NEARLY OVERCONVERGENT MODULAR FORMS

ERIC URBAN

ABSTRACT. We introduce and study finite slope nearly overconvergent (elliptic) modular forms. We give an application of this notion to the construction of the Rankin-Selberg p-adic L-function on the product of two eigencurves.

1. INTRODUCTION

The purpose of this paper is to define and give the basic properties of nearly overconvergent (elliptic) modular forms. Nearly holomorphic forms were introduced by G. Shimura in the 70's for proving algebraicity results for special values of L-functions [Sh76]. He defined the notion of algebraicity of those by evaluating them at CM-points. After introducing a sheaf-theoretic definition, it is also possible to give an algebraic and even integral structure on the space of nearly holomorphic forms, allowing to study congruences between them. This naturally leads to the notion of nearly overconvergent forms. For that matter, we can think that nearly overconvergent forms are to overconvergent forms what nearly holomorphic forms are to classical holomorphic modular forms. The notion of nearly overconvergent forms came to the author when working on his joint project with C. Skinner (see [Ur13] for an account of this work in preparation [SU]) and appears as a natural way to study certain p-adic families of nearly holomorphic forms and its application to Bloch-Kato type conjectures. In the aforementioned work where the case of unitary groups[1] is considered, the notion is not absolutely necessary but it is clearly in the background of our construction and keeping it in mind makes the strategy more transparent.

An important feature of nearly overconvergent forms is that its space is equipped with an action of the Atkin operator U_p and that this action is completely continuous. This allows to have a spectral decomposition and to study p-adic families of nearly overconvergent forms. Another remarkable fact which is not really surprising but useful is that this space embeds naturally in the space of p-adic forms. In particular, this allows us to define the p-adic q-expansion of these forms. All the tools and differential operators that are used in the classical theory are also available here thanks to the sheaf theoretic definition and the Gauss-Manin connection. In particular, we can define the Maass-Shimura differential operator for families and the overconvergent projection which

Date: October 15, 2013.

The author was supported by the NSF.

[1]It will be clear to the reader that it could be generalized to any Shimura variety of PEL type.

is a generalization in this context of the holomorphic projection of Shimura. Our theory is easily generalisable to general Shimura varieties of PEL type. To make this notion more appealing, I decided to include an illustration (which is not considered in [SU]) of its potential application to the construction of p-adic L-functions in the non-ordinary case. In the works of Hida [Hi85, Hi88] on the construction of 3-variable Rankin-Selberg p-adic L-functions attached to ordinary families, the fact that the ordinary idempotent is the p-adic equivalent notion of the holomorphic projector makes it play a crucial role in the construction of Hida's p-adic measure. Here the spectral theory of the U_p-operator on the space of nearly overconvergent forms and the overconvergent projection play that important role.

We now review quickly the content of the different sections. In the section 2, we recall the notion of nearly holomorphic forms and give its sheaf-theoretic definition. This allows to give an algebraic and integral version for nearly holomorphic forms and define their polynomial q-expansions as well as an arithmetic version of the classical differential operators of this theory. We also check that Shimura's rationality of nearly holomorphic forms is equivalent to ours. In section 3, we introduce the space of nearly overconvergent forms and we prove it embeds in the space of p-adic forms. We also study the spectral theory of U_p on them and give a q-expansion principle. Then we define the differential operators in families and the overconvergent projection. In the last section we apply the tools introduced before to make the construction of the Rankin-Selberg p-adic L-functions on the product of two eigencurve of tame level 1. When restricted to the ordinary locus, this p-adic L-function is nothing else but the 3-variable p-adic L-function of Hida.

After the basic material of this work was obtained, I learned from M. Harris that he had also given a sheaf theoretic definition[2] using the theory of jets which is valid for general Shimura varieties of Shimura's nearly holomorphic forms in [Ha85, Ha86] and the fact his definition is equivalent to Shimura's has been verified by his former student Mark Nappari in his Thesis [Na92]. It should be easy to see that our description is equivalent to his in the PEL case. However, Harris did not study nor introduce the nearly overconvergent version. I would like also to mention that some authors have introduced an ad hoc definition of nearly overconvergent forms as polynomials in E_2 with overconvergent forms as coefficients. However this definition cannot be generalized to other groups and is not convenient for the spectral theory of the U_p-operator (or the slope decomposition). Finally we recently learned from V. Rotger that H. Darmon and V. Rotger have independently introduced a definition similar to ours in [DR] using the work [CGJ].The reader will see that our work is independent of loc. cit. and recover the result of [CGJ] has a by-product and can therefore be generalized to any Shimura variety of PEL type.

This text grew out from the handwritten lecture notes of a graduate course the author gave at Columbia University during the Spring 2012. After, the work [SU] was presented

[2] His definition follows a suggestion of P. Deligne.

at various conferences[3] including the Iwasawa 2012 conference held in Heidelberg, several colleagues suggested him to write up an account in the $GL(2)$-case of this notion of nearly overconvergent forms. This text and in particular the application to the p-adic Rankin-Selberg L-function would not have existed without their suggestions.

The author would like to thank Giovanni Rosso and Chris Skinner for interesting conversations during the preparation of this work. He would like also to thank Pierre Colmez who encouraged him to write this note. He is also grateful to the organizers of the conference Iwasawa 2012 held in Heidelberg for their invitation and for giving the opportunity to publish this paper in the proceedings of this conference. This work was also lectured during the Postech winter school in january 2013. The author would like to thank the organizers of this workshop for their invitation. Finally the author would like to thank the Florence Gould Foundation for its support when he was a Member at the Institute for Advanced Studies and when some part of this work was conceived.

[3]The first half of this note was also presented in my lecture given at H. Hida's 60th birthday conference.

Contents

Notations. Throughout this paper p is a fixed prime. Let $\overline{\mathbf{Q}}$ and $\overline{\mathbf{Q}}_p$ be, respectively, algebraic closures of $\mathbf{Q}$ and $\mathbf{Q}_p$ and let $\mathbf{C}$ be the field of complex numbers. We fix embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Throughout we implicitly view $\overline{\mathbf{Q}}$ as a subfield of $\mathbf{C}$ and $\overline{\mathbf{Q}}_p$ via the embeddings ι_∞ and ι_p. We fix an identification $\overline{\mathbf{Q}}_p \cong \mathbf{C}$ compatible with the embeddings ι_p and ι_∞. For any irreducible rigid analytic space $\mathfrak{X}$ over a p-adic number field L, we denote respectively by $A(\mathfrak{X}), A^b(\mathfrak{X}), A^0(\mathfrak{X})$ and $F(\mathfrak{X})$ the rings of analytic function, bounded analytic function, the ring of analytic bounded by 1 and the analytic function field on $\mathfrak{X}$.

2. Nearly holomorphic modular forms

2.1. **Classical definition.** In this paragraph, for the purpose to set notations we recall[4] some classical definitions and operations on modular forms.

2.1.1. We recall that the subgroup of 2×2 matrices of positive determinant $GL_2(\mathbf{R})^+$ acts on on the Poincaré upper half plane

$$\mathfrak{h} := \{\tau = x + iy \in \mathbf{C} \mid y > 0\}$$

by the usual formula

$$\gamma.\tau = \frac{a\tau + b}{c\tau + d} \text{ for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbf{R})^+ \text{ and } \tau \in \mathfrak{h}$$

Let f be a complex valued function defined on $\mathfrak{h}$. For any integer $k \geq 0$ and $\gamma \in GL_2(\mathbf{R})^+$, we set

$$f|_k\gamma(\tau) := det(\gamma)^{k/2}(c\tau + d)^{-k} f(\gamma.\tau)$$

Let $r \geq 0$ be another integer. We say that f is a nearly holomorphic modular form of weight k and order $\leq r$ for an arithmetic group $\Gamma \subset SL_2(\mathbf{R})$ if f satisfies the following properties:

(a) f is C^∞ on $\mathfrak{h}$,
(b) $f|_k\gamma = f$ for all $\gamma \in \Gamma$,
(c) There are holomorphic functions $f_0, \dots, f_r$ on $\mathfrak{h}$ such that

$$f(\tau) = f_0(\tau) + \frac{1}{y} f_1(\tau) + \cdots + \frac{1}{y^r} f_r(\tau)$$

(d) f has a finite limit at the cusps.

If f is a C^∞ function on $\mathfrak{h}$, we set following Shimura's $\epsilon.f$ the function of $\mathfrak{h}$ defined by

$$(\epsilon.f)(\tau) := 8\pi i y^2 \frac{\partial f}{\partial \bar{\tau}}(\tau) \tag{2.1.1.a}$$

It is easy to check that the condition (c) can be replaced by the condition

(c') $\epsilon^{r+1}.f = 0$

Moreover if f is nearly holomorphic of weight k and order $\leq r$, then $\epsilon.f$ is nearly holomorphic of weight $k-2$ and order $\leq r-1$. We denote by $\mathcal{N}_k^r(\Gamma, \mathbf{C})$ the space of nearly holomorphic form of weight k, order $\leq r$ and level Γ. For $r = 0$, this is the space of holomorphic modular form of weight k and level Γ and we will sometimes use the standard notation $\mathcal{M}_k(\Gamma, \mathbf{C})$ instead of $\mathcal{N}_k^0(\Gamma, \mathbf{C})$.

[4]Those facts are mainly due to Shimura

2.1.2. Recall also the Maass-Shimura differential operator δ_k on the space of nearly holomorphic forms of weight k defined by

$$\delta_k.f := \frac{1}{2i\pi}y^{-k}\frac{\partial}{\partial\tau}(y^k f) = \frac{1}{2i\pi}(\frac{\partial f}{\partial\tau} + \frac{k}{2iy}f)$$

An easy computation shows that $\delta_k.f$ is of weight $k+2$ and its degree of near holomorphy is increased by one. For a positive integer s, let δ_k^s be the differential operator defined by

$$\delta_k^s := \delta_{k+2s-2} \circ \cdots \circ \delta_k$$

The following easy lemma is due to Shimura and is proved by induction on r. We will generalize it to nearly overconvergent forms in the next section.

Lemma 2.1.3. *Let $f \in \mathcal{N}_k^r(\Gamma, \mathbf{C})$. Assume that $k > 2r$. Then there exist $g_0, \dots, g_r$ with $g_i \in \mathcal{M}_{k-2i}(\Gamma, \mathbf{C})$ such that*

$$f = g_0 + \delta_{k-2}g_1 + \cdots + \delta_{k-2r}^r g_r$$

It is easy to check, the forms g_i's are unique. When the hypothesis of this lemma are satisfied, Shimura defines the holomorphic projection $\mathcal{H}(f)$ of f as the holomorphic form defined by

$$\mathcal{H}(f) := g_0$$

Remark 2.1.4. The conclusion of this lemma is wrong if the assumption $k > 2r$ is not satisfied. The most important example is given by the Eisenstein series E_2 of weight 2 and level 1.

2.2. **Sheaf theoretic definition.**

2.2.1. Let $Y = Y_\Gamma := \Gamma\backslash\mathcal{H}$ and $X = X_\Gamma := \Gamma\backslash(\mathfrak{h} \sqcup \mathbf{P}^1(\mathbf{Q}))$ be respectively the open modular curve and complete modular curve of level Γ. Let $\mathbf{E} = \bar{\mathbf{E}} \times_{X_\Gamma} Y_\Gamma$ be the universal elliptic curve over Y_Γ and let $\mathbf{p} : \bar{\mathbf{E}} \to X_\Gamma$ be the Kuga-Sato compactification of the universal elliptic curve over X_Γ. We consider the sheaf of invariant relative differential forms with logarithmic poles along $\partial\bar{\mathbf{E}} = \bar{\mathbf{E}}\backslash\mathbf{E}$ which is a normal crossing divisor of $\bar{\mathbf{E}}$.

$$\omega := \mathbf{p}_*\Omega^1_{\bar{\mathbf{E}}/X}(log(\partial\bar{\mathbf{E}}))$$

It is a locally free sheaf of rank one in the holomorphic topos of X. We also consider

$$\mathcal{H}^1_{dR} := R^1\mathbf{p}_*\Omega^\bullet_{\bar{\mathbf{E}}/X}(log(\partial\bar{\mathbf{E}}))$$

the sheaf of relative degree one de Rham cohomology of $\bar{\mathbf{E}}$ over X with logarithmic poles along $\partial\bar{\mathbf{E}}$. The Hodge filtration induces the exact sequence

$$0 \to \omega \to \mathcal{H}^1_{dR} \to \omega^\vee \to 0 \qquad (2.2.1.a)$$

and in the C^∞-topos, this exact sequence splits to give the Hodge decomposition:

$$\mathcal{H}^1_{dR} = \omega \oplus \overline{\omega}$$

2.2.2. More concretely, let π be the projection $\mathfrak{h} \to Y_\Gamma$ and $\pi^*\mathbf{E}$ be the pull back of $\mathbf{E}$ by π. We have

$$\pi^*\mathbf{E} = (\mathbf{C} \times \mathfrak{h})/\mathbf{Z}^2$$

where the action of $\mathbf{Z}^2$ on $\mathbf{C}\times\mathfrak{h}$ is defines by $(z,\tau).(a,b) = (z+a+b\tau,\tau)$ for $(z,\tau) \in \mathbf{C}\times\mathfrak{h}$ and $(a,b) \in \mathbf{Z}^2$. The fiber E_τ of $\pi^*\mathbf{E}$ at $\tau \in \mathfrak{h}$ can be identified with $\mathbf{C}/L_\tau$ with $L_\tau = \mathbf{Z} + \tau\mathbf{Z} \subset \mathbf{C}$. We have

$$\pi^*\omega = \mathcal{O}_{\mathfrak{h}} dz$$

with $\mathcal{O}_{\mathfrak{h}}$ the sheaf of holomorphic function on $\mathfrak{h}$. Note also that

$$\mathbf{E} = \Gamma\backslash\mathbf{C} \times \mathfrak{h}/\mathbf{Z}^2$$

with the action of Γ on $\mathbf{C} \times \mathfrak{h}/\mathbf{Z}^2$ given by $\gamma.(z,\tau) = ((c\tau+d)^{-1}z, \gamma.\tau)$. We therefore have

$$\gamma^* dz = (c\tau+d)^{-1} dz$$

From this relation and the condition at the cusps, it is easy and well known to see that

$$H^0(X_\Gamma, \omega^{\otimes^k}) \cong \mathcal{M}_k(\Gamma, \mathbf{C})$$

Let $\mathcal{C}^\infty_{\mathfrak{h}}$ the sheaf of C^∞ functions on $\mathfrak{h}$. Then the Hodge decomposition of $\pi^*\mathcal{H}^1_{dR}$ reads

$$\pi^*\mathcal{H}^1_{dR} \otimes \mathcal{C}^\infty_{\mathfrak{h}} = \mathcal{C}^\infty_{\mathfrak{h}} dz \oplus \mathcal{C}^\infty_{\mathfrak{h}} d\overline{z}$$

On the other hand, by the Riemann-Hilbert correspondence, we have

$$\pi^*\mathcal{H}^1_{dR} = \pi^* R^1\mathbf{p}_*\mathbf{Z} \otimes \mathcal{O}_{\mathfrak{h}} = Hom(R_1\mathbf{p}_*\mathbf{Z}, \mathcal{O}_{\mathfrak{h}}) = \mathcal{O}_{\mathfrak{h}}\alpha \oplus \mathcal{O}_{\mathfrak{h}}\beta$$

where α, β is the basis of horizontal sections inducing on $H_1(E_\tau, \mathbf{Z}) = L_\tau$ the linear forms $\alpha(a+b\tau) = a$ and $\beta(a+b\tau) = b$ so that we have

$$dz = \alpha + \tau\beta \text{ and } d\overline{z} = \alpha + \overline{\tau}\beta$$

From the action of Γ on the differential form dz, it is then easy to see that

$$\gamma^* \cdot \begin{pmatrix} dz \\ \beta \end{pmatrix} = \begin{pmatrix} (c\tau+d)^{-1} & 0 \\ -c & (c\tau+d) \end{pmatrix} \begin{pmatrix} dz \\ \beta \end{pmatrix}$$

We define the holomorphic sheaf of X_Γ:

$$\mathcal{H}^r_k := \omega^{\otimes^{k-r}} \otimes Sym^r(\mathcal{H}^1_{dR})$$

Then we have the following proposition.

