# Algebraic Curves - Homework 2

1.25$b$, 1.33 (for ($b$), also show that $V$ is an algebraic set in $\mathbf{C}^3$), 1.39, 1.40

1. ($i$) If $K$ is a field and $A$ is a finite-dimensional $K$-algebra which is a domain, prove that $A$ is a field.

($ii$) If $R$ is a non-zero ring and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are distinct maximal ideals in $R$, prove that the natural map $R/\cap\,\mathfrak{m}_i \rightarrow \prod(R/\mathfrak{m}_i)$ is an isomorphism (hint: show that $(1, 0, \dots, 0)$ is hit).

2. Let $\varphi : A \rightarrow B$ be a map of rings. Prove that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal if $\mathfrak{p}$ is a prime ideal and show by example that $\varphi^{-1}(\mathfrak{m})$ need not be a maximal ideal if $\mathfrak{m}$ is a maximal ideal. Interpret as statements about subrings of domains and fields. However, show that if $\varphi$ is a $k$-algebra map between finitely generated $k$-algebras where $k$ is a field, then $\varphi^{-1}(\mathfrak{m})$ *is* a maximal ideal whenever $\mathfrak{m}$ is a maximal ideal.

3. Let $A$ be a ring. For $a \in A$, define $A_a = A[X]/(1 - aX)$. This is naturally an $A$-algebra, and is sometimes denoted $A[1/a]$ (but we do not require $A$ to be a domain, so the fraction notation is merely suggestive).

($i$) Note that $a$ is a unit in $A$ (i.e., admits a multiplicative inverse). Formulate a universal mapping property for the $A$-algebra $A_a$, and prove that $b \in A$ maps to 0 in $A_a$ if and only if $a^n b = 0$ in $A$ for some positive integer $n$. Conclude that $A_a \neq 0$ if and only if $a$ is not nilpotent. Also show that if $A$ is a domain with fraction field $K$, then there is a canonical map $A_a \rightarrow K$ which is *injective* (so $A_a$ is a domain) and describe the image.

($ii$) For every $f \in A_a$, show that for some large integer $n$, $a^n f \in A_a$ is in the image of the natural map $A \rightarrow A_a$. Conclude that if $A$ has no non-zero nilpotents, then $A_a$ has the same property. When $A = k[X, Y]/(XY)$ and $a = Y$ with $k$ a field, show that $A_a \simeq k[Y, Y^{-1}] = k[Y]_Y$ as $k$-algebras and that the (non-injective!) natural map $A \rightarrow A_a$ induces an injective map of sets

$$\mathrm{Hom}_{k-\mathrm{alg}}(A_a, k) \rightarrow \mathrm{Hom}_{k-\mathrm{alg}}(A, k);$$

interpret this map geometrically when $k$ is algebraically closed.

($iii$) Let $I$ be an ideal in $A$. Use the universal property to define a natural map $A_a \rightarrow (A/I)_a$ and prove that this induces an isomorphism $A_a/(I \cdot A_a) \simeq (A/I)_a$ (hint: use mapping properties to construct the inverse map by pure thought).

($iv$) Show in general that the natural map $\varphi : A \rightarrow A_a$ sets up an inclusion-preserving bijection between prime ideals $\mathfrak{p}$ of $A$ not containing $a$ and prime ideals $\mathfrak{q}$ of $A_a$, via $\mathfrak{p} \mapsto \mathfrak{p} \cdot A_a$ and $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$.

4. Here is some general business concerning noetherian topological spaces. Let $X$ be a topological space. We say that $X$ is *quasi-compact* if every open covering has a finite subcovering (this terminology simply emphasizes that we do not assume the space to be Hausdorff).

($i$) Show that if $X$ is noetherian, then it is quasi-compact and all subspaces are noetherian. Conversely, if all open subsets in $X$ are quasi-compact, then show that $X$ is noetherian.

($ii$) If $X \rightarrow Y$ is a surjective continuous map and $X$ is an irreducible (resp. noetherian) space, then so is $Y$.

($iii$) If $X$ is an irreducible topological space, prove that any two non-empty opens have non-empty intersection (so all non-empty opens are dense). Conclude that all non-empty opens are irreducible, and quite generally that a noetherian space in which any two points admit disjoint open neighborhoods (i.e., a noetherian Hausdorff space) must be a finite set with the discrete topology.

5. Let $n, m \geq 1$ and let $M_{m,n} = k^{mn} = \{(x_{ij})\}$ denote the 'space' of $m$ by $n$ matrices over an algebraically closed field $k$. Prove that for any $r$, the subset corresponding to matrices with rank $\leq r$ is an affine algebraic set in this $k^{mn}$. When $m = n$, what can you say about the subset corresponding to invertible matrices?