An Introduction to the Volume Conjecture, I

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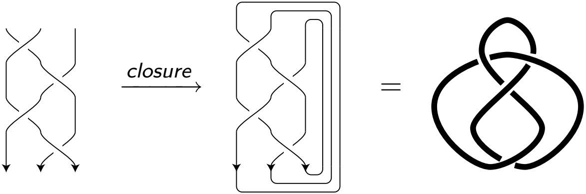
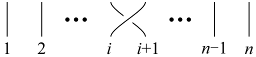
Braid presentation of a link

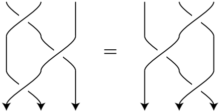
Theorem (J.W. Alexander)

Any knot or link can be presented as the closure of a braid.



n-braid group has

- generators: σ_i $(i = 1, 2, \ldots, n-1)$: 

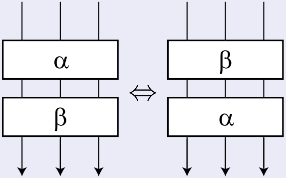
- relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ $(|i-j| > 1)$, 

Markov's theorem

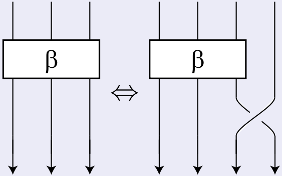
Theorem (A.A. Markov)

β and β' give equivalent links $\Leftrightarrow$ β and β' are related by

- *conjugation ($\alpha\beta \Leftrightarrow \beta\alpha$):*



- *stabilization ($\beta \Leftrightarrow \beta\sigma_n^{\pm 1}$):*



Yang–Baxter operator

- V: a N-dimensional vector space over $\mathbb{C}$.
- $R\colon V \otimes V \to V \otimes V$ (R-matrix), $\mu\colon V \to V$: isomorphisms,
- a, b: non-zero complex numbers.

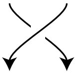
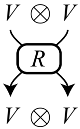
Definition (V. Turaev)

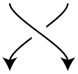
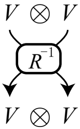
(R, μ, a, b) is called an enhanced Yang–Baxter operator if it satisfies

- $(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)$, (Yang–Baxter equation)
- $R(\mu \otimes \mu) = (\mu \otimes \mu)R$,
- $\mathrm{Tr}_2\big(R^{\pm}(\mathrm{Id}_V \otimes \mu)\big) = a^{\pm 1} b \, \mathrm{Id}_V$.

$\mathrm{Tr}_2\colon V \otimes V \to V$ is the operator trace. (For $M \in \mathrm{End}(V \otimes V)$ given by a matrix M^{ij}_{kl}, $\mathrm{Tr}_2(M)$ is given by $\sum_m M^{im}_{km}$.)

Braid ⇒ endomorphism

Replace  with 

, and  with 

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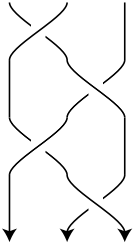
n-braid β ⇒ homomorphism $\Phi(\beta)\colon V^{\otimes n} \to V^{\otimes n}$



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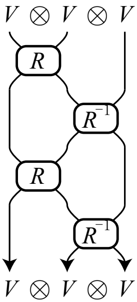


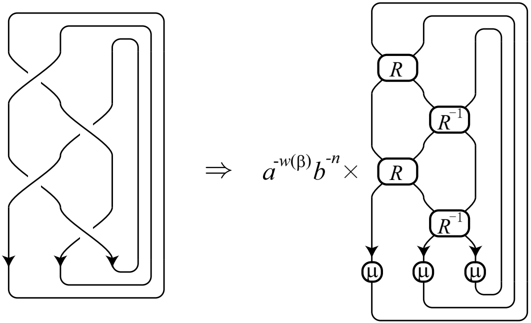
Definition of an invariant

Definition

n-braid $\beta \Rightarrow$ a link L.