Proposition 2.2.3. *The Hodge decomposition induces a canonical isomorphism*

$$H^0(X_\Gamma, \mathcal{H}^r_k) \cong \mathcal{N}^r_k(\Gamma, \mathbf{C})$$

Proof. Let $\eta \in H^0(X_\Gamma, \mathcal{H}^r_k)$. Then $\pi^*\eta(\tau) = \sum_{l=0}^r f_l(\tau) dz^{\otimes^{k-l}} \beta^{\otimes^l}$ where the f_l's are holomorphic functions on $\mathfrak{h}$. Since we have $\beta = \frac{1}{2iy}(dz - d\overline{z})$, we deduce:

$$\pi^*\eta(\tau) = \sum_{l=0}^r \frac{f_l(\tau)}{(2iy)^l} \sum_{i=0}^l (-1)^i \binom{l}{i} dz^{\otimes^{k-i}} d\overline{z}^{\otimes^i}$$

The projection of $\pi^*\eta$ on the $(k,0)$-component of $H^0(\mathfrak{h},\pi^*\mathcal{H}_k^r)$ is therefore given by $f(\tau)dz^{\otimes^k}$ with
$$f(\tau)=\sum_{l=0}^{r}\frac{f_l(\tau)}{(2iy)^l}$$
It is clearly a nearly holomorphic form. It is useful to remark that the projection on the $(k,0)$-component is injective[5]. Conversely, if $f(\tau)=\sum_{l=0}^{r}\frac{f_l(\tau)}{y^l}$ is a nearly holomorphic form of weight k and order $\leq r$, using the injectivity of the projection onto the $(k,0)$-component, it is straightforward to see that $\sum_{l=0}^{r}(2i)^l f_l(\tau)dz^{\otimes^{k-l}}\beta^{\otimes^l}$ is invariant by Γ and defines an element of $H^0(X_\Gamma,\mathcal{H}_k^r)$ projecting onto $f(\tau)dz^{\otimes^k}$ via the Hodge decomposition. ∎

The quotient by the first step of the de Rham filtration of $\mathcal{H}_k^r$ induces by Poincaré duality the following canonical exact sequence

(2.2.3.a) $$0\to\omega^{\otimes^k}\to\mathcal{H}_k^r\to\mathcal{H}_{k-2}^{r-1}\to 0$$

The map $\mathcal{H}_k^r\to\mathcal{H}_{k-2}^{r-1}$ induces the morphism ϵ of (2.1.1.a).

2.3. **Rational and integral structures.**

2.3.1. Let N be a positive integer and let us assume $\Gamma=\Gamma_1(N)$ with $N\geq 3$. Then $X_\Gamma=X_1(N)$ is defined over $\mathbf{Z}[\frac{1}{N}]$ as well as ω, $\mathcal{H}^1_{dR}$ and $\mathcal{H}_k^r$. Recall that $Y_\Gamma=Y_1(N)$ classifies the isomorphism classes of pairs $(E,\alpha_N)_{/S}$ where $E_{/S}$ is an elliptic scheme over an $\mathbf{Z}[\frac{1}{N}]$-schemes S and α_N is a $\Gamma_1(N)$-level structure for E (i. e. an injection of group scheme: $\mu_{N/S}\hookrightarrow E[N]_{/S}$). Moreover the generalized universal elliptic curve is defined over $X_1(N)$ and we can define the sheaves ω, $\mathcal{H}^1_{dR}$ and $\mathcal{H}_k^r$ over $X_1(N)_{/\mathbf{Z}[\frac{1}{N}]}$ as in the previous paragraph. The exact seqeunce (2.2.3.a) is also defined over $\mathbf{Q}$. For any $\mathbf{Z}[\frac{1}{N}]$-algebra A, we define $\mathcal{M}_k(N,A)$ and $\mathcal{N}_k^r(N,A)$ respectively as the global section of $\omega^{\otimes k}_{/A}$ and $\mathcal{H}^r_{k/A}$. This gives integral and rational definitions of the space of nearly modular forms.

2.3.2. *Nearly holomorphic forms as functors.* For any ring R, we denote by $R[X]_r$ the R-module of polynomial in X of degree $\leq r$. Let B the Borel subgroup of SL_2 of upper triangular matrices. Then we consider the representation ρ_k^r of $B(R)$ on $R[X]_r$ defined by
$$\rho_k^r(\left(\begin{smallmatrix} a & b\\ 0 & a^{-1}\end{smallmatrix}\right)).P(X)=a^kP(a^{-2}X+ba^{-1})$$
We write $R[X]_r(k)$ to stress that it is given this action of $B(R)$.

It follows from the exact sequence (2.2.1.a) that $\mathcal{H}^1_{dR}$ is locally free over $Y_1(N)_{\mathbf{Z}[1/N]}$. A similar result would hold for general Shimura varieties of PEL type. In that case the torsion-freeness result from the basic properties of relative de Rham cohomology (for example see [Ka70]). We may therefore consider $\mathcal{T}$ the B-torsor over $(Y_1(N))_{Zar}$ of

[5] We will see a similar fact in the p-adic case. See Proposition 3.2.4.

isomorphisms $\psi_U : \mathcal{H}^1_{dR/U} \cong \mathcal{O}_U \oplus \mathcal{O}_U$ inducing an isomorphism $\psi^1_U : \omega_{/U} \cong \mathcal{O}_U \oplus \{0\}$ such that the isomorphism $(\mathcal{H}^1_{dR}/\omega)_{/U} \cong \mathcal{O}_U$ induced by ψ_U is dual to ψ^1_U for all Zariski open subset $U \subset Y_\Gamma$. Then over $Y_1(N)$, we have

$$\mathcal{T} \times^B A[X]_r(k) \cong \mathcal{H}^r_{k/A}$$

This implies the isomorphism above since the formation of both left and right hand sides commute to base change.

For an elliptic scheme $E_{/R}$ we consider a basis (ω, ω') of $\mathcal{H}^1_{dR}(E/R)$ such that ω is a basis of $\omega_{E/R} = H^0(E, \Omega^1_{E/R})$ and $\langle \omega, \omega' \rangle_{dR} = 1$. Then $f \in \mathcal{N}^r_k(N, A)$ can be seen as a functorial rule assigning to any A-algebra R and a quadruplet $(E, \alpha_N, \omega, \omega')_{/R}$ a polynomial

$$f(E, \alpha_N, \omega, \omega')(X) = \sum_{l=0}^{r} b_l X^l \in R[X]_r$$

defined such that the pull-back of f to $\mathcal{H}^1_{dR}(E/R)$ is $\sum_{l=0}^r b_l \omega^{\otimes^{k-l}} \otimes \omega'^{\otimes^l}$. For any $a \in R^\times, b \in R$, we have

$$f(E, \alpha_N, a\omega, a^{-1}\omega' + b\omega)(X) = a^{-k} f(E, \alpha_N, \omega, \omega')(a^2 X - ab)$$

The condition that f is finite at the cusps is expressed in terms of q-expansion as usual. It will be defined in the next paragraph.

Proposition 2.3.3. *Let $f \in \mathcal{N}^r_k(N, A)$ and $\epsilon.f \in \mathcal{N}^{r-1}_{k-2}(N, A)$ the image of f by the projection (2.2.3.a). Then for any quadruplet $(E, \alpha_N, \omega, \omega')_{/R}$, we have*

$$(\epsilon.f)(E, \alpha_N, \omega, \omega')(X) = \frac{d}{dX} f(E, \alpha_N, \omega, \omega')(X)$$

Proof. For any ring R, we have the exact sequence:

$$0 \to R[X]_0(k) \to R[X]_r(k) \to R[X]_{r-1}(k-2)$$

where the right hand side map is given by $P(X) \mapsto P'(X)$. It is clearly $B(R)$-equivariant. By taking the contracted product of this exact sequence with $\mathcal{T}$, we obtain the exact sequence of sheaves (2.2.3.a) which implies our claim. Notice that the map of sheaves inducing ϵ is surjective only if $r!$ is invertible in A. ■

2.3.4. *Polynomial q-expansions.* We consider $Tate(q)$ the Tate curve over $\mathbf{Z}[\frac{1}{N}]((q))$ with its canonical invariant differential form ω_{can} and canonical $\Gamma_1(N)$ level structure $\alpha_{N,can}$. We have the Gauss-Manin connection:

$$\nabla : \mathcal{H}^1_{dR}(Tate(q)/\mathbf{Z}[\frac{1}{N}]((q))) \to \mathcal{H}^1_{dR}(Tate(q)/\mathbf{Z}[\frac{1}{N}]((q))) \otimes \Omega^1_{\mathbf{Z}[\frac{1}{N}]((q)))/\mathbf{Z}[\frac{1}{N}]}$$

and let

$$u_{can} := \nabla(q\frac{d}{dq})(\omega_{can})$$

Then (ω_{can}, u_{can}) form a basis of $\mathcal{H}^1_{dR}(Tate(q)/\mathbf{Z}[\frac{1}{N}]((q)))$ and u_{can} is horizontal, moreover $<\omega_{can}, u_{can}>_{dR}= 1$ (see for instance the appendix 2 of [Ka72]). For any $\mathbf{Z}[\frac{1}{N}]$-algebra A and $f \in \mathcal{N}^r_k(N, A)$, we consider

$$f(q, X) := f(Tate(q)_{/A((q))}, \alpha_{can}, \omega_{can}, u_{can})(X) \in A[[q]][X]_r.$$

We call it the polynomial q-expansion of the nearly holomorphic form f.

Remark 2.3.5. We can think of the variable X as $\frac{-1}{4\pi y}$.

2.3.6. *The nearly holomorphic form E_2.* It is well-known that the Eisenstein series of weight 2 and level 1 is a nearly holomorphic form of order 1. Its given by

$$E_2(\tau) = -\frac{1}{24} + \frac{1}{8\pi y} + \sum_{n=1}^{\infty} \sigma^1_n q^n$$

where σ^1_n is the sum of the positive divisor of n and $q = e^{2i\pi\tau}$.

We can define E_2 as a functorial rule. Let R be a ring with $\frac{1}{6} \in R$ and E be an elliptic curve over R. Recall (see for instance [Ka72, Appendix 1]) that any basis $\omega \in \omega_{E/R}$ defines a Weierstrass equation for E:

$$Y^2 = 4X^3 - g_2X - g_3$$

such that $\omega = \frac{dX}{Y}$ and $\eta = X\frac{dX}{Y}$ form a R-basis of $H^1_{dR}(E/R)$. Moreover if we replace ω by $\lambda\omega$, η is replaced by $\lambda^{-1}\eta$. Therefore $\omega \otimes \eta \in \omega_{E/R} \otimes H^1_{dR}(E/R)$ is independant of the choice of ω. It therefore defines a section of $\mathcal{H}^1_2$ over $Y_{/\mathbf{Z}[\frac{1}{6}]}$. If (ω, ω') is a basis of $H^1_{dR}(E/R)$ such that $<\omega, \omega'>_{dR}= 1$ then we can put

$$\tilde{E}_2(E, \omega, \omega') = <\eta, \omega'>_{dR} + X<\omega, \eta>_{dR}$$

where $<\cdot, \cdot>_{dR}$ stands for the Poincaré pairing on $H^1_{dR}(E/R)$.

Its polynomial q-expansion is given by

$$\tilde{E}_2(q, X) = \tilde{E}_2(Tate(q), \omega_{can}, u_{can})(X) = \frac{P(q)}{12} + X$$

because $u_{can} = -\frac{P(q)}{12}\omega_{can} + \eta_{can}$ and $<\omega_{can}, \eta_{can}>_{dR}= 1$ and where $P(q)$ is defined in [Ka72] by

$$P(q) = 1 - 24\sum_{n=1}^{\infty} \sigma^1_n q^n$$

Since the q-expansion of $\tilde{E}_2$ is finite at $q = 0$, $\tilde{E}_2$ defines a section of $\mathcal{H}^1_2$ over X. From the q-expansion, it is easy to see that $\tilde{E}_2 = -2E_2$. We clearly have $\epsilon.\tilde{E}_2 = 1$ (i.e. is the constant modular form of weight 0 taking the value 1).

Remark 2.3.7. We can see easily that multiplying by $\tilde{E}_2$ is useful to get a splitting of $\mathcal{H}^1_k$:

$$0 \to \omega^{\otimes^k} \to \mathcal{H}^1_k \to \omega^{\otimes^{k-2}} \to 0$$

In particular, that shows that nearly holomorphic forms are polynomial in E_2 with holomorphic forms as coefficients. As mentioned in the introduction, this provides a way to give an ad hoc definition of nearly overconvergent forms.

2.4. **Differential operators.** Recall that the Gauss-Manin connection

$$\nabla:\ \mathcal{H}^1_{dR} \to \mathcal{H}^1_{dR} \otimes \Omega_{X_1(N)/\mathbf{Z}[\frac{1}{N}]}(log(Cusp))$$

induces the Kodaira-Spence isomorphism $\omega^{\otimes^2} \cong \Omega_{X_1(N)/\mathbf{Z}[\frac{1}{N}]}(log(Cusp))$ and a connection

$$\nabla:\ Sym^k(\mathcal{H}^1_{dR}) \to Sym^k(\mathcal{H}^1_{dR}) \otimes \Omega_{X_1(N)/\mathbf{Z}[\frac{1}{N}]}(log(Cusp)) \cong Sym^k(\mathcal{H}^1_{dR}) \otimes \omega^{\otimes^2}$$

The Hodge filtration of $Sym^k(\mathcal{H}^1_{dR})$ is given by $Fil^0 \supset \cdots \supset Fil^k \supset Fil^{k+1} = \{0\}$ with

$$Fil^{k-r} = \mathcal{H}^r_k = \omega^{\otimes^{k-r}} \otimes Sym^r(\mathcal{H}^1_{dR}) \ \text{ for } \ 0 \leq r \leq k$$

Since ∇ satisfies Griffith transversality, when $k \geq r \geq 0$, it sends Fil^{k-r} into $Fil^{k-r-1} \otimes \Omega_{X_1(N)/\mathbf{Z}[\frac{1}{N}]}(log(Cusp))$. We therefore get the sheaf theoretic version of the Maass-Shimura operator[6]:

$$\delta_k:\ \mathcal{H}^r_k \to \mathcal{H}^{r+1}_{k+2} \quad \text{for} \quad 0 \leq r \leq k$$

We still denote δ_k the corresponding operator

$$\delta_k:\ \mathcal{N}^r_k(N, A) \to \mathcal{N}^{r+1}_{k+2}(N, A)$$

The following proposition gives the effect of δ_k on the polynomial q-expansion.

