Algebraic Curves - Homework 1

Ch 1: 1.4, 1.5, 1.10, 1.17, 1.22

1. Find all solutions to $x^2 + 2y^2 = 3$ with $x, y \in \mathbf{Q}$, and prove that $x^2 + 3y^2 = 2$ has no solutions in $\mathbf{Q}$. State and prove a generalization for $ax^2 + by^2 = c$ with $a, b, c \in k^\times$ and k an arbitrary field not of characteristic 2 (why is characteristic 2 more difficult?). Draw pictures.

2. Let k be an algebraically closed field. Give an example of affine algebraic sets Z_1, Z_2 in k^2 with $\underline{I}(Z_1 \cap Z_2) \neq \underline{I}(Z_1) + \underline{I}(Z_2)$. What is the geometric significance? Draw a picture.

3. This exercise develops basic facts for manipulating polynomials in several variables.

(*i*) Let R be a ring. Define $R[X_1, \ldots, X_n]$ in terms of 'sequences of coefficients', define on it a structure of commutative R-algebra, and prove that it has the following universal mapping property: for any R-algebra A and any $a_1, \ldots, a_n \in A$, there is a unique map of R-algebras $R[X_1, \ldots, X_n] \to A$ which sends X_i to a_i. The image of f under this map is called the *value* of f at $(a_1, \ldots, a_n)$. Note that when $R = 0$, the only R-algebra is R itself (e.g., $R[X] = R$ for $R = 0$).

(*ii*) If I is the ideal in $R[X_1, \ldots, X_n]$ generated by elements f_α, then state and prove a universal mapping property for the R-algebra $R[X_1, \ldots, X_n]/I$. Interpret this in the special case $I = (X_1 - r_1, \ldots, X_n - r_n)$ for $r_j \in R$. Conclude that $R[X]$ is not isomorphic to R as an R-algebra if $R \neq 0$, but give an example of a non-zero ring R for which there is an isomorphism $R[X] \simeq R$ as abstract rings.

(*iii*) For $f \in R[X]$, $g \in R[Y]$, prove that there are unique isomorphisms of R-algebras

$$(R[Y]/g)[X]/(f) \simeq R[X,Y]/(f,g) \simeq (R[X]/f)[Y]/(g)$$

determined by "$X \mapsto X$" and "$Y \mapsto Y$". Generalize for any finite number of variables, with (f) and (g) replaced by any ideals in the corresponding polynomial rings.

4. (*i*) If A is a UFD, prove that $A[X_1, \ldots, X_n]$ is a UFD (e.g., $A = \mathbf{Z}$ or A a field). Prove rigorously that $k[X, Y, Z, W]/(XY - ZW)$ is a domain but is not a UFD, where k is an algebraically closed field.

(*ii*) Prove that if k is a field and $f \in k[X]$ with positive degree is a product of distinct irreducible polynomials, then $Y^2 - f \in k[X, Y]$ is irreducible. For $n > 1$, prove that $X^n + Y^n - 1 \in k[X, Y]$ is irreducible if the characteristic of k does not divide n, but is reducible otherwise.

5. It is a basic fact that the 'symmetric function' polynomials $S_1, \ldots, S_n \in \mathbf{Z}[T_1, \ldots, T_n]$ with

$$S_i := \sum_{\substack{\{a_1, \ldots, a_i\} \\ \subset \{1, \ldots, n\}}} \prod_{k=1}^{i} T_{a_k}$$

(ex: if $n = 3$, $S_1 = T_1 + T_2 + T_3$, $S_2 = T_1T_2 + T_1T_3 + T_2T_3$ and $S_3 = T_1T_2T_3$) are algebraically independent over $\mathbf{Q}$ (i.e., the canonical map $\mathbf{Q}[X_1, \ldots, X_n] \to \mathbf{Q}[T_1, \ldots, T_n]$ sending X_i to S_i is injective) and $\mathbf{Q}[S_1, \ldots, S_n]$ is the subring of $\mathfrak{S}_n$-invariants in $\mathbf{Q}[T_1, \ldots, T_n]$ (consult Lang's *Algebra*, 3rd ed., Ch IV, §6 for a self-contained proof).

(*i*) Let $d \geq 1$. Prove the existence of a 'universal discriminant' polynomial $\Delta_d \in \mathbf{Z}[A_0, \ldots, A_{d-1}]$, unique up to sign, with the property that if k is any algebraically closed field and $f = \sum a_i T^i$ is a monic polynomial of degree d, then f is a product of d distinct linear factors if and only if $\Delta_d(a_0, \ldots, a_{d-1}) \neq 0$.

(*ii*) Let $f \in k[X, Y]$ be a non-constant polynomial and k an algebraically closed field. If f has distinct irreducible factors $f_1, \ldots, f_n$, prove that $Z(f)$ is the union of the $Z(f_i)$'s, with each $Z(f_i)$ infinite and all $Z(f_i) \cap Z(f_j)$ finite for $i \neq j$. Prove that for any irreducible f of degree $d > 1$, all lines in k^2 meet f in $\leq d$ points (what if $d = 1$?), and the only lines $y = ax + b$ in k^2 which fail to meet f in exactly d distinct points are those for which $(a, b) \in k^2$ satisfy a certain non-trivial polynomial relation (depending on f). In particular, there are infinitely many such exceptional lines. For $f = Y - X^2$, what is the geometric meaning of this exceptional set of lines? How about $f = Y^2 - X^3$? Draw pictures.