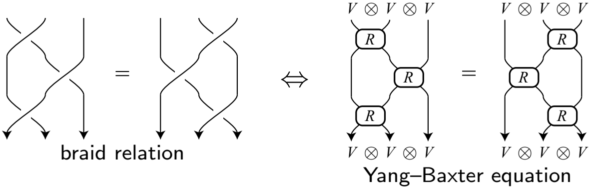
$$T_{(R,\mu,a,b)}(L) := a^{-w(\beta)} b^{-n} \operatorname{Tr}_1\Bigl(\operatorname{Tr}_2\bigl(\cdots(\operatorname{Tr}_n(\Phi(\beta)\mu^{\otimes n}))\cdots\bigr)\Bigr),$$

where $\operatorname{Tr}_k \colon V^{\otimes k} \to V^{\otimes(k-1)}$ is defined similarly.

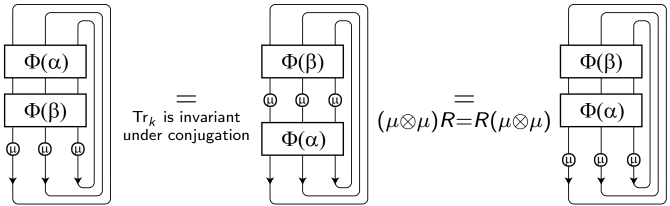


Invariance of $T_{(R,\mu,a,b)}(L)$ under braid relation and conjugation

- Invariance under the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i = \sigma_{i+1}$.

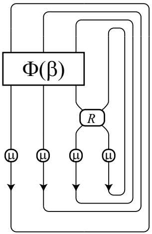


- invariance under conjugation

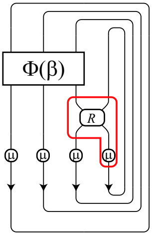


Invariance of $T_{(R,\mu,a,b)}(L)$ under stabilization

- invariance under stabilization



- invariance under stabilization



Quantum $(\mathfrak{g}, V)$ invariant

- $\mathfrak{g}$: a Lie algebra,
- V: its representation

$\Rightarrow$
an enhanced Yang–Baxter operator (R, μ, a, b).
$\Rightarrow$
quantum $(\mathfrak{g}, V)$ invariant.

Definition

The quantum $(sl(2, \mathbb{C}), V_N)$ invariant is called the N-dimensional colored Jones polynomial $J_N(L; q)$. (q is a complex parameter.)

- V_N: N-dimensional irreducible representation of $sl(2, \mathbb{C})$.
- $J_2(L; q)$ is the ordinary Jones polynomial.
- $J_N(\text{unknot}; q) = 1$.

Precise definition of the colored Jones polynomial

- $V := \mathbb{C}^N$.
- $$R^{ij}_{kl} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m}\frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$
  with $\{m\} := q^{m/2} - q^{-m/2}$ and $\{m\}! := \{1\}\{2\}\cdots\{m\}$.
- $\mu^i_j := \delta_{i,j}q^{(2i-N+1)/2}$.
- $R(e_k \otimes e_l) := \sum_{i,j=0}^{N-1} R^{ij}_{kl} e_i \otimes e_j$ and $\mu(e_j) := \sum_{i=0}^{N-1} \mu^i_j e_i$.

$\Rightarrow$
$(R, \mu, q^{(N^2-1)/4}, 1)$ gives an enhanced Yang–Baxter operator.

Definition

$J_N(L; q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(K) \times \frac{\{1\}}{\{N\}}$: colored Jones polynomial.

Volume conjecture

Conjecture (Volume Conjecture, R. Kashaev, J. Murakami+H.M.)

K: knot

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K; \exp(2\pi\sqrt{-1}/N))|}{N} = \mathrm{Vol}(S^3 \setminus K).$$

Definition (Simplicial volume (Gromov norm))

$$\mathrm{Vol}(S^3 \setminus K) := \sum_{H_i\text{:hyperbolic piece}} \text{Hyperbolic Volume of } H_i.$$

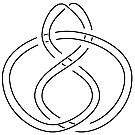
Definition (Jaco–Shalen–Johannson decomposition)

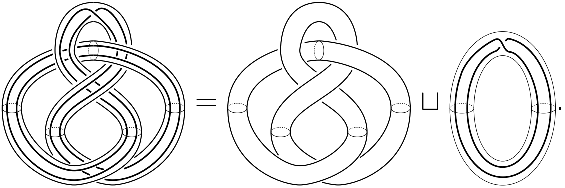
$S^3 \setminus K$ can be uniquely decomposed as

$$S^3 \setminus K = \left(\bigsqcup H_i\right) \sqcup \left(\bigsqcup E_j\right)$$

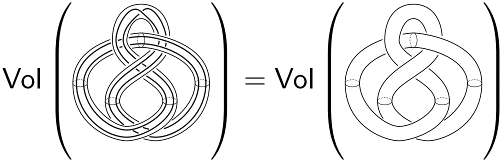
with H_i hyperbolic and E_j Seifert-fibered.