Proposition 2.4.1. *Let $f \in \mathcal{N}^r_k(N, A)$. Then the polynomial q-expansion of $\delta_k f$ is given by:*

$$(\delta_k f)(q, X) = X^k D(X^{-k} f(q, X))$$

where D is the differential operator on $A[[q]][X]$ given by $D = q\frac{\partial}{\partial q} - X^2 \frac{\partial}{\partial X}$. In other words, if $f(q, X) = \sum_{i=0}^r f_i(q) X^i$, we have

$$(\delta_k f)(q, X) = \sum_{i=0}^{r} q\frac{d}{dq} f_i(q) X^i + (k-i) f_i(q) X^{i+1}$$

Proof. The pull-back of f to $\phi^* \mathcal{H}^r_k$ with $\phi : \mathrm{Spec}\,(A((q))) \to X_1(N)_{/A}$ corresponds to $(Tate(q), \alpha_{N,can})_{/A((q))}$ is given by

$$\phi^* f = \sum_{i=0}^{r} f_i(q) \omega_{can}^{\otimes^{k-i}} \otimes u_{can}^{\otimes^i}$$

Since ∇ induces $\frac{dq}{q} \cong \omega^2_{can}$, $\nabla(q\frac{d}{dq})(\omega_{can}) = u_{can}$ and $\nabla(q\frac{d}{dq})(u_{can}) = 0$,, we have:

$$\nabla(q\frac{d}{dq})(\phi^* f) = \sum_{i=0}^{r} \frac{d}{dq} f_i(q) \omega_{can}^{\otimes^{k-i+2}} \otimes u_{can}^{\otimes^i} + (k-i) f_i(q) \omega_{can}^{\otimes^{k-i-1}} \otimes u_{can}^{\otimes^{i+1}}$$

[6]We leave it as an exercise to check that this operator corresponds to the classical Maass-Shimura operator via the isomorphism of Proposition 2.2.3.

This implies our claim by the definition of the polynomial q-expansion.∎

Remark 2.4.2. We can rewrite the formula for δ_k in the following way:

$$\delta_k = D + kX$$

Using the relation $[D, X] = -X^2$, we easily show by induction that

$$\delta_k^r = \sum_{j=0}^{r} \binom{r}{j} \frac{\Gamma(k+r)}{\Gamma(k+r-j)}.X^j D^{r-j} \tag{2.4.2.a}$$

Notice in particular that for $s \leq r$ and h holomorphic, we have

$$(\epsilon_{k+2r}^s \circ \delta_k^r)h = \sum_{j=s}^{r} \binom{r}{j} \frac{\Gamma(k+r)\Gamma(j+1)}{\Gamma(k+r-j)\Gamma(j+1-s)}.X^{j-s}(q\frac{d}{dq})^{r-j}h \tag{2.4.2.b}$$

2.5. **Hecke operators.**

2.5.1. Let $R_1(N)$ be the abstract Hecke algebra attached the pair $(\Gamma_1(N), \Delta_1(N))$ by Shimura [Sh73, chapt. 3]. This algebra is generated over $\mathbf{Z}$ by the operators T_n for n running in the set of natural integers. If ℓ is a prime dividng N, the operator T_ℓ is sometimes called U_ℓ. These operators act on the space of nearly holomorphic forms by the usual standard formulas and preserve the weight and degree of nearly holomorphy. Moreover the Hecke operators respect the rationality and integrality of nearly holomorphic forms.

2.5.2. For any ring $A \subset \mathbf{C}$, we then denote by $h_k^r(N, A) \subset End_{\mathbf{C}}(\mathcal{N}_k^r(N, \mathbf{C}))$ the subalgebra generated over A by the image of the T_n's. If $\mathbf{Z}[\frac{1}{N}] \subset A \subset B$, then the above remark shows that we have

$$h_k^r(N, A) \otimes_A B \cong h_k^r(N, B) \tag{2.5.2.a}$$

2.5.3. An easy computation shows that, for each integer n and $f \in \mathcal{N}_k^r(N, \mathbf{C})$ we have:

$$(\delta_k.f)|_{k+2}T_n = n.\delta_k(f|_k T_n) \tag{2.5.3.a}$$

$$\epsilon.(f|_k T_n) = n.(\epsilon.f)|_{k-2}T_n \tag{2.5.3.b}$$

Remark 2.5.4. From these formulae, we see that if f is a holomorphic eigenform of weight k. Then $\delta_k^r f$ is a nearly holomorphic eigenform of weight $k+2r$. Moreover the system of Hecke eigenvalues of $\delta_k^r f$ is different than the one of any holomorphic Hecke eigenform of weight $k+2r$ if $r > 0$.

2.6. **Rationality and CM-points.**

2.6.1. We review quickly the rationality notion introduced by Shimura. For $K \subset \overline{\mathbf{Q}}$ an imaginary quadratic field and $\tau \in \mathfrak{h} \cap \mathcal{K}$, the elliptic curve E_τ has complex multiplication by $\mathcal{K}$ and is therefore defined over $K^{ab} \subset \overline{\mathbf{Q}}$ the maximal abelian extension of K by the theory of Complex Multiplication. We then denote by ω_τ an invariant Kähler differential of E_τ defined over $\mathcal{K}^{ab}$ and we denote by Ω_τ the corresponding CM period defined by

$$\omega_\tau = \Omega_\tau dz$$

Then $(\omega_\tau, \overline{\omega}_\tau)$ forms a basis of $H^1_{dR}(E_\tau/\mathcal{K}^{ab})$. Let E be a number field and $f \in \mathcal{N}_k^r(N, E)$. Let $\alpha_{N,\tau}$ the $\Gamma_1(N)$-level structure of E_τ induced by $\frac{1}{N}\mathbf{Z}/\mathbf{Z} \subset \mathbf{C}/L_\tau$. Then the polynomial $f(E_\tau, \alpha_{N,\tau}, \omega_\tau, \overline{\omega}_\tau)(X)$ belongs to $EK^{ab}[X]$. Since we have $\overline{\omega}_\tau = \overline{\Omega}_\tau d\overline{z}$, we deduce that $f(E_\tau, \alpha_{N,\tau}, \omega_\tau, \overline{\omega}_\tau)(0)$ is the left hand side of (2.6.1.a) and that we therefore have

$$\frac{f(\tau)}{\Omega_\tau^k} \in E\mathcal{K}^{ab} \tag{2.6.1.a}$$

According to Shimura, a nearly holomorphic form is defined as rational if and only if it satisfies (2.6.1.a) for any imaginary quadratic field K and almost all $\tau \in \mathcal{H} \cap K$. It can be easily seen his definition is equivalent to our sheaf theoretic definition.

Proposition 2.6.2. *Let $f \in \mathcal{N}_k^r(N, \mathbf{C})$ and E be a number field such that for any imaginary quadratic field $\mathcal{K} \subset \overline{\mathbf{Q}}$ and almost all $\tau \in \mathcal{K} \cap \mathfrak{h}$, we have*

$$\frac{f(\tau)}{\Omega_\tau^k} \in E.\mathcal{K}^{ab},$$

then, $f \in \mathcal{N}_k^r(N, E\mathbf{Q}^{ab})$.

Proof. We just give a sketch under the assumption $k > 2r$ since the general case can be deduced after multiplying f by E_2. Thanks to a Galois descent argument, we may assume E contains the eigenvalues of all Hecke operators acting on $\mathcal{N}_k^r(N, \mathbf{C})$. By Lemma 2.1.3, we can decompose f as

$$f = f_0 + \delta_{k-2} f_1 + \cdots + \delta_{k-2r}^r f_r$$

with f_i holomorphic of weight $k - 2i$. Now we remark that if T is a Hecke operator defined over E, then $f|_k T$ satisfies (2.6.1.a). This follows easily from the definition of the action of Hecke operators using isogenies. Moreover from Remark 2.5.4, the system of Hecke eigenvalues of nearly holomorphic forms $\delta^i h$ and $\delta^{i'} h'$ are distinct for any two holomorphic forms h and h' when $i \neq i'$; we deduce that $\delta^i f_i$ satisfy (2.6.1.a). We may assume therefore $f = \delta_{k-2r}^r g$ for an holomorphic form g of weight $k - 2r$. In fact using a similar argument, we may even assume g is an eigenform. Then $g = \lambda . g_0$ with g_0 defined over E. Since $\delta_{k-2r}^r g_0$ is defined over E, we deduce from §2.6.1 that $\delta_{k-2r}^r g_0$ satisfies (2.6.1.a) and therefore $\lambda \in EK^{ab}$ and $f \in \mathcal{N}_k^r(N, EK^{ab})$. Since this can be done for any K the result follows. ∎

3. Nearly overconvergent forms

In this section, we introduce our definition of nearly overconvergent modular forms and show they are p-adic modular forms of a special type. We use the spectral theory of the Atkin U_p-operator on them and we define p-adic families of such forms. We study also the effect of differential operators on them and define an analogue of the holomorphic projection. These tools are useful to study certain p-adic families of modular forms and also to study p-adic L-functions.

3.1. **Katz p-adic modular forms.**

3.1.1. We fix p a prime. Let $X_{\mathrm{r}ig}$ be the generic fiber in the sense of rigid geometry of the formal completion of $X_{/\mathbf{Z}_p}$ along its special fiber. Let $A \in H^0(X_{/\mathcal{F}_p}, \omega^{\otimes^{p-1}})$ be the Hasse invariant and let $\tilde{A}^q$ be a lifting of A^q to characteristic 0 for q sufficiently large. For $\rho \in p^{\mathbf{Q}} \cap [p^{-1/p+1}, 1]$, we write $X^{\geq \rho}$ for the rigid affinoid subspace of $X_{\mathrm{r}ig}$ defined as the set of $x \in X_{\mathrm{r}ig}$ satisfying $|\tilde{A}^q(x)|_p \geq \rho^q$. For $\rho = 1$ we get the ordinary locus of $X_{\mathrm{r}ig}$ and we denote it X_{ord}. The space of p-adic modular forms of weight k is defined as

$$M_k^{p-adic}(N) := H^0(X_{\mathrm{ord}}, \omega^{\otimes^k})$$

The space of overconvergent forms of weight k is the subspace of p-adic forms which are defined on some strict neighborhood of X_{ord} so:

$$M_k^{\dagger}(N) := \varinjlim_{\rho<1} H^0(X^{\geq \rho}, \omega^{\otimes^k})$$

3.1.2. Let $\varphi : X_{\mathrm{ord}} \to X_{\mathrm{ord}}$ the lifting of Frobenius induced on Y_{ord} by $(E, \alpha_N) \mapsto (E^{(\varphi)}, \alpha_N^{(\varphi)})$ where $E^{(\varphi)} := E/E[p]^{\circ}$ and $\alpha_N^{(\varphi)}$ is the composition of α_N and the Frobenius isogeny $E \to E^{(\varphi)}$. We get a φ^*-linear morphism obtained as the composite

$$\Phi : \mathcal{H}^1_{dR} \to \varphi^*\mathcal{H}^1_{dR} = \mathcal{H}^1_{dR}(\mathbf{E}^{\varphi}/X_{\mathrm{ord}}) \to \mathcal{H}^1_{dR}$$

This morphism stabilizes the Hodge filtration of $\mathcal{H}^1_{dR}$ and we know by Dwork that there is a unique Φ-stable splitting, called the unit root splitting:

$$\mathcal{H}^1_{dR/X_{\mathrm{ord}}} = \omega_{/X_{\mathrm{ord}}} \oplus \mathcal{U}_{/X_{\mathrm{ord}}}$$

such that Φ is invertible on $\mathcal{U}$ and $\mathcal{U}$ is a free sheaf of rank 1 generated by its sub-sheaf of horizontal sections for the Gauss-Manin connection. This unit root splitting induces a splitting of $\mathcal{H}^r_{k/X_{\mathrm{ord}}}$ and therefore of a canonical projection

$$\mathcal{H}^r_{k/X_{\mathrm{ord}}} \to \omega^{\otimes^k}_{/X_{\mathrm{ord}}} \tag{3.1.2.a}$$

We now recall the definition of the Theta operator on the space of p-adic modular forms of weight k. At the level of sheaves, Θ is defined as the composite of the following maps of sheaves over the ordinary locus:

$$\omega^{\otimes^k}_{/X_{\mathrm{ord}}} \xrightarrow{\delta_k} \mathcal{H}^1_{k+2/X_{\mathrm{ord}}} \to \omega^{\otimes^{k+2}}_{/X_{\mathrm{ord}}}$$

where he second arrow is the one given by (3.1.2.a) for $r = 1$. This defines

$$\Theta : M_k^{p-adic}(N) \to M_{k+2}^{p-adic}(N)$$

It follows from the Proposition 2.4.1 and Proposition 3.2.4 below that on the level of q-expansion, we have:

$$\Theta(f)(q) = q\frac{d}{dq}f(q)$$

for all $f \in M_k^{p-adic}(N)$. The following proposition will be useful in the next paragraph.

Proposition 3.1.3. *For any $\rho < 1$ and any Zariski open $V \subset X^{\geq \rho}$, the unit root splitting on $V_{\mathrm{ord}} := U \cap X_{\mathrm{ord}}$ does not extend to a splitting of the Hodge filtration of $\mathcal{H}^1_{dR}$ over any finite cover of V.*

Proof. We show it by contradiction. Let us assume that this splitting extends to some finite cover S of V for some $\rho < 1$.

$$\mathcal{H}^1_{dR}(\mathbf{E}/S) = \omega_{\mathbf{E}/S} \oplus \mathcal{U}(\mathbf{E}/S).$$

Let $S_{\mathrm{ord}} = S \times_{V_{\mathrm{ord}}} V$. Since $\mathcal{U}(\mathbf{E}/S) \otimes \mathcal{O}_{S_{\mathrm{ord}}}$ is stable by Φ so is $\mathcal{U}(\mathbf{E}/S)$. Since V is a strict neighborhood of V_{ord}, we can find an unramified extension L of $\mathbf{Q}_p$ and $x \in S(L) \backslash S_{\mathrm{ord}}(L)$. Then, we will obtain a splitting $H^1_{dR}(\mathbf{E}_x/L) = \omega_{\mathbf{E}_x/L} \oplus \mathcal{U}(\mathbf{E}_x/L)$ with Φ_x inducing a semi-linear invertible endomorphism of $\mathcal{U}(\mathbf{E}_x/L)$. Let k_L be the residue field of L. By the results of [BO, §7.4 and §7.5], the pair $(H^1_{dR}(\mathcal{E}_x/L), \Phi_x)$ is isomorphic to $(H^1_{crys}(\mathbf{E}_{x,0}/O_L), F^*)$ where $H^1_{crys}(\mathbf{E}_{x,0}/O_L)$ stands for the crystalline cohomology of the special fiber $\mathbf{E}_{x,0/k_L}$ of $\mathbf{E}_x$ and where $\Phi_x \otimes 1_{k_L} = F^*$ is induced by the Frobenius isogeny $F : \mathbf{E}_{x,0} \to \mathbf{E}_{x,0}^{(p)}$. Since it has a splitting of the form $H^1_{crys}(\mathbf{E}_{x,0}/O_L) = Fil^1 \oplus U$ with F^* inversible on U, $\mathbf{E}_{x,0}$ has to be ordinary which is a contradiction since $x \notin S_{\mathrm{ord}}(L)$. ■

3.2. **Nearly overconvergent forms as p-adic modular forms.**

3.2.1. *Definition of nearly overconvergent forms.* For each ρ, $H^0(X^{\geq\rho}, \mathcal{H}_k^r)$ is naturally a $\mathbf{Q}_p$-Banach space for the Supremum norm $|\cdot|_\rho$ and if $\rho' < \rho < 1$, the map $H^0(X^{\geq\rho'}, \mathcal{H}_k^r) \hookrightarrow H^0(X^{\geq\rho}, \mathcal{H}_k^r)$ is completely continuous. We define the space of nearly overconvergent forms of weight k and order $\leq r$ by

$$\mathcal{N}_k^{r,\dagger}(N) := \varinjlim_{\rho<1} \mathcal{N}_k^{r,\rho}(N)$$

with $\mathcal{N}_k^{r,\rho}(N) := H^0(X^{\geq\rho}, \mathcal{H}_k^r)$. We can define the operators δ_k and ϵ on nearly overconvergent forms since they are defined at the level of sheaves. Moreover we can define the polynomial q-expansion of a nearly overconvergent form and it is straightforward to check that the action of δ_k and ϵ on this q-expansion is the same as for nearly holomorphic forms.

