An Introduction to the Volume Conjecture, I

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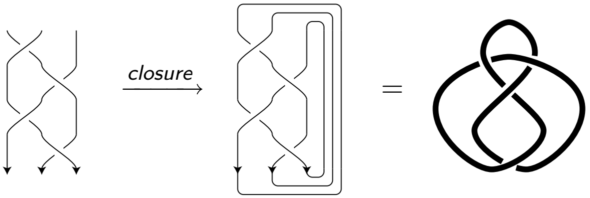

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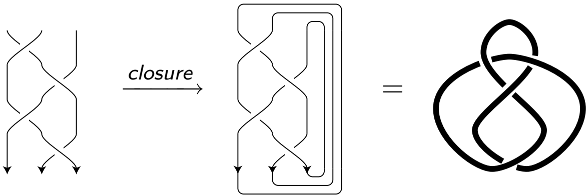

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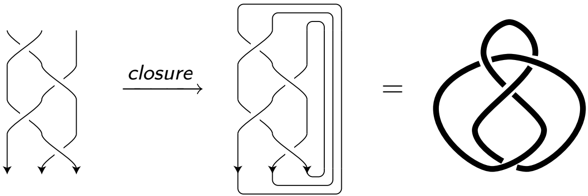

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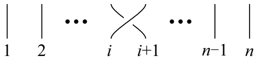
- generators: σ_i $(i = 1, 2, \ldots, n-1)$: 

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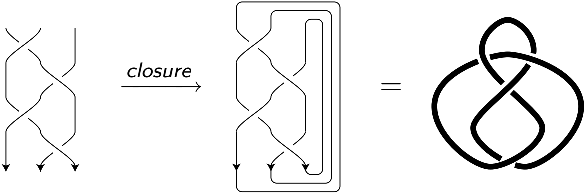

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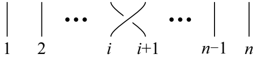
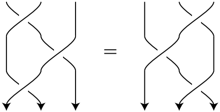
- generators: σ_i $(i = 1, 2, \ldots, n-1)$: 

- relations: $\sigma_i \sigma_j = \sigma_j \sigma_i$ $(|i - j| > 1)$, 

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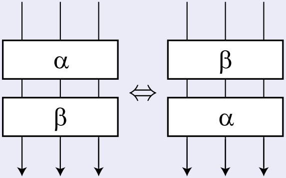

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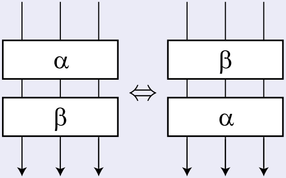
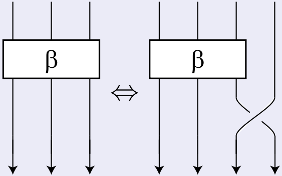


- *stabilization ($\beta \Leftrightarrow \beta\sigma_n^{\pm 1}$):*



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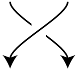
$\mathrm{Tr}_2\colon V\otimes V\to V$ is the operator trace. (For $M\in\mathrm{End}(V\otimes V)$ given by a matrix M^{ij}_{kl}, $\mathrm{Tr}_2(M)$ is given by $\sum_m M^{im}_{km}$.)

Braid $\Rightarrow$ endomorphism

Replace

Braid ⇒ endomorphism

Replace  with

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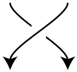
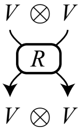
Replace  with 

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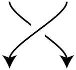
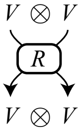
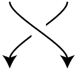
Replace  with 

, and  with

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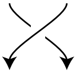
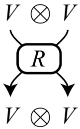
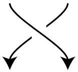
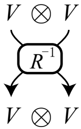
Replace  with 

, and  with 

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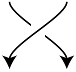
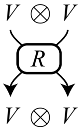
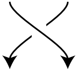
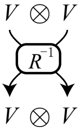
Replace  with 

, and  with 

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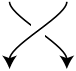
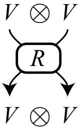
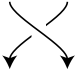
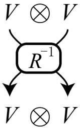
Replace  with 

