\sqrt{-1}H(u) - \pi u + uv(u)\sqrt{-1}/4 - \pi\kappa(\gamma_u)/2),$$

The Chern–Simons invariant is obtained by T. Yoshida's formula.

Generalization of VC to hyperbolic knots

Conjecture

For any hyperbolic knot K, the following limit exists

$$\lim_{N\to\infty} \frac{\log J_N(K;\exp((u+2\pi\sqrt{-1})/N))}{N}$$

for small u. Put

$$H(K;u) := (u+2\pi\sqrt{-1}) \times \text{(the limit above).}$$

- $H(K;u)$ *is differentiable,*
- $v(K;u) := 2\dfrac{d\,H(K;u)}{d\,u} - 2\pi\sqrt{-1}$ *satisfies the following.*

$$\operatorname{Vol}(K_u) = \operatorname{Im} H(K;u) - \pi \operatorname{Re} u - \operatorname{Re} u \operatorname{Im} v(K;u)/2.$$

Small parameter

The previous conjecture should be compared with:

Theorem (S. Garoufalidis and T. Lê)

For any K, $\exists\varepsilon$ s.t. if $|a| < \varepsilon$

$$\lim_{N\to\infty} J_N(K;\exp(a/N)) = \frac{1}{\Delta(K;\exp a)},$$

where $\Delta(K;t)$ is the Alexander polynomial.

What happens between $2\pi\sqrt{-1}$ and 0?

FAQs

Q1. Is Jun Murakami your relative?

A1. No!

Q2. How about Haruki Murakami?

A2. Never!