Remark 3.2.2. For any nearly overconvergent form f of weight k and order at most r, we can easily show that there exist overconvergent forms $g_0, \dots, g_r$ such that

$$f = g_0 + g_1.E_2 + \cdots + g_r.E_2^r$$

where for $i = 0, \dots, r$, g_i is of weight $k - 2i$. This could be used as an ad hoc definition of nearly overconvergent forms but it would be uneasy to show this space is stable by the action of Hecke operators and that it has a slope decomposition as we will see in the next sections. Moreover, this definition would not be suited for generalization to higher rank reductive groups.

3.2.3. We now want to consider nearly overconvergent modular forms as p-adic modular forms. Using (3.1.2.a), we get a map

$$H^0(X^{\geq \rho}, \mathcal{H}_k^r) \to H^0(X_{\mathrm{ord}}, \mathcal{H}_k^r) \to H^0(X_{\mathrm{ord}}, \omega^{\otimes k}) \tag{3.2.3.a}$$

We have the following

Proposition 3.2.4. *The maps* (3.2.3.a) *induces a canonical injection*

$$\mathcal{N}_k^{r,\dagger}(N) \hookrightarrow M_k^{p-adic}(N) \tag{3.2.4.a}$$

fitting in the commutative diagram:

$$\begin{tikzcd} \mathcal{N}_k^{r,\dagger}(N) \arrow[r, hook] \arrow[d, hook] & M_k^{p-adic}(N) \arrow[d, hook] \\ \mathbf{Q}_p[[q]][X]_r \arrow[r] & \mathbf{Q}_p[[q]] \end{tikzcd}$$

where the bottom map is induced by evaluating $X = 0$.

Proof. The fact that the diagram commutes follows from the fact that u_{can} belongs to the fiber of the unit root sheaf $\mathcal{U}$ at the $\mathbf{Z}_p((q))$-point defining $Tate(q)$. Indeed, it is explained in the appendix 2 of [Ka72] that u_{can} is fixed by Frobenius. We are left with proving the injectivity. We consider f in the kernel of this map. Let $U \subset X^{\geq r} \cap Y_{rig}$ be an irreducible affinoid. It is the generic fiber in the sense of Raynaud of an affine formal scheme $Spf(R)$ with R a p-adically complete domain. Let $\mathcal{E}_{/R}$ the universal elliptic curve over R. Let us choose a basis (ω, ω') of $\mathcal{H}^1_{dR}(\mathcal{E}/R)$ as in the previous section and such that $\langle \omega, \omega' \rangle_{dR} = 1$. Let $h \in R$ be a lifting of the Hasse invariant of $\mathcal{E} \times_R Spec(R/pR)$ and let $S := \widehat{R[1/h]}$ where the hat here stands for p-adic completion. Then $\mathcal{E}/S$ has ordinary reduction and the unit root splitting over S defines a basis (ω, u) of $\mathcal{H}^1_{dR}(\mathcal{E}/S)$. We must have

$$u = \omega' + \lambda.\omega$$

with $\lambda \in S$. But U is a strict neighborhood of $U_{\mathrm{ord}} = U \cap X_{\mathrm{ord}}$ which is the generic fiber of $Spf(S)$, we know by the previous proposition that λ is not algebraic over R. Let $Q(X) := f(\mathcal{E}_{/R}, \alpha_N, \omega, \omega')(X) \in R[X]_r$. We want to show that $Q(X) = 0$. By assumption, we know that $Q(\lambda) = f(\mathcal{E}_{/S}, \alpha_N, \omega, \omega' + \lambda\omega)(0) = f(\mathcal{E}_{/S}, \alpha_N, \omega, u)(0) = 0$. Since λ is not algebraic over R, this is possible only if $Q(X) = 0$. Since this can be done for any pair (ω, ω') we conclude that $f \equiv 0$. ∎

If f is a nearly overconvergent form, the p-adic q-expansion of f is by definition the q-expansion of the image of f in the space of p-adic forms. The following corollary can be thought of as a *polynomial q-expansion principle* for the degree of near overconvergence.

Corollary 3.2.5. *Let $f \in \mathcal{N}_k^{r,\dagger}(N)$. If $f(q,X)$ is of degree r then there is no $g \in \mathcal{N}_k^{r-1,\dagger}(N)$ having the same p-adic q-expansions.*

Proof. We prove this by contradiction. Let us assume that such a g exists. Let $h = f - g$. Since $f(q,X)$ is of degree r and $g \in \mathcal{N}_k^{r-1,\dagger}(N)$, $h(q,X)$ is still of degree r and therefore h is non-zero. However, by assumption $h(q,0) = 0$. This implies that $h = 0$ by the diagram of the previous proposition, which is a contradiction.∎

3.3. E_2**, Θ and overconvergence.** In the following two corollaries, we recover the main results of [CGJ] using the polynomial q-expansion principle. It can be easily generalized to modular forms for other Shimura varieties.

Corollary 3.3.1. *The p-adic modular form E_2 is not overconvergent.*

Proof. By the Corollary 3.2.5, this is immediate since the polynomial q expansion of E_2 is of degree 1.∎

Corollary 3.3.2. *If f is overconvergent of weight k and $k \neq 0$, then $\Theta.f$ is not overconvergent.*

Proof. It follows from Proposition 2.4.1 and Proposition 3.2.4, that $\Theta.f$ is the image of the nearly overconvergent form $\delta_k.f$ in the space of p-adic modular forms by the map (3.2.4.a). Moreover

$$(\delta_k.f)(q,X) = q\frac{d}{dq}f + kXf(q)$$

It is therefore of degree 1 since $k \neq 0$ and the result follows from Corollary 3.2.5.∎

3.3.3. *Overconvergent projection.* We give a p-adic version of the holomorphic projector.

Lemma 3.3.4. *Let f be a nearly overconvergent form of weight k and order $\leq r$ such that $k > 2r$. Then for each $i = 0, \cdots, r$, there exists a unique overconvergent form g_i of weight $k - 2i$ such that*

$$f = \sum_{i=1}^{r} \delta_{k-2i}^{i}.g_i$$

Proof. This is special case of the proof of Proposition 3.5.4 below. ∎

We define the overconvergent projection $\mathcal{H}^\dagger(f)$ of f by:

$$\mathcal{H}^\dagger(f) := g_0$$

It is an overconvergent version of the holomorphic projection since if f is holomorphic, then we clearly have:

$$\mathcal{H}^\dagger(f) = \mathcal{H}(f)$$

which means that $\mathcal{H}^\dagger(f)$ is holomorphic.

Remark 3.3.5. Let $f \in \mathcal{N}_k^{r,\dagger}(N, \mathbf{Q}_p)$ and $g \in \mathcal{N}_l^{s,\dagger}(N, \mathbf{Q}_p)$ such that $k + l > 2s + 2r$. Then the following holds $\mathcal{H}^\dagger(f\delta_l^m g) = (-1)^m \mathcal{H}^\dagger(g\delta_k^m f)$. One can also show that when a Hecke equivariant p-adic Petersson inner product is defined then δ and ϵ are very close to be adjoint operators. This implies a formula of the type $\langle f, g\rangle_{p-adic} = \langle f, \mathcal{H}^\dagger(g)\rangle_{p-adic}$ when f is overconvergent. We hope to come back to this in a future paper.

3.3.6. *Action of the Atkin-Hecke operator U_p.* If $\rho > p^{-1/p+1}$, it follows from the theory of the canonical subgroup (Katz-Lubin) that we can extend canonically φ on $X_{\rm ord}$ into

$$\varphi : X^{\geq\rho} \to X^{\geq\rho^p}.$$

Let $\mathbf{E}/X^{\geq\rho^p}$ be the generalized universal elliptic curve over $X^{\geq\rho^p}$ and let $\mathbf{E}^{(\varphi)}/X^{\geq\rho}$ be its pullback by φ. We have degree p isogeny

$$\mathbf{E} \overset{F_\varphi}{\to} \mathbf{E}^{(\varphi)}$$

over $X^{\geq\rho}$ and we denote $V_\varphi : \mathbf{E}^{(\varphi)} \to \mathbf{E}$ the dual isogeny. On the level of sheaves, the operator U_p is defined as the composition of the following maps.

$$\mathcal{H}^r_{k/X^{\geq\rho}} \xrightarrow{V_\varphi^*} \mathcal{H}^{r,(\varphi)}_k{}_{/X^{\geq\rho}} = \mathcal{H}^r_{k/X^{\geq\rho^p}} \otimes_{\varphi^*} \mathcal{O}_{/X^{\geq\rho}} \overset{Id\otimes\frac{1}{p}\cdot Tr}{\to} \mathcal{H}^r_{k/X^{\geq\rho^p}} \overset{j}{\to} \mathcal{H}^r_{k/X^{\geq\rho}}$$

where j is induced by the completely continuous inclusion $\mathcal{O}_{X^{\geq\rho^p}} \to \mathcal{O}_{X^{\geq\rho}}$ defined by the restriction of analytic function on $X^{\geq\rho^p}$ to $X^{\geq\rho}$ and Tr is induced by the trace of the degree p map $\varphi^* : \mathcal{O}_{X^{\geq\rho^p}} \to \mathcal{O}_{X^{\geq\rho}}$. Since j is completely continuous, U_p induces a completely continuous endomorphism of $\mathcal{N}_k^{r,\rho}(N, \mathbf{Q}_p)$. The following proposition is easy to prove.

Proposition 3.3.7. *Let $f \in \mathcal{N}_k^{r,\dagger}(N, \mathbf{Q}_p)$. Let us write its polynomial q-expansion as:*

$$f(q, X) = \sum_{n=0}^{\infty} a(n, f)(X)q^n$$

Then we have:

(i) $(f|U_p)(q, X) = \sum_{n=0}^{\infty} a(np, f)(pX)q^n$
(ii) $\epsilon(f|U_p) = p.(\epsilon f)|U_p$
(iii) $(\delta_k f)|U_p = p\delta_k(f|U_p)$

Proof. (i) follows from a standard computation and (ii) and (iii) follow from (i) and the effect of δ_k and ϵ on the polynomial q-expansion explained in Section 2.■

3.3.8. Let $\widehat{\mathcal{N}}_k^{\infty,\rho}(N,\mathbf{Q}_p)$ be the p-adic completion of

$$\mathcal{N}_k^{\infty,\rho}(N,\mathbf{Q}_p) := \bigcup_{r\geq 0} \mathcal{N}_k^{r,\rho}(N,\mathbf{Q}_p).$$

Then we have:

Corollary 3.3.9. *The action of U_p on $\widehat{\mathcal{N}}_k^{\infty,\rho}(N,\mathbf{Q}_p)$ is completely continuous.*

Proof. It follows easily from the lemma below for the sequence $M_i = \mathcal{N}_k^{i,\rho}(N)$ and the relation (ii) of the previous proposition.∎

Lemma 3.3.10. *Let M_i be an increasing sequence of Banach modules over a p-adic Banach algebra A. Let u be an endomorphism on $M := \bigcup_i M_i$ such that*

(i) *u induces a completely continuous endomorphism on each of the M_i's.*
(ii) *Let α_i be the norm of the operator on the Hausdorff quotient of M_i/M_{i-1} induced by u. Then the sequence α_i converges to 0.*

Then u induces a completely continuous operator on the p-adic completion of M.

Proof. This is an easy exercise which is left to the reader.∎

Remark 3.3.11. We can give a sheaf theoretic definition of $\hat{\mathcal{N}}_k^{\infty,\rho}$. Let $\mathcal{A}$ be the ring of analytic functions defined over $\mathbf{Q}_p$ on the closed unit disc. It is isomorphic to the power series in X with $\mathbf{Q}_p$-coefficient converging to 0. We denote it $\mathcal{A}_k$ if one equips it with the representation of the standard Iwahori subgroup of $SL_2(\mathbf{Z}_p)$ defined by:

$$(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).f)(X) = (a+cX)^k f(\frac{b+dX}{a+cX})$$

If we restrict this representation to the Borel subgroup we get a representation that contains $\mathbf{Z}_p[X]_r(k)$ for all r. In fact, it is the p-adic completion of $\mathbf{Z}_p[X](k) = \bigcup_r \mathbf{Z}_p[X]_r(k)$. It is not difficult to see, one can define a sheaf of Banach spaces on $X^{\geq\rho}$ (in the sense of [AIP]) by considering the contracted product

$$\mathcal{H}_k^{\infty} := \mathcal{T} \times^B \mathcal{A}_k$$

Then, we easily see that

$$\hat{\mathcal{N}}_k^{\infty,\rho} = H^0(X^{\geq\rho}, \mathcal{H}_k^{\infty})$$

3.3.12. *Slopes of nearly overconvergent forms.* Let $f \in \mathcal{N}_k^{r,\dagger}(N,\overline{\mathbf{Q}}_p)$ which is an eigenform for U_p for the eigenvalue α. If $\lambda \neq 0$, we say f is of finite slope $\alpha = v_p(\lambda)$. The following proposition compares the slope and the degree of near overconvergence and extends the classicity result of Coleman to nearly overconvergent forms.

Proposition 3.3.13. *Let $f \in \mathcal{N}_k^{\infty,\dagger}(N,\overline{\mathbf{Q}}_p)$, then the following properties hold*

(i) *If f is of slope α, then its degree of overconvergence r satisfies $r \leq \alpha$,*

(ii) *If f is of degree r and slope $\alpha < k-1-r$, then f is a classical nearly holomorphic form.*

Proof. The part (i) is easy. If r is the degree of near overconvergence of f, then $g = \epsilon^r.f$ is a non trivial overconvergent form and by Proposition 3.3.7 its slope is $\alpha - r$. Since a slope has to be non-negative, (i) follows. The part (ii) is a straightforward generalization of the result for $r = 0$ which is a theorem of Coleman [Co96]. We may assume that r is the exact degree of near overconvergence. By the point (i), we therefore have $\alpha \geq r$. From the assumption, we deduce $k - 1 - r \geq r$. Therefore $k > 2r$ and we may apply Lemma 3.3.4:

$$f = \sum_{i=o}^{r} \delta_{k-2i}^i g_i$$

with g_i overconvergent of weight $k-2i$ for each i. By uniqueness of the g_i's, we see easily that that the $\delta_{k-2i}g_i$ are eigenforms for U_p with the same eigenvalue as f. So for each i, g_i is of slope $\alpha - i < k-1-r-i \leq (k-2i)-1$. Therefore it is classical by the theorem of Coleman. This implies f is classical nearly holomorphic.■

3.4. **Families of nearly overconvergent forms.**

3.4.1. *Weight space.* Let $\mathfrak{X}$ be the rigid analytic space over $\mathbf{Q}_p$ such that for any p-adic field $L \subset \overline{\mathbf{Q}}_p$ $\mathfrak{X}(L) = Hom_{cont}(\mathbf{Z}_p^\times, L^\times)$. Any integer $k \in \mathbf{Z}$ can be seen as the point $[k] \in \mathfrak{X}(\mathbf{Q}_p)$ defined as $[k](x) = x^k$ for all $x \in \mathbf{Z}_p^\times$. Recall we have the decomposition $\mathbf{Z}_p = \Delta \times 1 + q\mathbf{Z}_p$ where $\Delta \subset \mathbf{Z}_p^\times$ is the subgroup of roots of unity contained in $\mathbf{Z}_p^\times$ and $q = p$ if p odd and $q = 4$ if $p = 2$. We can decompose $\mathfrak{X}$ as a disjoint union

$$\mathfrak{X} = \bigsqcup_{\psi \in \hat{\Delta}} B_\psi$$

where $\hat{\Delta}$ is the set of characters of Δ and B_ψ is identified to the open unit disc of center 1 in $\overline{\mathbf{Q}}_p$. If $\kappa \in \mathfrak{X}(L)$ then it correspond to $u_\kappa \in B_\psi(L)$ if $\kappa|_\Delta = \psi$ and $\kappa(1+q) = u_\kappa$.