, and  with 

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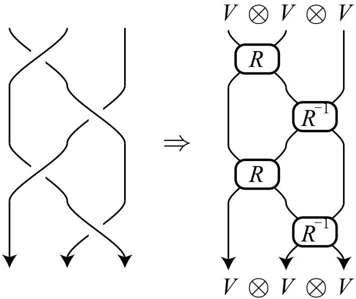

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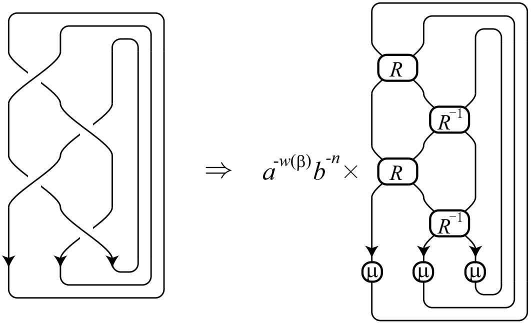

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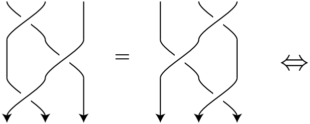

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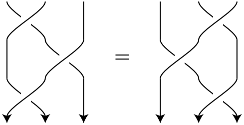

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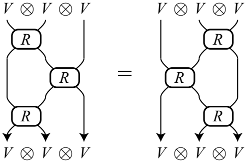

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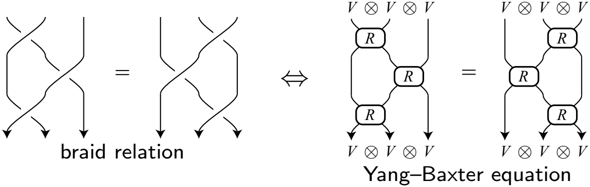

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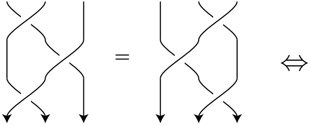

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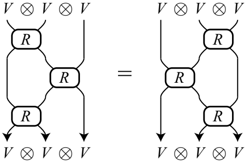

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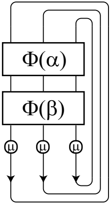

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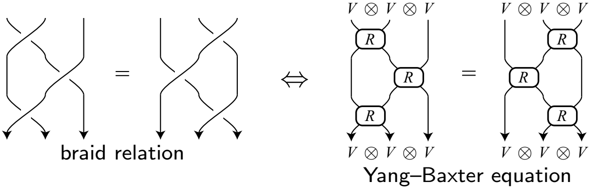

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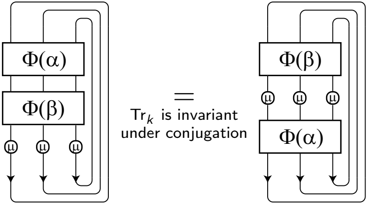

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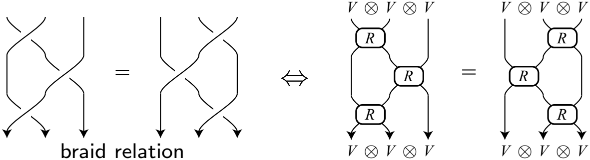

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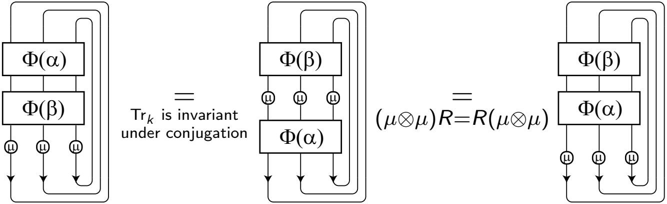

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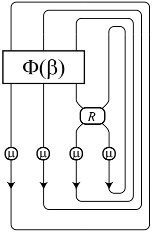