Example of JSJ decomposition





hyperbolic Seifert fibered



Colored Jones polynomial of 

Proof of the VC for  is given by T. Ekholm.

Theorem (K. Habiro, T. Lê)

$J_N\Big($  $;q\Big) = \sum_{j=0}^{N-1}\prod_{k=1}^{j}\left(q^{(N-k)/2}-q^{-(N-k)/2}\right)\left(q^{(N+k)/2}-q^{-(N+k)/2}\right).$

$q \mapsto \exp(2\pi\sqrt{-1}/N)$

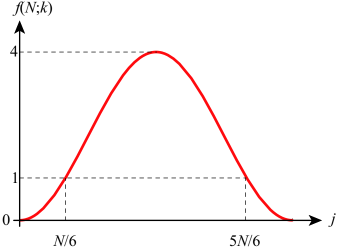
$J_N\Big($  $;\exp(2\pi\sqrt{-1}/N)\Big) = \sum_{j=0}^{N-1}\prod_{k=1}^{j} f(N;k)$

with $f(N;k) := 4\sin^2(k\pi/N)$.

Find the maximum of the summands

$J_N\Big($  $; e^{2\pi\sqrt{-1}/N}\Big) = \displaystyle\sum_{j=0}^{N-1}\prod_{k=1}^{j} f(N;k)$ with $f(N;k) := 4\sin^2(k\pi/N)$.

Graph of $f(N;k)$



Put $g(N;j) := \prod_{k=1}^{j} f(N;k)$.

j	0	$\cdots$	$N/6$	$\cdots$	$5N/6$	$\cdots$	1
$f(N;k)$		<1	1	>1	1	<1	
$g(N;j)$	1	$\searrow$		$\nearrow$	maximum	$\searrow$	

Limit of the sum is the limit of the maximum

- Maximum of $\{g(N;j)\}_{0\le j\le N-1}$ is $g(N;5N/6)$.
- $J_N\Big($  $;\exp(2\pi\sqrt{-1}/N)\Big)=\sum_{j=0}^{N-1} g(N;j)$.

$$\Downarrow$$

$g(N;5N/6)\le J_N\Big($  $;\exp(2\pi\sqrt{-1}/N)\Big)\le N\times g(N;5N/6)$

$$\Downarrow$$

$$\frac{\log g(N;5N/6)}{N}\le\frac{\log J_N}{N}\le\frac{\log N}{N}+\frac{\log g(N;5N/6)}{N}$$

$$\Downarrow$$

$$\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}\le\lim_{N\to\infty}\frac{\log J_N}{N}\le\lim_{N\to\infty}\frac{\log N}{N}+\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}$$

$$\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}\le\lim_{N\to\infty}\frac{\log J_N}{N}\le\lim_{N\to\infty}\frac{\log N}{N}+\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}$$

$$\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}\le\lim_{N\to\infty}\frac{\log J_N}{N}\le\lim_{N\to\infty}\frac{\log N}{N}+\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}$$

$$\Downarrow$$

$$\lim_{N\to\infty}\frac{\log J_N}{N}=\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}$$

Calculation of the limit of the maximum

$$\lim_{N\to\infty}\frac{\log J_N\left(\;\cdot\;;\exp(2\pi\sqrt{-1}/N)\right)}{N}=\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}$$

$$=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{5N/6}\log f(N;k)=2\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{5N/6}\log\bigl(2\sin(k\pi/N)\bigr)$$

$$=\frac{2}{\pi}\int_0^{5\pi/6}\log\bigl(2\sin x\bigr)\,dx=-\frac{2}{\pi}\Lambda(5\pi/6)=0.323066\dots,$$

where $\Lambda(\theta):=-\int_0^{\theta}\log|2\sin x|\,dx$ is the Lobachevsky function.
What is $\Lambda(5\pi/6)$?