3.4.2. *Families.* Let $\mathfrak{U} \subset \mathfrak{X}$ be an affinoid subdomain. It is known from the works of Coleman-Mazur [CM98] and Pilloni [Pi10], that there exist $\rho_\mathfrak{U} \in [0,1) \cap p^{\mathbf{Q}}$ such that for all $\rho \geq \rho_\mathfrak{U}$, there exists an orthonomalizable Banach space $\mathcal{M}_\mathfrak{U}^\rho$ over $A(\mathfrak{U})$ such that for all $\kappa \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$, we have

$$\mathcal{M}_\mathfrak{U}^\rho \otimes_\kappa \overline{\mathbf{Q}}_p \cong H^0(X^{\geq\rho}, \omega_\kappa)$$

We consider the sheaf[7] $\Omega_\mathfrak{U}$ over $\mathfrak{U} \times X^{\geq\rho}$ associated to the $A(\mathfrak{U} \times X^{\geq\rho})$-module $\mathcal{M}_\mathfrak{U}^\rho$ and we put

$$\mathcal{H}_\mathfrak{U}^r := \Omega_\mathfrak{U} \otimes \mathcal{H}_0^r$$

For any weight $k \in \mathbf{Z}$ such that $[k] \in \mathfrak{U}(\mathbf{Q}_p)$, we recover $\mathcal{H}_k^r$ by the *pull-back*

$$([k] \times id_{X^{\geq\rho}})^* \mathcal{H}_\mathfrak{U}^r \cong \mathcal{H}_k^r \qquad \text{(3.4.2.a)}$$

[7] In [Pi10], this sheaf is constructed in a purely geometric way and the existence of $\mathcal{M}_\mathfrak{U}^\rho$ is deduced from it.

For general weight $\kappa \in \mathfrak{U}(L)$, we define the sheaf $\mathcal{H}^r_\kappa = \omega_\kappa \otimes \mathcal{H}^r_0$ where ω_κ is the invertible sheaf defined in [Pi10, §3]. We define the space of nearly overconvergent forms of weight κ by

$$\mathcal{N}^{r,\rho}_\kappa(N, L) := H^0(X^{\geq \rho}, \mathcal{H}^r_{\kappa/L})$$

and the space of $\mathfrak{U}$-families of nearly overconvergent forms:

$$\mathcal{N}^{r,\rho}_{\mathfrak{U}}(N) := H^0(X^{\geq \rho}, \mathcal{H}^r_{\mathfrak{U}})$$

We also define $\mathcal{N}^{r,\dagger}_\kappa(N)$ and $\mathcal{N}^{r,\dagger}_{\mathfrak{U}}(N)$ the spaces we obtain by taking the inductive limit over ρ. The space $\mathcal{N}^{r,\rho}_{\mathfrak{U}}(N)$ is a Banach module over $A(\mathfrak{U})$ and for any weight $\kappa \in \mathfrak{U}(L)$, we have

$$\mathcal{N}^{r,\rho}_{\mathfrak{U}}(N) \otimes_\kappa L = \mathcal{N}^{r,\rho}_\kappa(N, L)$$

This follows easily from (3.4.2.a) and the fact that $X^{\geq \rho}$ is an affinoid.

As in the previous section, one can define an action of U_p on these spaces and show it is completely continuous. For any integer r, any affinoid $\mathfrak{U} \subset \mathfrak{X}$ and $\rho \geq \rho_{\mathfrak{U}}$, we may consider the Fredholm determinant

$$P^r_{\mathfrak{U}}(X) = P^r_{\mathfrak{U}}(\kappa, X) := det(1 - X.U_p | \mathcal{N}^{r,\rho}_{\mathfrak{U}}) \in A(\mathfrak{U})[[X]]$$

because one can show as in [Pi10] that $\mathcal{N}^{r,\rho}_{\mathfrak{U}}(N)$ is $A(\mathfrak{U})$-projective. A standard argument shows this Fredholm determinant is independent of ρ. For $\mathfrak{U} = \{\kappa\}$, we just write $P^r_\kappa(X)$. If $r = 0$, we omit r from the notation.

For any integer m, we consider the map $[m] :\ \mathfrak{X} \to \mathfrak{X}$ defined by $\kappa \mapsto \kappa.[m]$ and we denote $\mathfrak{U}[m]$, the image of $\mathfrak{U}$ by this map. We easily see from its algebraic definition, that the operator ϵ can be defined in families and induces a short exact sequence:

$$0 \to \mathcal{M}^\rho_{\mathfrak{U}} \to \mathcal{N}^{r,\rho}_{\mathfrak{U}} \to \mathcal{N}^{r-1,\rho}_{\mathfrak{U}[-2]} \to 0$$

From Proposition 3.3.7 and the above exact sequence one easily sees by induction on r that

$$P^r_{\mathfrak{U}}(\kappa, X) = \prod_{i=0}^{r} P_{\mathfrak{U}[-2i]}(\kappa.[-2i], p^i.X)$$

Let us define $\widehat{\mathcal{N}^{\infty,\rho}_{\mathfrak{U}}}(N)$ as the p-adic completion of $\mathcal{N}^{\infty,\rho}_{\mathfrak{U}}(N) := \bigcup_{r \geq 0} \mathcal{N}^{r,\rho}_{\mathfrak{U}}(N)$. Then it follows from Lemma 3.3.10 again that U_p acts completely continuously on it with the Fredholm determinant given by the converging product

$$P^\infty_{\mathfrak{U}}(\kappa, X) = \prod_{i=0}^{\infty} P_{\mathfrak{U}[-2i]}(\kappa.[2i], p^i X)$$

Definition 3.4.3. *Let $\mathfrak{U} \subset \mathfrak{X}$ be an affinoid subdomain and $Q(X) \in A(\mathfrak{U})[X]$ be a polynomial of degree d such that $Q(0) = 1$. The pair $(Q, \mathfrak{U})$ is said admissible for nearly overconvergent forms (reps. for overconvergent forms) if there is a factorization*

$$P^\infty_{\mathfrak{U}}(X) = Q(X)R(X) \quad (resp. \quad P_{\mathfrak{U}}(X) = Q(X)R(X)\ )$$

with P and Q relatively prime and $Q^*(0) \in A(\mathfrak{U})^\times$ with $Q^*(X) := X^d Q(1/X)$.

If $(Q,\mathfrak{U})$ is admissible for nearly overconvergent forms, it results from Coleman-Riesz-Serre theory [Co] that there is a unique U_p-stable decomposition

$$\mathcal{N}_{\mathfrak{U}}^{\infty,\rho} = \mathcal{N}_{Q,\mathfrak{U}} \oplus \mathcal{S}_{Q,\mathfrak{U}}$$

such that $\mathcal{N}_{Q,\mathfrak{U}}$ is projective of finite rank over $A(\mathfrak{U})$ with

(i) $det(1-U_pX|\mathcal{N}_{Q,\mathfrak{U}}) = P(X)$
(ii) $Q^*(U_p)$ is invertible on $\mathcal{S}_{Q,\mathfrak{U}}$

It is worth noticing also that the projector $e_{Q,\mathfrak{U}}$ of $\mathcal{N}_{\mathfrak{U}}^{\infty,\rho}$ onto $\mathcal{N}_{Q,\mathfrak{U}}$ can be expressed as $S(U_p)$ for some entire power series $S(X) \in XA(\mathfrak{U})X$. If we have two admissible pairs $(Q,\mathfrak{U})$ and $(Q',\mathfrak{U}')$, we write $(Q,\mathfrak{U}) < (Q',\mathfrak{U}')$ if $\mathfrak{U} \subset \mathfrak{U}'$ and if Q divides the image $Q'|_{\mathfrak{U}}(X)$ of $Q'(X)$ by the canonical map $A(\mathfrak{U}')[X] \to A(\mathfrak{U})[X]$. When this happens, we easily see from the properties of the Riesz decomposition that

$$e_{Q,\mathfrak{U}} \circ (e_{Q',\mathfrak{U}'} \otimes_{A(\mathfrak{U}')} 1_{A(\mathfrak{U})}) = e_{Q,\mathfrak{U}} \tag{3.4.3.a}$$

We have dropped ρ from the notation in $\mathcal{N}_{Q,\mathfrak{U}}$ since this space is clearly independant of ρ by a standard argument. We define $\mathcal{N}_{\mathfrak{U}}^{fs}$ as the inductive limit of the $\mathcal{N}_{Q,\mathfrak{U}}$ over the Q's. Since $\mathcal{N}_{Q,\mathfrak{U}}$ is of finite rank and U_p-stable it is easy to see that there exists r such that

$$\mathcal{N}_{Q,\mathfrak{U}} \subset \mathcal{N}_{\mathfrak{U}}^{r,\dagger}$$

Remark 3.4.4. If $\alpha_{Q,\mathfrak{U}}$ is the maximal[8] valuation taken by the values of the analytic function $Q^*(0) \in A(\mathfrak{U})$ on $\mathfrak{U}$, then one can easily see that $r \leq \alpha_{Q,\mathfrak{U}}$ by the point (i) of Proposition 3.3.13.

3.4.5. *Families of q-expansions and polynomial q-expansions.* By evaluating at the Tate object we have defined in section 2, w e can define the polynomial q-expansion of an element $F \in \mathcal{N}_{\mathfrak{U}}^{r,\rho}(N)$ that we write $F(q,X) \in A(\mathfrak{U})[X]_r[[q]]$. The evaluation $F_\kappa(q,X)$ at κ of $F(q,X)$ is the polynomial q-expansion of the nearly overconvergent form of weight κ obtained by specializing F at κ. We also denote $F_\kappa(q) = F_\kappa(q,0)$ the p-adic q-expansion of the specialization of F at κ. In what follow, we show that when the slope is bounded a family of q-expansion of nearly overconvergent forms is equivalent to a family of nearly overconvergent forms.

Let $F(q) \in A(\mathfrak{U})[[q]]$ and $\Sigma \subset \mathfrak{U}(\overline{\mathbf{Q}}_p)$ a Zariski dense subset of points. We say that $F(q)$ is a Σ-family of q-expansions of nearly overconvergent form of type $(Q,\mathfrak{U})$ if for all but finitely many $\kappa \in \Sigma$ the evaluation $F_\kappa(q)$ of $F(q)$ at κ is the p-adic q-expansion of a nearly overconvergent form of weight κ and type Q_κ (i.e. is annihilated by $Q_\kappa^*(U_p)$). Let $\mathcal{N}_{Q,\mathfrak{U}}^{\Sigma}$ be the $A(\mathfrak{U})$-module of families of q-expansion of nearly overconvergent forms of type $(Q,\mathfrak{U})$. Similarly, we can define $\mathcal{N}_{Q,\mathfrak{U}}^{\Sigma,pol} \subset A(\mathfrak{U})[X][[q]]$ the subspace of polynomial q-expansion satisfying a smiler property for specialization at points in Σ with an obvious map:

$$\mathcal{N}_{Q,\mathfrak{U}}^{\Sigma,pol} \to \mathcal{N}_{Q,\mathfrak{U}}^{\Sigma}$$

[8] This maximum is $< \infty$ since $Q^*(0) \in A(\mathfrak{U})^\times$

given by the evaluation X at 0.

Then we have:

Lemma 3.4.6. *The q-expansion map and polynomial expansion maps induce the isomorphisms*

$$\mathcal{N}_{Q,\mathfrak{U}} \cong \mathcal{N}_{Q,\mathfrak{U}}^{\Sigma,pol} \cong \mathcal{N}_{Q,\mathfrak{U}}^{\Sigma}.$$

Proof. From Proposition 3.2.4, it suffices to show that the q-expansion map induces:

$$\mathcal{N}_{Q,\mathfrak{U}} \cong \mathcal{N}_{Q,\mathfrak{U}}^{\Sigma}.$$

The argument to prove this is well-known but we don't know a reference for it. We therefore sketch it below. Notice first that for any $\kappa_0 \in \mathfrak{U}(\overline{\mathbf{Q}}_p)$ the evaluation map at κ_0 induces an injective map:

$$\mathcal{N}_{Q,\mathfrak{U}}^{\Sigma} \otimes_{\kappa_0} \overline{\mathbf{Q}}_p \hookrightarrow \overline{\mathbf{Q}}_p[[q]] \tag{3.4.6.a}$$

Indeed if $F \in \mathcal{N}_{Q,\mathfrak{U}}^{\Sigma}$ is such that $F_{\kappa_0}(q) = 0$ then if $\varpi_{\kappa_0} \in A(\mathfrak{U})$ is a generator of the ideal of the elements of $A(\mathfrak{U})$ vanishing at κ_0, we have $F(q) = \varpi_{\kappa_0}.G(g)$ for some $G \in A(\mathfrak{U})[[q]]$. Clearly for any $\kappa \in \Sigma\backslash\{\kappa_0\}$, we have $G_\kappa(q) = \frac{1}{\varpi_{\kappa_0}(\kappa)} F_\kappa(q)$ is the q-expansion of a nearly over convergent for of weight κ and type Q_κ. Therefore $G \in \mathcal{N}_{Q,\mathfrak{U}}^{\Sigma}$ and our first claim is proved. Now let $\kappa \in \Sigma$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{N}_{Q,\mathfrak{U}} \otimes_\kappa \overline{\mathbf{Q}}_p \arrow[r, "(3)"] \arrow[d, "(1)"'] & \mathcal{N}_{Q,\mathfrak{U}}^{\Sigma} \otimes_\kappa \overline{\mathbf{Q}}_p \arrow[d, "(2)"] \\
\mathcal{N}_{Q_\kappa,\kappa}^{\Sigma} \arrow[r, "(4)"] & \overline{\mathbf{Q}}_p[[q]]
\end{tikzcd}$$

Since (2) and (4) are injectives and (1) is an isomorphism, we deduce (3) is injective. Now since the image of (2) is included in the image of (4) and (1) is surjective, we deduce that (3) is an isomorphism of finite vector spaces. Since $\mathcal{N}_{Q,\mathfrak{U}}^{\Sigma}$ is torsion free over $A(\mathfrak{U})$ such that $\mathcal{N}_{Q,\mathfrak{U}}^{\Sigma} \otimes_\kappa \overline{\mathbf{Q}}_p$ has bounded dimension when κ runs in Σ, a standard argument shows that $\mathcal{N}_{Q,\mathfrak{U}}^{\Sigma}$ is of finite type over $A(\mathfrak{U})$ (see for instance [Wi, §1.2]). Notice that the injectivity of (3) below is true for all κ and therefore we deduce that the map

$$\mathcal{N}_{Q,\mathfrak{U}} \to \mathcal{N}_{Q,\mathfrak{U}}^{\Sigma} \tag{3.4.6.b}$$

is injective with a torsion cokernel of finite type. We want now to prove the surjectivity. Let $F(q) \in \mathcal{N}_{Q,\mathfrak{U}}^{\Sigma}$ and let $I_F \subset A(\mathfrak{U})$ be the ideal of element a such that $a.F(q)$ is in the image of (3.4.6.b) and let a_F be a generator of I_F. Let $G \in \mathcal{N}_{Q,\mathfrak{U}}$ whose image is $a_F.F(q)$. For any κ_0 such that $a_F(\kappa_0) = 0$, we get that $G_{\kappa_0} = 0$. By the isomorphism (1), G is therefore divisible by ϖ_{κ_0} and thus $\frac{a_F}{\varpi_{\kappa_0}} \in I_F$ which contradicts the fact that a_F is a generator of I_F. Therefore a_F does not vanish on $\mathfrak{U}$ and F is in the image of (3.4.6.b). This proves the surjectivity we have claimed.∎

More generally, for any $\mathbf{Q}_p$-Banach space M, we can define $\mathcal{N}^{\Sigma}_{Q,\mathfrak{U}}(M)$ the subspace of elements $F \in A(\mathfrak{U})\hat{\otimes}M[[q]]$ such that for almost all $\kappa \in \Sigma$, the evaluation F_κ at κ of $F(q)$ is the q-expansion of an element of $\mathcal{N}_{Q_\kappa,\kappa}(M) = \mathcal{N}_{Q_\kappa,\kappa} \otimes M$. Similarly, one defines $\mathcal{N}^{\Sigma,pol}_{Q,\mathfrak{U}}(M)$. Then it is easy to deduce the following:

Corollary 3.4.7. *We have the isomorphisms:*

$$\mathcal{N}_{Q,\mathfrak{U}}\hat{\otimes}M \cong \mathcal{N}^{\Sigma,pol}_{Q,\mathfrak{U}}(M) \cong \mathcal{N}^{\Sigma}_{Q,\mathfrak{U}}(M).$$

Proof. Left to the reader.∎

3.5. **Maass-Shimura operator and overconvergent projection in p-adic families.** The formula for the action of the Maass-Shimura operators on the q-expansion suggests it behaves well in families. We explain this here using the lemma 3.4.6.b. This could be avoided but it would take more time than we want to devote to this here. We explain this in a remark below.

