Volume and topology II
(Paradoxical Decompositions)

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Theorem (W. Thurston). *Suppose M_1 and M_2 are orientable hyperbolic 3-manifolds and $f : M_1 \to M_2$ has non-zero degree d. If $\mathrm{vol}\, M_1 = |d| \,\mathrm{vol}\, M_2$ then f is homotopic to a covering map of degree d.*

This involves extending the argument to the situation where $\tilde{f}_\infty$ is only a measurable function.

Thurston also defined a relative Gromov norm, which he used to show:

Theorem (W. Thurston). *If M_1 is a non-compact orientable hyperbolic 3-manifold of finite volume, and M_2 is obtained by Dehn-filling at least one cusp of M_1 then $\mathrm{vol}\, M_1 > \mathrm{vol}\, M_2$.*

Suppose Γ is a discrete subgroup of $\mathbb{H}^3$. For $x \in \mathbb{H}^3$ set $\Gamma_x(\epsilon) = \{\gamma \in \mathrm{Isom}_+\mathbb{H}^3 \mid \mathrm{dist}(x, \gamma \cdot x) < \epsilon\}$

Lemma (Special case of Margulis' lemma). *There exists a constant ϵ_0 with the following property:*

- *If $\Gamma < Isom_+{}^+\mathbb{H}^3$ is a discrete group and $x \in \mathbb{H}^3$ then $\langle \Gamma_x(\epsilon_0) \rangle$ is virtually nilpotent.*

If Γ is torsion-free, i.e. if $\mathbb{H}^3/\Gamma$ is a manifold, the discrete, torsion-free, virtually nilpotent subgroups of Γ are actually abelian. There are three types:

- Cyclic groups generated by a loxodromic isometry;
- Cyclic groups generated by a parabolic isometry;
- Rank 2 free abelian groups generated by two parabolics.

The middle case can not arise if $\mathbb{H}^3/\Gamma$ has finite volume.

Definition. The *ϵ-thin part* $M_{(0,\epsilon]}$ of an orientable hyperbolic manifold M is the set of points $p \in M$ such that there is a geodesic loop of length $\leq \epsilon$ based at p. The ϵ-thick part is $M_{(\epsilon,\infty)} = M - M_{(0,\epsilon]}$

Suppose $x \in M_{(0,\epsilon]}$, and let $\tilde{x}$ be a lift of x to $\mathbb{H}^3$. Then there exists $\gamma \in \Gamma$ such that $\mathrm{dist}(\tilde{x}, \gamma \cdot \tilde{x}) \leq \epsilon$.

For $G \subset \mathrm{Isom}_+\mathbb{H}^3$, define

$$C_\epsilon(G) = \{x \in \mathbb{H}^3 : \ \mathrm{dist}(\tilde{x}, g \cdot \tilde{x}) \leq \epsilon \text{ for some } g \in G\}.$$

If $G \cong \mathbb{Z}$ is generated by a loxodromic isometry, then $C_\epsilon(G)$ is a banana (or empty). In this case $C_\epsilon(G)/G$ is a geometric tubular neighborhood of a geodesic.

If $G \cong \mathbb{Z}^2$ is generated by parabolic isometries, then $C_\epsilon(G)$ is a horoball and $C_\epsilon(G)/G$ is a cusp neighborhood.

So, if M has finite volume and $\epsilon < \epsilon_0$ then $M_{(0,\epsilon]}$ is a union of cusp neighborhoods and tubes around short geodesics.

Theorem (Jørgensen). *For each $C > 0$ there exists a finite set $\{M_1, \ldots, M_k\}$ of finite-volume orientable hyperbolic 3-manifolds such that every orientable hyperbolic 3-manifold M with $\operatorname{vol} M < C$ is constructed by Dehn-filling some cusps of one of the M_i.*

The idea is that there are only finitely many possible homeomorphism types for $M_{(\mu,\infty)}$ when $\operatorname{vol} M < C$. If $x_1, \ldots, x_n$ are points of $M_{(\mu,\infty)}$ with $\operatorname{dist}(x_i, x_j) > \mu$ then the balls $B(x_i\mu/2)$ are pairwise disjoint, so $n < C/v$ where $v = \operatorname{vol} B(x_i\mu/2)$. If $\{x_1, \ldots, x_n\}$ is maximal then every point of $M_{(\mu,\infty)}$ is within distance 2μ of some x_i. Thus there is a Delaunay "triangulation" of $\overline{M}_{(\mu,\infty)}$ with a bounded number of cells. (Lifted to $\mathbb{H}^3$, the 3-cells are convex hulls of sets of ≥ 4 points that lie on a sphere containing no lifts of x_i in its interior.