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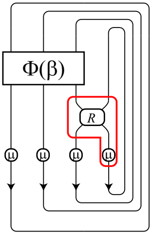

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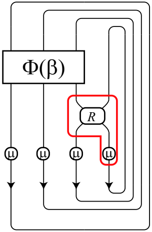

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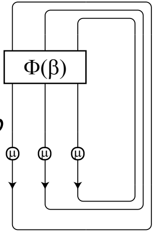

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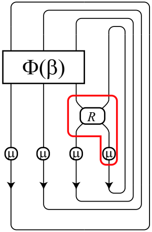

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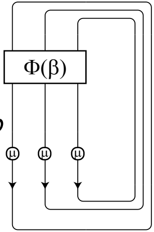

Theorem (Turaev)

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Definition

The quantum $(sl(2, \mathbb{C}), V_N)$ invariant is called the N-dimensional colored Jones polynomial $J_N(L; q)$. (q is a complex parameter.)

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The quantum $(sl(2, \mathbb{C}), V_N)$ invariant is called the N-dimensional colored Jones polynomial $J_N(L; q)$. (q is a complex parameter.)

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- $J_N(\text{unknot}; q) = 1$.

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Definition

$J_N(L; q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(K) \times \frac{\{1\}}{\{N\}}$: colored Jones polynomial.

Volume conjecture

Conjecture (Volume Conjecture, R. Kashaev, J. Murakami+H.M.)

K: *knot*

$$2\pi \lim_{N\to\infty} \frac{\log|J_N(K;\exp(2\pi\sqrt{-1}/N))|}{N} = \mathrm{Vol}(S^3 \setminus K).$$

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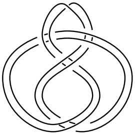
Definition (Jaco–Shalen–Johannson decomposition)

$S^3 \setminus K$ can be uniquely decomposed as

$$S^3 \setminus K = \left(\bigsqcup H_i\right) \sqcup \left(\bigsqcup E_j\right)$$

with H_i hyperbolic and E_j Seifert-fibered.

Example of JSJ decomposition



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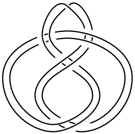

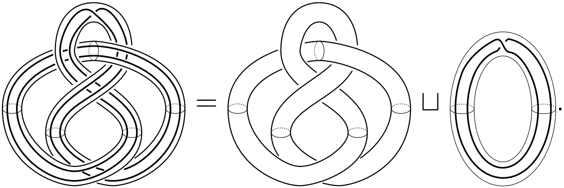

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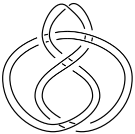

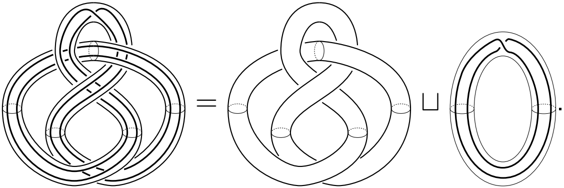

hyperbolic

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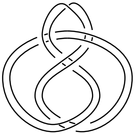

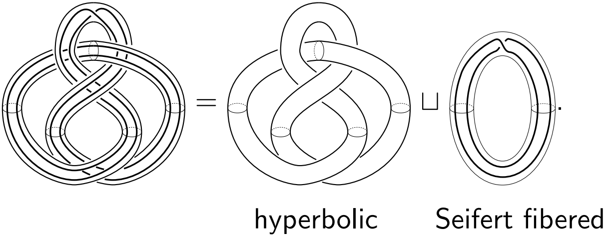

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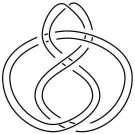
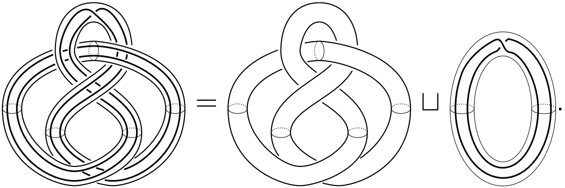
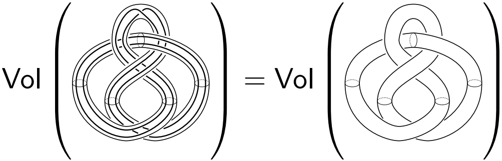




hyperbolic Seifert fibered



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$q \mapsto \exp(2\pi\sqrt{-1}/N)$