3.5.1. We defined the analytic function $Log(\kappa)$ on $\mathfrak{X}$ by the formula:

$$Log(\kappa) = \frac{log_p(\kappa(1+q)^t)}{log_p((1+q)^t)}$$

where log_p is the p-adic logarithm defined by the usual Taylor expansion $log_p(x) = -\sum_{n=1}^{\infty}\frac{(1-x)^n}{n}$ for all $x \in \mathbf{C}_p$ such thar $|x-1|_p \leq p^{-1}$ and t is an integer greater than $1/v_p(\kappa(1+q))$. Of course, from the definition we have $Log([k]) = k$. Moreover Log is clearly an analytic function on $\mathfrak{X}$.

3.5.2. If $F(q,X) = \sum_{n=0}^{\infty} a_n(X,F)q^n \in A(\mathfrak{U})[X]_r[[q]]$, we define

$$\delta F(q,X) := D.F(q,X) + Log(\kappa)XF(q,X)$$

where

$$D = q\frac{\partial}{\partial q} - X^2\frac{\partial}{\partial X}.$$

If $F(q,X) \in \mathcal{N}^{\Sigma,pol}_{Q,\mathfrak{U}}$ for a Zariski-dense set of classical weight Σ, it is clear that $\delta.F(q,X) \in \mathcal{N}^{\tilde{\Sigma},pol}_{\tilde{Q},\mathfrak{U}}$ with $\tilde{\Sigma} = \Sigma[2]$ and $\tilde{Q}(\kappa, X) = Q(\kappa.[-2], pX)$. We therefore thanks to Lemma 3.4.6 deduce we have a map

$$\delta : \mathcal{N}^{\dagger,r,\mathrm{fs}}_{\mathfrak{U}} \to \mathcal{N}^{\dagger,r+1,\mathrm{fs}}_{\mathfrak{U}[2]}$$

Remark also, it is straightforward to see that the effect of the operator ϵ on the polynomial q-expansion of families is the partial differentiation with respect to X:

$$(\epsilon.F)(q,X) = \frac{\partial}{\partial X}F(q,X) \quad \forall F \in \mathcal{N}^{\dagger,r,\mathrm{fs}}_{\mathfrak{U}}$$

Remark 3.5.3. Like in Remark 3.3.11, we can give a sheaf theoretic definition of $\hat{\mathcal{N}}_{\mathfrak{U}}^{\infty,\rho}$. For simplicity, let us assume that all the p-adic characters in $\mathfrak{U}$ are analytic on $\mathbf{Z}_p$. Let $\mathcal{A}_{\mathfrak{U}} := A(\mathfrak{U})\hat{\otimes}\mathcal{A}$. Elements in $\mathcal{A}_{\mathfrak{U}}$, can be seen as rigid analytic functions on $\mathfrak{U}\times\mathbf{Z}_p$. It is equipped with the representation of the standard Iwahori subgroup of $SL_2(\mathbf{Z}_p)$ defined by:

$$((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}).f)(\kappa, X) = \kappa(a+cX)f(\frac{b+dX}{a+cX})$$

Again as in remark 3.3.11, one can define but now using the technics of [Pi10] a sheaf of Banach spaces on $X^{\geq\rho}\times\mathfrak{U}$

$$\mathcal{H}_{\mathfrak{U}}^{\infty} := \mathcal{T}\times^{B}\mathcal{A}_{\mathfrak{U}} \cong \omega_{\mathfrak{U}}\otimes\mathcal{H}_0^{\infty}$$

and show that we have:

$$\hat{\mathcal{N}}_{\mathfrak{U}}^{\infty,\rho} = H^0(X^{\geq\rho}\times\mathfrak{U}, \mathcal{H}_{\mathfrak{U}}^{\infty})$$

Since $\mathcal{A}_{\mathfrak{U}}$ is a representation of the Lie algebra of sl_2, it would be possible to define a connection using the BGG formalism like in the algebraic case (see for instance [Ti11, §3.2])

$$\mathcal{H}_{\mathfrak{U}}^{\infty} \to \mathcal{H}_{\mathfrak{U}[2]}^{\infty}$$

One would then obtain the Maass-Shimura operator in family without the finite slope condition:

$$\delta : \hat{\mathcal{N}}_{\mathfrak{U}}^{\rho,\infty} \to \hat{\mathcal{N}}_{\mathfrak{U}[2]}^{\rho,\infty}$$

It is then easy to verify that $\delta(\mathcal{N}_{\mathfrak{U}}^{\rho,r}) \subset \mathcal{N}_{\mathfrak{U}[2]}^{\rho,r+1}$. We leave the details of this construction for another occasion or to the interested reader.

Finally, we want to mention that Robert Harron and Liang Xiao [Xi] have also given a geometric construction of this operator in family using a splitting of the Hodge filtration and showing the definition is independent of the chosen splitting. The above sketched construction can be done without such a choice but it probably boils down to a similar argument.

Now we have the following proposition.

Proposition 3.5.4. *Let $\mathfrak{U}\subset\mathfrak{X}$ be an open affinoid subdomain and $F\in\mathcal{N}_{\mathfrak{U}}^{\dagger,r,\mathrm{f}s}$, then for each $i\in\{0,\dots,r\}$ there exists*

$$G_i \in \frac{1}{\prod_{j=2}^{2r}(Log(\kappa)-j)}\mathcal{N}_{\mathfrak{U}[-2i]}^{\dagger,0,\mathrm{f}s}$$

such that

$$F = G_0 + \delta.G_1 + \cdots + \delta^r.G_r$$

Moreover, this decomposition is unique.

If $\mathfrak{U}=\{\kappa\}$ such that $Log(\kappa)\notin\{2,3,\dots 2r\}$, the result holds as well.

Proof. It is sufficient to prove this when $\mathfrak{U}$ is open since we can obtain the general result after specialization. We prove this by induction on r. Notice that for $G \in \mathcal{N}^{\dagger,0,\mathrm{f}s}_{\mathfrak{U}[-2r]}$, we have by (2.4.2.a)

$$\epsilon^r.\delta^r.G = r!.\prod_{i=1}^{r}(Log(\kappa) - r - i).G$$

since the left hand and right hand sides coincide after evaluation at classical weights bigger than $2r$. We put

$$G_r := \frac{1}{r!.\prod_{i=1}^{r}(Log(\kappa) - r - i)}\epsilon^r F$$

then $G_r \in \frac{1}{\prod_{i=1}^{r}(Log(\kappa)-r-i)}\mathcal{N}^{\dagger,0,\mathrm{f}s}_{\mathfrak{U}[-2r]}$ and $F - \delta^r.G_r$ is by construction of degree of nearly overconvergent less or equal to $r-1$. We conclude by induction. ∎

Then we define

$$\mathcal{H}^\dagger(F) := G_0$$

This is the overconvergent (or holomorphic) projection in family since it clearly coincides with the holomorphic projection for nearly holomorphic forms of weight $k > 2r$.

Lemma 3.5.5. *For any nearly overconvergent family of finite slope $F \in \mathcal{N}^{\dagger,r,\mathrm{f}s}_{\mathfrak{U}}$, and Heke operator T, we have*

$$\mathcal{H}^\dagger(T.F) = T.\mathcal{H}^\dagger(F).$$

In particular, for any admissible pair $(Q, \mathfrak{U})$ and we have

$$e_{Q,\mathfrak{U}}(\mathcal{H}^\dagger(F)) = \mathcal{H}^\dagger(e_{Q,\mathfrak{U}}(F)).$$

Proof. It follows easily from the relation $\delta^j(T(n).F) = n^jT(n).\delta^j(F)$ and the uniqueness of the G_i's in the decomposition of Proposition 2.4.2.a.∎

4. Application to Rankin-Selberg p-adic L-functions

Let $\mathcal{E}$ be the eigencurve of level 1 constructed by Coleman and Mazur. in [CM98]. In this section, we give the main lines of a construction of a p-adic L-function on $\mathcal{E} \times \mathcal{E} \times \mathfrak{X}$. The general case of arbitrary tame level can be done exactly the same way. The restriction of our p-adic L-function to the ordinary part of the eigencurve, gives Hida's p-adic L-function constructed in [Hi88] and [Hi93]. Our method follows closely Hida's construction for ordinary families of eigenforms. We are able to treat the general case using the framework of nearly overconvergent forms. We will omit the details of computation that are similar to Hida's construction and will focus on how we get rid of the ordinary assumptions. We don't pretend to any originality here. We just want to give an illustration of the theory of nearly overconvergent forms to the construction of p-adic L-functions in the non-ordinary case.

4.1. **Review on Rankin-Selberg L-function for elliptic modular forms.** We recall the definition and integral representations of the Rankin-Selberg L-function of two elliptic modular forms and its critical values. Let f and g be two elliptic normalized newforms of weights k and l with $k > l$ and nebentypus ψ and ξ respectively of level M. We denote their Fourier expansion by:

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n q^n$$

Shimura is probably the first one to study in [Sh76] the algebraicities of the critical values of

$$D_M(s,f,g) := L(\psi\xi, k+l-2s-2)(\sum_{n=1}^{\infty} a_n b_n n^{-s})$$

More precisely he proved that for every integer $m \in \{0,\dots,k-l-1\}$, then

$$\frac{D(l+m,f,g)}{\pi^{l+2m+1}\langle f,f\rangle_M} \in \overline{\mathbf{Q}}$$

Here $\langle f,f\rangle_M$ is the Petersson inner product of f with itself. Recall it is defined by the formula

$$\langle f,g\rangle_M = \int_{\Gamma_1(M)\backslash\mathfrak{h}} \overline{f(\tau)} g(\tau) y^{k-2} dx dy$$

When $0 \leq 2m < k-l$, the essential ingredient in the proof of Shimura was to establish a formula of the type

$$D_M(l+m,f,g) = \langle f, g\delta^m_{k-l-2m}E\rangle_M = \langle f, \mathcal{H}(g\delta^m_{k-l-2m}E)\rangle_M$$

where E is a suitable holomorphic Eisenstein series of weight $k-l-2m$.

When f and g vary in Hida families and m is also allowed to vary p-adically, Hida has constructed a 3-variable p-adic L-function interpolating a suitable p-normalization these numbers. We now recall the precise formula that is used to interpolate these special values in [Hi88]. We first need some standard notations. For any integer M, we put $\tau_M = \left(\begin{smallmatrix} 0 & -1 \\ M & 0 \end{smallmatrix}\right)$ and for any modular form h, we dente by h^ρ the form defined by $h^\rho(\tau) = \overline{h(-\bar{\tau})}$ for $\tau \in \mathfrak{h}$. For any Dirichlet character χ of level M and any integer $j \geq 2$ such that $\chi(-1) = (-1)^j$, we denote by $E_j(\chi)$ the Eisenstein series of level M, nebentypus χ and weight j whose q-expansion is given by

$$E_j(\chi)(\tau) = \frac{L(1-m,\chi)}{2} + \sum_{n=1}^{\infty}(\sum_{\substack{d|n \\ (d,M)=1}} \chi(d)d^{j-1})q^n$$

Proposition 4.1.1. [Hi88, Thm 6.6] *Let L be an integer such that f and g are of level Lp^β, then we have*

$$D_{Lp}(l+m,f,g) = t.\pi^{l+2m+1}\langle f^\rho|_k\tau_{Lp^\beta}, \mathcal{H}(g|_l\tau_{Lp^\beta}\delta^m_{k-l-2m}(E_{k-l-2m,Lp}(\psi\xi)\rangle_{Lp^\beta}$$

with
$$t = \frac{2^{k+l+2m}(Lp^\beta)^{\frac{k-l}{2}-m-1}i^{l-k}}{m!(l+m-1)!}.$$
and $m \in \mathbf{Z}$ *with* $0 \leq m < (k-l)/2$.

4.2. **The p-adic Petersson inner product.**

4.2.1. For simplicity, we assume the tame level is 1. Fix and admissible pair $(R, \mathfrak{V})$ for overconvergent forms and let $\mathcal{M}_{R,\mathfrak{V}}$ the corresponding associated space of $\mathfrak{V}$-families of overconvergent forms. Let $\mathbf{T}_{R,\mathfrak{V}}$ the Hecke algebra acting on $\mathcal{M}_{R,\mathfrak{V}}$. A standard argument using the q-expansion principle shows that the pairing
$$\mathcal{M}_{R,\mathfrak{V}} \otimes_{A(\mathfrak{U})} \mathbf{T}_{R,\mathfrak{V}} \to A(\mathfrak{V})$$
given by
$$(T, f) := a(1, f|T)$$
is a perfect duality. Since the level[9] is 1, we also know that $\mathbf{T}_{R,\mathfrak{V}}$ is reduced. Therefore, the trace map induces a non-degenerate pairing on $\mathbf{T}_{Q,\mathfrak{U}}$ with ideal discriminant $\mathfrak{d}_{R,\mathfrak{V}} \subset A(\mathfrak{V})$ whose set of zeros is the set of weight where the map $\mathcal{E}_{Q,\mathfrak{U}} \to \mathfrak{U}$ is ramified. In particular, we have a canonical isomorphism:
$$\mathcal{M}_{R,\mathfrak{V}} \otimes F(\mathfrak{V}) \cong T_{R,\mathfrak{V}} \otimes F(\mathfrak{V}) \tag{4.2.1.a}$$

4.2.2. From this, we deduce a Hecke-equivariant pairing
$$(-,-)_{R,\mathfrak{V}} : \ \mathcal{M}_{R,\mathfrak{V}} \otimes_{A(\mathfrak{V})} \otimes \mathcal{M}_{R,\mathfrak{V}} \longrightarrow F(\mathfrak{V})$$
Let now $\mathcal{F}$ be a Galois extension of $F(\mathfrak{V})$ the field of fraction of $A(\mathfrak{V})$. We assume that for each irreducible component $\mathcal{C}$ of $\mathcal{E}_{R,\mathfrak{V}}$, $\mathcal{F}$ contains the function field $F(\mathcal{C})$ of $\mathcal{C}$. For each irreducible component $\mathcal{C}$, we define the corresponding idempotent $1_\mathcal{C} \in T_{R,\mathfrak{V}} \otimes \mathcal{F}$ and we write $F_\mathcal{C}$ for the element defined by
$$F_\mathcal{C} := \sum_{n=1}^\infty \lambda_\mathcal{C}(T_n) q^n \in \mathcal{F}[[q]]$$
where $\lambda_\mathcal{C}$ is the character of the Hecke algebra defined by $T.1_\mathcal{C} \otimes id_\mathcal{F} = 1_\mathcal{C} \otimes \lambda_\mathcal{C}(T)$. If we denote by $(-,-)_\mathcal{F}$ the scalar extension of $(-,-)_{R,\mathfrak{V}}$ to $\mathcal{F}$ then the Hecke invariance of the inner product implies that
$$a(1, 1_\mathcal{C}.G) = (F_\mathcal{C}, G)_\mathcal{F}$$
is the coefficient of $F_\mathcal{C}$ when one writes G as a linear decomposition of the eigen families $F_\mathcal{C}$'s.