The proof of Thurston's hyperbolic Dehn-filling theorem implies:

Let M be an orientable finite-volume hyperbolic 3-manifold. Fix a set of cusps of M. For any $\epsilon > 0$, all but finitely many manifolds M' obtained by Dehn-filling these cusps have $|\operatorname{vol} M - \operatorname{vol} M'| < \epsilon$.

Theorem. *The set of volumes of orientable hyperbolic 3-manifolds forms a well-ordered subset of $\mathbb{R}$, and there are only finitely many distinct manifolds with each volume.*

Suppose $\operatorname{vol} M_1 > \operatorname{vol} M_2 > \cdots$. By passing to a subsequence we may assume each M_n is constructed by Dehn-filling of a given set of cusps of a manifold M. By Gromov's Theorem we have $\operatorname{vol} M > \operatorname{vol} M_n$ for all n. Thus $|\operatorname{vol} M - \operatorname{vol} M_n| > |\operatorname{vol} M - \operatorname{vol} M_1|$ for all $n > 1$. Contradiction.

In view of Jørgensen's theorem:

Given a topological property P of finite volume hyperbolic 3-manifolds, we can ask "what is the first volume of a manifold with property P?"

Cao and Meyerhoff answered this for $P =$ "M is non-compact and orientable". They also found #2.

More recently Gabai, Meyerhoff and Milley extended this list up to #10. The list consists of the first 10 manifolds in the cusped census provided with Jeff Weeks' SnapPea program.

They also answered this question for $P =$ "M is closed and orientable." The answer is: Week's manifold, m003(-3,1), which has volume $0.942707362\ldots$

Definition. Let M be a finite-volume hyperbolic 3-manifold. Say that ϵ is a *Margulis number* for M if $M_{(0,\epsilon]}$ is a disjoint union of tubes and cusp neighborhoods.

If $\mathcal{F}$ is some family of finite-volume hyperbolic 3-manifolds we define the *Margulis constant* of $\mathcal{F}$ to be

$$\mu(\mathcal{F}) = \sup\{\epsilon : \epsilon \text{ is a Margulis number for all } M \in \mathcal{F}\}.$$

Observe that $\mu(M) \doteq \mu(\{M\})$ is a topological invariant of M, by Mostow rigidity.

The Margulis constant for the class of all closed orientable hyperbolic manifolds is unknown. The best known lower bound (0.104) is due to Meyerhoff. It appears that $\mu(m003(-3,1))$ is about 0.774.

Shalen has recently shown that 0.3925 is a Margulis number for $M = \mathbb{H}^3/\Gamma$ if the trace field is quadratic, Γ has integral traces, and there is no torsion of order 2, 3 or 7 in $H_1(M)$.

The following purely topological result applies to all closed hyperbolic 3-manifolds.

Theorem (Jaco-Shalen). *Suppose that M is a closed irreducible 3-manifold such that $\pi_1(M)$ does not have a subgroup isomorphic to $\mathbb{Z}^2$. Then every 2-generator subgroup which has infinite index in $\pi_1(M)$ is free.*

The proof uses the Compact Core theorem and Stallings' five term exact sequence. The theorem implies that all closed hyperbolic 3-manifolds fall into two classes:

- Manifolds which have a finite cover with a 2-generator fundamental group;
- Manifolds M such that $\pi_1(M)$ is 2-free, meaning that every 2-generator subgroup of $\pi_1(M)$ is a free group.

We will see that $\log 3$ is a lower bound for the Margulis constant of the second class of manifolds. It may be feasible to classify hyperbolic manifolds with 2-generator fundamental group.