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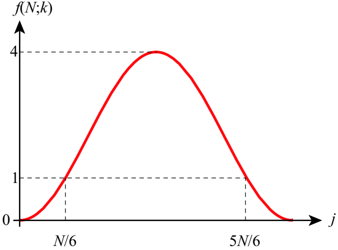

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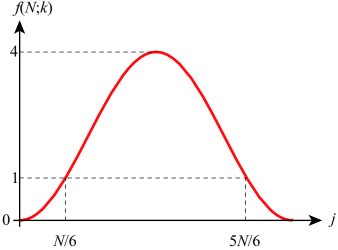

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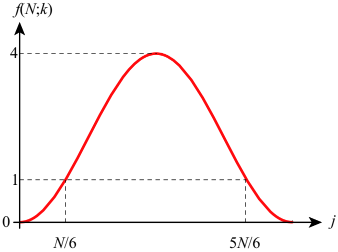

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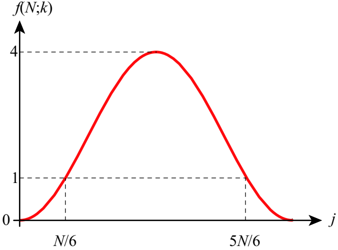

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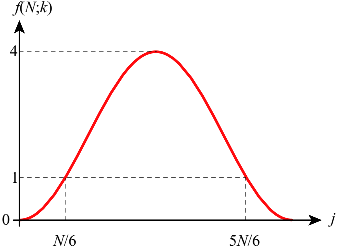

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Calculation of the limit of the maximum

$$\lim_{N\to\infty} \frac{\log J_N\Big($$

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$$= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N;k) = 2 \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log\bigl(2\sin(k\pi/N)\bigr)$$

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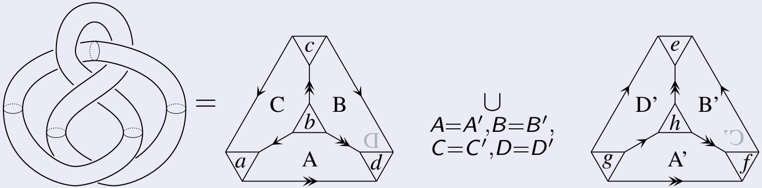

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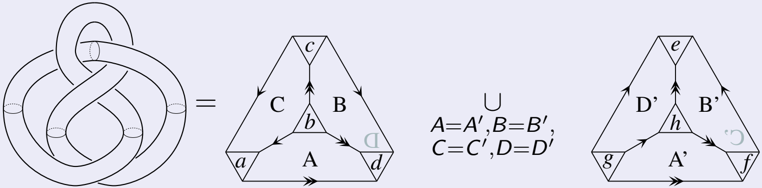

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$\Rightarrow S^3 \setminus$  possesses a complete hyperbolic structure.

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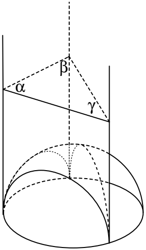

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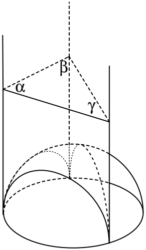
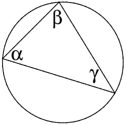


Top view



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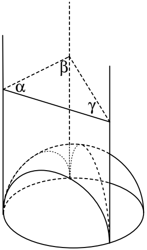

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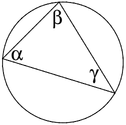

Ideal hyperbolic tetrahedron is defined (up to isometry) by the similarity class of this triangle.

$$\mathrm{Vol}(\Delta(\alpha, \beta, \gamma)) = \Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma).$$

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