Remark 4.2.3. This construction can be easily extended to the space $\mathcal{N}_{R,\mathfrak{V}}$ for any admissible psi $(R, \mathfrak{V})$ for nearly overconvergent forms.

[9] When the level is not 1, one uses the theory of primitive forms which described the maximal semi-simple direct factor of $T_{R,\mathfrak{V}} \otimes F(\mathfrak{V})$

4.3. **The nearly overconvergent Eisenstein family.** We consider the Eisenstein family $E(q) \in A^0(\mathfrak{X})[[q]]$ such that for each weight $\kappa \in \mathfrak{X}(\overline{\mathbf{Q}}_p)$, its evaluation at κ is given by

$$E(\kappa, q) = \sum_{n=1}^{\infty} a(n, E, \kappa) q^n := \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \sum_{d|n} \langle d \rangle_\kappa . d^{-1} q^n.$$

where for any $m \in \mathbf{Z}_p^\times$, $\langle m \rangle_\kappa \in A(\mathfrak{X})$ stands for the analytic function of $\mathfrak{X}$ defined by

$$\kappa \mapsto \kappa(m)$$

In particular, when $\kappa = [k].\psi$ with ψ a ramified finite order character of $\mathbf{Z}_p^\times$, then $E(\kappa, q)$ is the q-expansion of

$$E_k^{(p)}(\psi)(\tau) := E_k(\psi)(\tau) - E_k(\psi)(p\tau).$$

We define the nearly overconvergent Eisenstein family $\Theta.E \in A^0(\mathfrak{X} \times \mathfrak{X})[[q]]$ by

$$\Theta.E(\kappa, \kappa') := \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \langle n \rangle_\kappa a(n, E, \kappa') q^n$$

Lemma 4.3.1. *If $\kappa = [r]$ and $\kappa' = [k]\psi$, the evaluation at (κ, κ') of $\Theta.E$ is*

$$\Theta.E(\kappa, \kappa') = \Theta^r . E_k^{(p)}(\psi)(q).$$

It is the p-adic q-expansion of the nearly holomorphic Eisenstein series $\delta_k^r E_k^{(p)}(\psi)$.

Proof. The first part is obvious and the second part follows from the formula (2.4.2) and the canonical diagram of Proposition 3.2.4.∎

4.4. **Construction of the Rankin-Selberg L-function on $\mathcal{E} \times \mathcal{E} \times \mathfrak{X}$.**

4.4.1. *Some preparation.* Let $(Q, \mathfrak{U})$ be an admissible pair for overconvergent forms of tame level 1 and let $T_{Q,\mathfrak{U}}$ be the corresponding Hecke algebra over $A(\mathfrak{U})$. By definition it is the ring of analytic function on the affinoid subdomain $\mathcal{E}_{Q,\mathfrak{U}}$ sitting over the affinoid subdomain $Z_{Q,\mathfrak{U}}$ associated to $(Q, \mathfrak{U})$ of the spectral curve of the U_p-operator. Recall that

$$Z_{Q,\mathfrak{U}} = Max(A(\mathfrak{U})[X]/Q^*(X)) \subset Z_{U_p} \subset \mathbf{A}^1_{\mathbf{r}ig} \times \mathfrak{U}$$

where Z_{U_p} is the spectral curve attache to U_p (i.e. the set of points $(\alpha, \kappa) \in \mathbf{A}^1_{\mathbf{r}ig} \times \mathfrak{U}$ such that $P_\kappa^0(\alpha) = 0$) and

$$T_{Q,\mathfrak{U}} = A(\mathcal{E}_{Q,\mathfrak{U}}) \text{ with } \mathcal{E}_{Q,\mathfrak{U}} = \mathcal{E} \otimes_{Z_{U_p}} Z_{Q,\mathfrak{U}}$$

The universal family of overconvergent modular eigenforms of type $(Q, \mathfrak{U})$ is given by

$$G_{Q,\mathfrak{U}} := \sum_{n=1}^{\infty} T(n) q^n \in T_{Q,\mathfrak{U}}[[q]]$$

Tautologically, for any point $x \in \mathcal{E}_{Q,\mathfrak{U}}$ of weight $\kappa_x \in \mathfrak{U}$, the evaluation $G_{Q,\mathfrak{U}}(x)$ at x of $G_{Q,\mathfrak{U}}$ is the overconvergent normalized eigenform f_x of weight κ_x associated to x.

Let

$$G^E_{Q,\mathfrak{U}} := G_{Q,\mathfrak{U}}.\Theta.E \in T_{Q,\mathfrak{U}} \otimes A^b(\mathfrak{X}\times\mathfrak{X})[[q]] = A^b(\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X}\times\mathfrak{X})[[q]]$$

The Fourier coefficients of this Fourier expansions are analytic functions on $\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X}\times\mathfrak{X}$. Let now $(R,\mathfrak{V})$ be an admissible pair for nearly overconvergent forms of tame level 1. Then we consider

$$G^E_{Q,\mathfrak{U},R,\mathfrak{V}}(q) \in A^b(\mathfrak{V}\times\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X})[[q]]$$

defined by

$$G^E_{Q,\mathfrak{U},R,\mathfrak{V}}(\kappa,y,\nu)(q) := e_{R,\mathfrak{V}}.G^E_{Q,\mathfrak{U}}(y,\nu,\kappa\kappa_y^{-1}\nu^{-2})(q)$$

and where $e_{R,\mathfrak{V}} = S(U_p)$ for some $S\in X.A(\mathfrak{V})[[X]]$ is the projector of $\mathcal{N}^{\infty,\mathrm{f}s}_{\mathfrak{V}}$ onto $\mathcal{N}_{R,\mathfrak{V}}$.

Proposition 4.4.2. *With the notation above $G^E_{Q,\mathfrak{U},R,\mathfrak{V}}(q)$ is the q-expansion of an element of $G^E_{Q,\mathfrak{U},R,\mathfrak{V}} \in \mathcal{N}_{R,\mathfrak{V}}\otimes_{\mathbf{Q}_p} A^b(\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X})$. Moreover if we have $(Q,\mathfrak{U}) < (Q',\mathfrak{U}')$ and $(R,\mathfrak{V}) < (R'\mathfrak{V}')$, then $G^E_{Q,\mathfrak{U},R,\mathfrak{V}}$ is the image of $G^E_{Q',\mathfrak{U}',R',\mathfrak{V}'}$ by the natural map*

$$\mathcal{N}_{R',\mathfrak{V}'}\otimes_{\mathbf{Q}_p} A^b(\mathcal{E}_{Q',\mathfrak{U}'}\times\mathfrak{X}) \to \mathcal{N}_{R,\mathfrak{V}}\otimes_{\mathbf{Q}_p} A^b(\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X})$$

Proof. By Corollary 3.4.7, with with $(Q,\mathfrak{U})$ replaced by $(R,\mathfrak{V})$ and with $M = A^b(\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X})$, it is sufficient to show that the specialization at a Zariski dense set of arithmetic points of $([k],x,[r])\in\mathfrak{V}\times\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X}$ is the q-expansion of a nearly holomorphic form of weight k annihilated by $R^*_{[k]}(U_p)$. It is sufficient to choose the triplet $([k],x,[r])$ such that $\kappa_x = [l]$ with $l\in\mathbf{Z}_{\geq 2}$, $r\geq 0$ such that $k-l-2r\geq 0$ since such triplets form a Zariski dense set of $\mathfrak{V}\times\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X}$. The evaluation at such a triplet is easily seen to be the p-adic q-expansion of $e_{R,k}(g_x.\Theta^r.E_{k-l-2r})$. By definition of $e_{R,k}$ it follows that this form belongs to $\mathcal{N}_{R,k}$. The second part of the proposition is a trivial consequence of (3.4.3.a). ∎

4.4.3. *A 3-variable p-adic meromorphic function.* Let $Z_{R,\mathfrak{V}}\subset A^1_{\mathrm{rig}}\times\mathfrak{V}$ the affinoid of the spectral curve $Z^\infty_{U_p}$ attached to $P^\infty_{\mathfrak{V}}$. This affinoid is a priori not contained in the spectral curve attached to U_p but the eigencurve is still sitting over it since $Z_{U_p}\subset Z^\infty_{U_p}$. We can therefore consider

$$\mathcal{E}_{R,\mathfrak{V}} = \mathcal{E}_N\times_{Z^\infty_{U-p}} Z_{R,\mathfrak{V}}$$

and the $\mathcal{E}_{R,\mathfrak{V}}$'s form an admissible covering of $\mathcal{E}_N$ when the $(R,\mathfrak{V})$ vary.

Let $\mathcal{C}\subset\mathcal{E}_{R,\mathfrak{V}}$ be an irreducible component. Then we set

$$D_{\mathcal{C},Q,\mathfrak{U}} := a(1,1_{\mathcal{C}}.\mathcal{H}^\dagger(G^E_{Q,\mathfrak{U},R,\mathfrak{V}}))\in F(\mathcal{C})\otimes F(\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X})$$

Remark 4.4.4. If $H_{\mathcal{C}}\subset A(\mathcal{C})$ is a denominator of $1_{\mathcal{C}}$, then the poles of $D_{\mathcal{C},Q,\mathfrak{U}}$ come from the zeros of $H_{\mathcal{C}}$ and the poles of the overconvergent projector. Therefore we have:

$$H_{\mathcal{C}}.\prod_{i=2}^{2r_{R,\mathfrak{V}}}(Log(\kappa)-i).D_{\mathcal{C},Q,\mathfrak{U}}\in A(\mathcal{C}\times\mathcal{E}_{Q,\mathfrak{U}}\times\mathfrak{X}) \tag{4.4.4.a}$$

We denote by $D_{R,\mathfrak{V},Q,\mathfrak{U}}$ the unique element of

$$F(\mathcal{E}_{R,\mathfrak{V}} \times \mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X}) = \prod_{\substack{\mathcal{C} \subset \mathcal{E}_{R,\mathfrak{V}} \\ irreducible}} F(\mathcal{C}) \otimes F(\mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X})$$

restricting to $D_{\mathcal{C},Q,\mathfrak{U}}$ on $\mathcal{C} \times \mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X}$ for each irreducible component $\mathcal{C}$ of $\mathcal{E}_{R,\mathfrak{V}}$. It can be constructed as the image of $\mathcal{H}^\dagger(G^E_{Q,\mathfrak{U},R,\mathfrak{V}})$ in $T_{R,\mathfrak{V}} \otimes F(\mathfrak{V}) \otimes F(\mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X}) = F(\mathcal{E}_{R,\mathfrak{V}} \times \mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X})$ by the map (4.2.1.a) tensored by $F(\mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X})$

We have the following result:

Lemma 4.4.5. *There exist a meromorphic function $\mathcal{D}$ on $\mathcal{E} \times \mathcal{E} \times \mathfrak{X}$ whose restriction to $F(\mathcal{E}_{R,\mathfrak{V}} \times \mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X})$ gives $D_{R,\mathfrak{V},Q,\mathfrak{U}}$ for any quadruplet $(R, \mathfrak{V}, Q, \mathfrak{U})$.*

Proof. If we have pairs $(R, \mathfrak{V}), (Q, \mathfrak{U}), (R', \mathfrak{V}'), (Q', \mathfrak{U})$ with $(R, \mathfrak{V}) < (R', \mathfrak{V}')$ and $(Q, \mathfrak{U}) < (Q', \mathfrak{U}')$ then we have by (3.4.3.a) $e_{R',\mathfrak{V}'}|_{\mathfrak{V}} \circ e_{R,\mathfrak{V}} = e_{R,\mathfrak{V}}$ and $e_{Q',\mathfrak{U}'}|_{\mathfrak{U}} \circ e_{Q,\mathfrak{U}} = e_{Q,\mathfrak{U}}$. Since the overconvergent projection is Hecke equivariant, we deduce that

$$e_{R,\mathfrak{V}} \otimes e_{Q,\mathfrak{V}}.(D_{R',\mathfrak{V}',Q',\mathfrak{U}'}|_{\mathfrak{V}\times\mathfrak{U}}) = D_{R,\mathfrak{V},Q,\mathfrak{U}}$$

and that the $D_{R,\mathfrak{V},Q,\mathfrak{U}}$'s glue to define a memormorphic function $\mathcal{D} \in F(\mathcal{E} \times \mathcal{E} \times \mathfrak{X})$.

4.4.6. *The interpolation property.* For $x \in \mathcal{E}(\overline{\mathbf{Q}}_p)$, we denote θ_x the corresponding character of the Hecke algebra. If x is attached to a classical form, we denote by f_x the eigenform attached to x. By definition,

$$\iota_p \circ \iota_\infty^{-1}(f_x(q)) = \sum_{n=1}^{\infty} \theta_x(T_n) q^n$$

We denote by k_x its weight, ψ_x its nebentypus and p^{m_x} its minimal level with m_x a positive integer. We will always assume $k_x \geq 2$ and that p^{m_x} is the conductor of ψ_x. We consider the complex number $W(f_x)$ defined by

$$h_x := f_x^\rho | \tau_{p^{m_x}} = W(f_x) f_x$$

It is a complex number of norm 1 called the root number of f_x.

If $\mathcal{E}$ is smooth at x, then there is only one irreducible component containing it and if $\mathcal{C}$ is the irreducible component of an affinoid $\mathcal{E}_{R,\mathfrak{V}}$ containing x, then

$$H_{\mathcal{C}}(\kappa_x) \neq 0 \tag{4.4.6.a}$$

In that case, we can define the specialization $1_x \in T_{R_{\kappa_x},\kappa_x}$ of $1_{\mathcal{C}}$ at κ_x and it satisfies:

$$T_n.1_x = \theta_x(T_n).1_x \quad \forall n$$

In general, $T_{R_{\kappa_x},\kappa_x}$ is not semi-simple so 1_x is not necessarily the (generalized) θ_x-eigenspace projector. But if the projection map $\mathcal{E} \to \mathfrak{X}$ is étale at x then it is. We know it is the case when x is non-critical; recall that x is said non-critical if

$$v_p(\theta_x(U_p)) < k_x - 1.$$

If all the slopes of R_{κ_x} are strictly less that $k_x - 1$ (something that we can assume after shrinking $\mathcal{E}_{R,\mathfrak{V}}$), the image of $1_{\mathcal{C}}$ into the Hecke algebra acting on the space of forms

$\mathcal{M}_{R_{\kappa_x},\kappa_x}$ is the projector 1_{f_x} attached to the new form f_x. Moreover we can show by the same computations as [Hi85, sect. 4] that

$$\text{(4.4.6.b)} \qquad a(1,1_{f_x}.g) = a(p,f_x)^{m_x-n}.p^{(n-m_x)(k_x/2-1)}\frac{\langle f_x^\rho|\tau_{p^n},g\rangle_{p^n}}{\langle h_x,f_x\rangle_{p^{m_x}}}$$

for any $g \in \mathcal{M}_{R_{\kappa_x},\kappa_x}$ of level p^n with $n \geq m_x$.