Theorem (C - Shalen). *Suppose Γ is a discrete subgroup of $Isom_+\mathbb{H}^3$ which has no parabolics elements and is freely generated by γ_1 and γ_2. If $p \in \mathbb{H}^3$ then*

$$\max\{\mathrm{dist}(p, \gamma_1 \cdot p), \mathrm{dist}(p, \gamma_2 \cdot p)\} > \log 3.$$

Theorem (Andersen, C, Canary, Shalen). *Suppose Γ is a discrete subgroup of $Isom_+\mathbb{H}^3$ which has no parabolic elements and is freely generated by $\gamma_1, \ldots, \gamma_n$. Let $p \in \mathbb{H}^3$ and set $d_i = \mathrm{dist}(p, \gamma_i \cdot p)$. Then*

$$\sum_{i=1}^{n} \frac{1}{1+e^{d_i}} \leq \frac{1}{2}.$$

These are much stronger than the versions proved in our papers, due to the proof of Marden's Tameness Conjecture by Agol and Calegari-Gabai.

The assumption of no parabolics can be dropped by applying Ohshika's proof of the general version of Bers Density Conjecture.

The free group F_2 on the letters x, y has a "paradoxical decomposition":

$$F_2 = \{1\} \cup X \cup Y \cup \overline{X} \cup \overline{Y},$$

where $(X,\ Y,\ \overline{X},\ \overline{Y})$ is the set of words that start with $(x,\ y,\ x^{-1},\ y^{-1})$. The "paradoxical" aspect (that leads to the Banach-Tarski paradox) is that left multiplication by x maps $\overline{X}$ onto $\{1\} \cup \overline{X} \cup Y \cup \overline{Y}$.

Think of F_2 as acting on a 4-valent tree T. Then both T and its Cantor set of ends T_∞, inherit decompositions with the same paradoxical property.

When $\Gamma = \langle x, y \rangle$ is a free group of isometries of $\mathbb{H}^3$ the analogue of T_∞ is the *limit set* $\Lambda_\Gamma = \overline{\Gamma \cdot z} - \Gamma \subset S^2_\infty$, $z \in \mathbb{H}^3$. (This does not depend on z.)

We will construct a paradoxical decomposition of Λ_Γ (using measures, instead of subsets).

Our construction of measures does depend on a choice of point $z \in \mathbb{H}^3$, but in a very controlled way. Really, it gives a *conformal density of dimension* D.

Example: For each point $z \in \mathbb{H}^3$, let ν_z be the "visual measure" on S^2_∞. Given a Borel set $X \subset S^2_\infty$, $\nu_z(X)$ is the measure of the solid angle subtended by X at the point z.

The relationship between ν_z and $\nu_{z'}$ is:

$$d\nu_{z'} = P^2_{z,z'} d\nu_z \quad \left(\text{i.e.} \int f \nu'_z = \int f \, P_{z,z'} \nu_z \right),$$

where $P_{z,z'}$ is a certain real-valued function on S^2_∞:

In the upper half space model, if z and z' are on the t-axis at heights 1 and t_0 then $P_{z,z'}(\infty) = 1/t_0$.

The family ν_z is a 2-dimensional conformal density. If $d\mu_{z'} = P^D_{z,z'} d\mu_z$, the family μ_z is D-dimensional.

Suppose Γ is a discrete free group generated by two isometries x and y. Take a point p in $\mathbb{H}^3$ and consider its orbit $\Gamma \cdot p$.

For any point $z \in \mathbb{H}^3$ and $s > 0$ we have a Poincaré series:

$$\Sigma(z, s) = \sum_{\gamma \in \Gamma} e^{-s\,\mathrm{dist}(z, \gamma \cdot p)}.$$

If s is larger than the exponential growth rate of $r \to |B(z, r) \cap \Gamma \cdot p|$, this series will converge, and if s is smaller than the growth rate the series will diverge.

So there is a *critical exponent* D such that $\Sigma(z, s)$ diverges if $s > D$ and converges if $s < D$. (The value of D does not depend on z.)

When $s = D$ the series may or may not converge. To avoid discussing "Patterson's trick" we will assume it diverges.

Fix $p \in \mathbb{H}^3$ and consider the orbit $\Gamma \cdot p$. Choose a decreasing sequence (s_n) converging to the critical exponent D of $\Sigma(z, s)$.