Lobachevsky function $\Lambda(\theta)$

Some properties of $\Lambda := -\int_0^\theta \log|2\sin x|\,dx$.

- Λ is an odd function and has period π.
- $\Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta+\pi/2)$. ($\Lambda(n\theta) = n\sum_{k=1}^{n-1}\Lambda(\theta + k\pi/n)$ in general.)

The first property is easy.

To prove the second, we use the double angle formula of sine:

$$\sin(2x) = 2\sin x\cos x.$$

$$\Rightarrow \qquad \log|2\sin(2x)| = \log|2\sin x| + \log|2\sin(x+\pi/2)|. \quad \square$$

So we have

$$\Lambda(5\pi/6) = -\Lambda(\pi/6)$$

$$\Lambda(\pi/3) = 2\Lambda(\pi/6) + 2\Lambda(2\pi/3) = 2\Lambda(\pi/6) - 2\Lambda(\pi/3)$$

$$\Rightarrow \qquad \Lambda(5\pi/6) = -\frac{3}{2}\Lambda(\pi/3).$$

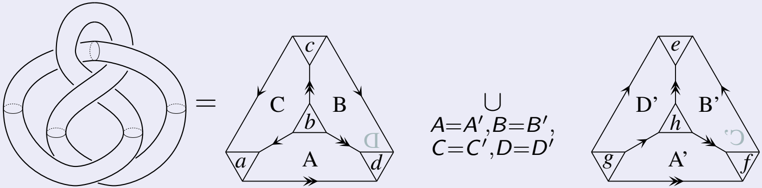
$$\Rightarrow \qquad 2\pi\lim_{N\to\infty}\log J_N\Bigg($$

$$;\exp(2\pi\sqrt{-1}/N)\Bigg)/N = 6\Lambda(\pi/3)$$

Decomposition of $S^3 \setminus$  into two tetrahedra

What is $6\Lambda(\pi/3)$?

Theorem (W. Thurston)



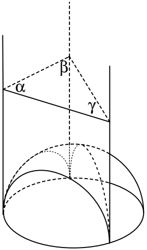
We can regard both pieces in the right hand side as regular ideal hyperbolic tetrahedra.

$\Rightarrow S^3 \setminus$  possesses a complete hyperbolic structure.

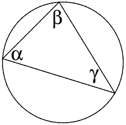
Ideal hyperbolic tetrahedron

- $\mathbb{H}^3 := \{(x, y, z) \mid z > 0\}$: with hyperbolic metric $ds := \frac{\sqrt{dx^2+dy^2+dz^2}}{z}$.
- Ideal hyperbolic tetrahedron : tetrahedron with geodesic faces with four vertices in the boundary at infinity.
- We may assume
  - One vertex is at (∞, ∞, ∞).
  - The other three are on xy-plane.

Ideal hyperbolic tetrahedron $\Delta(\alpha, \beta, \gamma)$



Top view



Ideal hyperbolic tetrahedron is defined (up to isometry) by the similarity class of this triangle.

$$\mathrm{Vol}(\Delta(\alpha, \beta, \gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Proof of VC - conclusion

$2\pi J_N\Big($  $;\exp(2\pi\sqrt{-1}/N)\Big) = 6\Lambda(\pi/3)$

$= 2\,\mathrm{Vol}(\text{regular ideal hyperbolic tetrahedron})$

$= \mathrm{Vol}\Big(S^3 \setminus$  $\Big)$

$\Rightarrow$ Volume Conjecture for  .

So far the Volume Conjecture is proved for

- torus knots (Kashaev and Tirkkonen)
- torus links of type $(2, 2m)$ (Hikami)
- figure-eight knot (Ekholm)
- 5_2 knot (hyperbolic) (Kashaev and Yokota)
- Whitehead doubles of torus knots (Zheng)
- twisted Whitehead links (Zheng)
- Borromean rings (Garoufalidis and Lê)
- Whitehead chains (van der Veen)