For each $z \in \mathbb{H}^3$ define

$$\mu_{z,n} = \frac{1}{\Sigma(z, s_n)} \sum_{\gamma \in \Gamma} e^{-s_n \operatorname{dist}(z, \gamma \cdot p)} \delta_{\gamma \cdot p}.$$

After passing to a subsequence we may assume that these measures converge weakly to μ_z. Note that the support of μ_z is contained in S^2_∞ and μ_z has total mass 1. It also follows formally that:

- for any other point z' $\mu_{z',n}$ converges (to $\mu_{z'}$).
- $\mu_z = \mu_{X,z} + \mu_{\overline{X},z} + \mu_{Y,z} + \mu_{\overline{Y},z}$, where $\mu_{X,z}$ is constructed by summing over $X \cdot z$ and taking the limit.
- μ_z, $\mu_{X,z}$, ... are D-dimensional conformal densities ($D \leq 2$).
- $\mu_{\overline{X},x(z)} = \mu_{\overline{X},z} + \mu_{Y,z} + \mu_{\overline{Y},z} = \mu_z - \mu_{X,z}$, etc.

Note that we also have $d\mu_{\overline{X},x(z)} = P_{z,x(z)} d\mu_{\overline{X},z}$.

We need to work in the space of purely loxodromic discrete free groups generated by x and y. We can now use several big (new) theorems about this space.

- $AH(F_2) = Hom(F_2, PSL_2(\mathbb{C}))/\sim$. Concretely, $AH(F_2)$ has 3 complex parameters.
- $\mathcal{D} \subset AH(F_2)$ is the set of conjugacy classes of discrete faithful reps. Chuckrow showed that $\mathcal{D}$ is closed.
- $\mathcal{GF} \subset \mathcal{D}$ is the set of *geometrically finite* (or Schottky) groups – those with finite-sided fundamental domains. Marden showed that $\mathcal{GF}$ is open.
- $\overline{\mathcal{GF}} = \mathcal{D}$ by Bers' Density Conjecture (Bromberg, Ohshika).
- Purely loxodromic groups are dense in $\mathcal{B} = \overline{\mathcal{GF}} - \mathcal{GF}$ (Bers).
- Canary, extending Bonahon, showed that if $\Gamma \in \mathcal{B}$ is topologically tame then any positive Γ-invariant function f with $\Delta(f) \leq 0$ is constant.
- By Marden's Tameness Conjecture (Agol, Calegari-Gabai), all groups in $\mathcal{D}$ are topologically tame.

Minimizing

We are minimizing $\max(\mathrm{dist}(z, x(z)), \mathrm{dist}(z, y(z)))$ over $\mathcal{D}$. The minimum cannot occur in the open set $\mathcal{GF}$ because we can always perturb a group in $\mathcal{GF}$ to reduce the displacements. So we only have to consider purely loxodromic, geometrically infinite, topologically tame groups in $\mathcal{B}$. (These have $\Lambda_\Gamma = S^2_\infty$.)

For any D-dimensional conformal density μ_z, and $z_0 \in \mathbb{H}^3$ then the function $u(z) = \int P_{z_0,z}\mu_z$ satisfies $\Delta(u) = -D(n - D - 1)u$. Since u must be constant in our case we have $D = 2$. This implies that $\mu_z = \nu_z$, the visual density.

By symmetrizing, we may assume $\mu_{X,z}(S^2_\infty) = \mu_{\overline{X},z}(S^2_\infty) \leq 1/4$. Recall that $\mu_{\overline{X},x(z)} = \mu_{\overline{X},z} + \mu_{Y,z} + \mu_{\overline{Y},z} = \mu_z - \mu_{X,z}$, and $d\mu_{\overline{X},x(z)} = P_{z,x(z)}d\mu_{\overline{X},z}$

Thus for some function f we have $\int f\mu_{\overline{X},x(z)} \leq 1/4$ but $\int f\, P_{z,x(z)}\mu_{\overline{X},x(z)} \geq 3/4$. A worst case analysis shows this can only happen if $\mathrm{dist}(z, x(z)) \geq \log 3$.