For any $\nu \in \mathfrak{X}(\overline{\mathbf{Q}}_p)$, we write $k_\nu := \log(\nu)$. We say ν is arithmetic if $k_\nu \in \mathbf{Z}$ and we denote ψ_ν the finite order character such that $\nu = [k_\nu].\psi_\nu$.

We have the following theorem.

Theorem 4.4.7. *Let L be a finite extension of $\overline{\mathbf{Q}}_p$ and $(x,y) \in \mathcal{E}\times\mathcal{E}(L)$ and any arithmetic $\nu = [r].\psi_\nu$ such that κ_x, κ_y are arithmetic and satisfy the following*

(i) *$k_x - k_y > r \geq 0$,*
(ii) *x is classical and non-critical,*
(iii) *y is classical,*
(iv) *The level of f_y^ρ equals the level of $f_y^\rho|\psi_\nu$*
(v) *ψ_x and ψ_y are ramified.*

Then we have

$$\text{(4.4.7.a)} \quad \mathcal{D}(x,y,\nu) = (-1)^{k_y}W(f_x)W(f_y^\rho)a(p,f_x)^{m_x-m_y}p^{m_x(1-\frac{k_x}{2})+m_y(r+\frac{k_y}{2})} \times\Gamma(k_y+r)\Gamma(k_x+1)\frac{D_{p^n}(f_x,f_y^\rho|\psi_\nu,k_y+r)}{(2i\pi)^{k_x+k_y+2r+1}\pi^{1-k_x}\langle f_x,f_x\rangle_{p^{m_x}}}$$

Proof. This computation follows closely those of Hida in [Hi93]. We treat the case $k_x - k_y > 2r \geq 0$. The case $k_x - k_y > r \geq (k_x-k_y)/2$ can be treated similarly (see for instance [Hi93]) and is obtained using the functional equation for the nearly holomorphic Eisenstein series. We also assume ψ_ν is trivial to lighten the notations.

By our hypothesis, we can choose a quadruplet $(R,\mathfrak{V},Q,\mathfrak{U})$ such that

(a) $(x,y) \in \mathcal{E}_{R,\mathfrak{V}}\times\mathcal{E}_{Q,\mathfrak{U}}(L)$
(b) The eigenvalues of $R^*_{\kappa_x}(X) \in L[X]$ are of valuation smaller than $k_x - 1$

Then, $\mathcal{D}(x,y,[r]) = D_{R,\mathfrak{V},Q,\mathfrak{U}}(x,y,[r])$. By the condition (b), we know that $T_{R_{\kappa_x},\kappa_x} = T_{R,\mathfrak{V}}\otimes_{\kappa_x} L$ is semi-simple and therefore the map $\mathcal{E}_{R,\mathfrak{V}}\to\mathfrak{V}$ is étale at x. In particular, $\mathcal{E}_{R,\mathfrak{V}}$ is smooth at x and x belongs to only one irreducible component $\mathcal{C}$ of $\mathcal{E}_{R,\mathfrak{V}}$. By construction, we have:

$$\begin{aligned} D_{R,\mathfrak{V},Q,\mathfrak{U}}(x,y,[r]) &= a(1,1_x\circ\mathcal{H}^\dagger(e_{R_{\kappa_x},\kappa_x}(G_{Q,\mathfrak{U}}(y)\Theta E([r],\kappa_x\kappa_y^{-1}[2r]^{-1})))) \\ &= a(1,1_{f_x}(\mathcal{H}^\dagger(e_{R_{\kappa_x},\kappa_x}(f_y\delta^r_{k_x-k_y-2r}E^{(p)}_{k_x-k_y-2r})))) \\ &= a(1,1_{f_x}\circ e_{R_{\kappa_x},\kappa_x}(\mathcal{H}^\dagger(g))) \end{aligned}$$

with $g = f_y \delta^r_{k_x-k_y-2r} E^{(p)}_{k_x-k_y-2r}(\psi_x\psi_y^{-1})$. Since g is nearly holomorphic of order $\leq r$ and weight $k_x > 2r$, we have $\mathcal{H}^\dagger(g) = \mathcal{H}(g)$ is holomorphic. Since $\mathcal{H}(g)$ is an holomorphic form of level p^n with $n = Max(m_x, m_y)$, we have

$$\begin{aligned} (4.4.7.b)\quad D_{R,\mathfrak{V},Q,\mathfrak{U}}(x,y,[r]) &= a(1, 1_{f_x} \circ e_{R_{\kappa_x},\kappa_x}\mathcal{H}(g)) \\ &= a(p,f_x)^{m_x-n}.p^{(n-m_x)(\frac{k_x}{2}-1)}.\frac{\langle f_x^\rho|\tau_{p^n}, \mathcal{H}(g)\rangle_{p^n}}{\langle h_x, f_x\rangle_{p^{m_x}}} \\ &= a(p,f_x)^{m_x-n}.p^{(n-m_x)(\frac{k_x}{2}-1)}.\frac{\langle f_x^\rho|\tau_{p^n}, g\rangle_{p^n}}{\langle h_x, f_x\rangle_{p^{m_x}}} \end{aligned}$$

As in [Hi93], we now transform f_y a little:

$$\begin{aligned} f_y &= (-1)^{k_y} f_y|\tau_{p^n}|\tau_{p^n} = (-1)^{k_y}.p^{\frac{k_y}{2}(n-m_y)}.f_y|\tau_{p^{m_y}}[p^{n-m_y}]|\tau_{p^n} \\ &= (-1)^{k_y} W(f_y^\rho).p^{\frac{k_y}{2}(n-m_y)}.f_y^c|[p^{n-m_y}]|\tau_{p^n} \end{aligned}$$

By replacing this in the expression above we get for (4.4.7.b):

$$\begin{aligned} (4.4.7.c)\qquad & a(p,f_x)^{m_x-n}.(-1)^{k_y}W(f_y^\rho).p^{(n-m_x)(\frac{k_x}{2}-1)}.p^{\frac{k_y}{2}(n-m_y)}. \times \\ & \frac{\langle f_x^\rho|\tau_{p^n}, f_y^c|[p^{n-m_y}]|\tau_{p^n} E^{(p)}_{k_x-k_y-2r}(\psi_x\psi_y^{-1})\rangle_{p^n}}{\langle h_x, f_x\rangle_{p^{m_x}}} = \\ & a(p,f_x)^{m_x-n}.(-1)^{k_y}W(f_y^\rho).p^{n(k_y+r)+m_x(1-\frac{k_x}{2})-m_y\frac{k_y}{2}} \times \\ & (k_x+k_y+r+1)!r!.\frac{D_{p^n}(f_x, f_y^\rho|[p^{n-m_y}], k_y+r)}{\pi^{k_y+2r+1}2^{k_x+k_y+2r}i^{k_y-k_x}\langle h_x, f_x\rangle_{p^{m_x}}} \end{aligned}$$

Now using the fact that for p dividing M, we have:

$$D_M(f, g[p^m], s) = a(p,f)^m p^{-ms} D_M(f,g,s))$$

we deduce that (4.4.7.b) is equal to

$$\begin{aligned} & a(p,f_x)^{m_x-m_y}.(-1)^{k_y}W(f_y^\rho).p^{m_x(1-\frac{k_x}{2})+m_y(r+\frac{k_y}{2})} \times \\ & (k_y+r-1)!r!.\frac{D_{p^n}(f_x, f_y^\rho|[p^{n-m_y}], k_y+r)}{\pi^{k_y+2r+1}2^{k_x+k_y+2r}i^{k_y-k_x}\langle h_x, f_x\rangle_{p^{m_x}}} \end{aligned}$$

and the specialization formula stated in the theorem follows.■

Remark 4.4.8. a) This result is still true if x is classical and critical if it is not θ-critical. The condition (iv) and (v) are not necessary and could be removed at the expanse to modify the formula by adding some Euler factors at p.

b) From the construction, we see that this meromorphic function has possible poles along certain hypersurfaces of $\mathcal{E}\times\mathcal{E}\times\mathfrak{X}$ corresponding to intersections of the irreducible components of the first variable and also along certain hypersurfaces created by the overconvergent projection. This happens when the overconvergent form f_x is at the same time the specialization of a family of overconvergent forms and a family of positive order nearly overconvergent forms. It is easy to see that implies x is θ-critical. In the next section, we review the definition of a θ-critical point and compute the residue of $\mathcal{D}$ when the weight map at this point is étale.

4.4.9. *Residue at an étale θ-critical point.* Let $x_1 \in \mathcal{E}(L)$ of classical weight $k_1 \geq 2$ and slope $k_1 - 1$. We say that x is θ-critical if there exist x_0 of weight $k_0 = 2 - k_1$ such that $f_{x_1} = \Theta^{k_1-1} f_{x_0}$. Here we denote f_{x_0} the ordinary form of weight $k_0 = 2 - k_1$ attached to x_0. We then write $x_1 = \theta(x_0)$. We have the following result.

Theorem 4.4.10. *Let x_0 and x_1 as above. Assume that $\kappa : \mathcal{E} \to \mathfrak{X}$ is étale at $x_1 = \theta(x_0)$ or equivalently that $\mathcal{E}$ is smooth at x_0. Then the order of the pole of $\mathcal{D}(x, y, \nu)$ at x_1 is at most one and*

$$Res|_{x=x_1}(\mathcal{D}(x,y,\nu)) = \frac{\prod_{j=0}^{k_1-2}(Log(\nu\kappa_y) - j)(Log(\nu) - j)}{(k_1-1)!}\mathcal{D}(x_0, y, \nu[1-k_1]) \tag{4.4.10.a}$$

for all $(y, \nu) \in \mathcal{E} \times \mathfrak{X}$

Proof. The fact that $\mathcal{E} \xrightarrow{\kappa} \mathfrak{X}$ is étale at x_1 is equivalent to $\mathcal{E}$ smooth at x_0 is well-known and follows from R. Coleman's work.

We choose $(R_0, \mathfrak{V}_0)$ such that $x_0 \in \mathcal{E}_{R_0,\mathfrak{V}_0}(L)$. Consider the pair $(R_1, \mathfrak{V}_1)$ with $R_1(\kappa, X) = R_0(\kappa[2-2k_1], p^{k_1-1}X)$ and $\mathfrak{V}_1 = \mathfrak{V}_0[2k_1 - 2]$.

For $i = 0, 1$, let $\mathcal{C}_i$ be the (unique) irreducible component of $\mathcal{E}_{R_i,\mathfrak{V}_i}$ containing x_i and let consider $F_i = F_{\mathcal{C}_i}$ the corresponding Coleman family. Let $G = G^E_{R_1,\mathfrak{V}_1,Q,\mathfrak{U}}$ for some admissible pair $(Q, \mathfrak{U})$.

Let $\mathcal{F}$ be an extension of $F(\mathfrak{V})$ as in §4.2.2. Then by definition of $D = D_{C_1,Q,\mathfrak{U}}$ and of the overconvergent projection, we have

$$G = D.F_1 + D'\delta^{k_1-1}F_0 + H \tag{4.4.10.b}$$

with some $D' \in \mathcal{F} \otimes A^b(\mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X})$ and $H \in \mathcal{N}_{R_1,\mathfrak{V}_1} \otimes A^b(\mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X})$ such that $(H, F_i)_{\mathcal{F}} = 0$ for $i = 0, 1$ where $(-,-)_{\mathcal{F}}$ is the p-adic Petersson inner product defined in §4.2.2. Notice that by our hypothesis, $F_1(x_1) = f_{x_1} = \delta^{k_1-1}F_0(x_0)$. Since G is regular at $[k_1]$ and F_1 and $\delta^{k_1-1}F_0$ are the only families of nearly overconvergent forms of finite slope specializing to f_{x_1}, this implies that $D + D'$ is regular at x_1. Therefore in particular the order of the pole of D at x_1 is the same as the order of the pole of D' at x_1 and

$$Res|_{x=x_1}(D(x,y,\nu)) = -Res|_{x=x_1}(D'(x,y,\nu)) \tag{4.4.10.c}$$

From (4.4.10.b), we have

$$\epsilon^{k_1-1}G(\kappa_x, y, \nu) = \prod_{j=0}^{k_1-2}(2 - Log(\kappa_x) + j)D'(x,y,\nu).F_0 + \epsilon^{k_1-1}H \tag{4.4.10.d}$$

Since the eigencurve is smooth at x_0, this implies that $\prod_{j=0}^{k_1-2}(2 - Log(\kappa_x) + j)D'$ is regular at $x = x_1$ and therefore the pole of D' at x_1 is at most simple. Moreover, we get

$$Res|_{x=x_1}(D'(x,y,\nu)) = \frac{(-1)^{k_1}}{(k_1-1)!}.a(1, 1_{\mathcal{C}_0}.\epsilon^{k_1-1}G([k_1], y, \nu)) \tag{4.4.10.e}$$

Now we want to evaluate $\epsilon^{k_1-1}G(\kappa, y, \nu)$. For any classical triplet $(\kappa, y, \nu) = ([k], y, [r])$ with $k - k_y > 2r \geq 0$ and $\psi = \psi_y \psi_{x_1}$, we deduce from the evaluation of (2.4.2.b) at $X = 0$ that

$$\begin{aligned} \epsilon^{k_1-1}G(x, y, \nu)(q) &= \epsilon^{k_1-1}e_{R_1,\mathfrak{V}_1} f_y \Theta^r.E_{k-k_y-2r}(\psi)(q) \\ &= e_{R_0,\mathfrak{V}_0}\epsilon^{k_1-1} f_y \Theta^r.E_{k-k_y-2r}(\psi)(q) \\ &= e_{R_0,\mathfrak{V}_0} f_y \epsilon^{k_1-1}\Theta^r.E_{k-k_y-2r}(\psi)(q) \\ &= \tfrac{\Gamma(k-k_y-r)r!}{\Gamma(k-k_y-r-k_1+1)(r-k_1+1)!} e_{R_0,\mathfrak{V}_0} f_y \Theta^{r-k_1+1}E_{k-k_y-2r}(\psi)(q) \end{aligned}$$

We deduce that

$$\epsilon^{k_1-1}G(\kappa, y, \nu) = \prod_{j=1}^{k_1-1}(Log(\kappa\kappa_y^{-1}\nu^{-1}) - j)(Log(\nu) - j + 1)G^E_{R_0,\mathfrak{V}_0,Q,\mathfrak{U}}(\kappa[2-2k_1], y, \nu[1-k_1])$$

since the left and right hand sides of the above have the same evaluations on a Zariski dense set of point of $\mathfrak{X} \times \mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X}$. Evaluating at $\kappa = [k_1]$ gives:

(4.4.10.f)

$$\epsilon^{k_1-1}G([k_1], y, \nu) = \prod_{j=0}^{k_1-2}(j - Log(\nu\kappa_y))(Log(\nu) - j)G^E_{R_0,\mathfrak{V}_0,Q,\mathfrak{U}}([2-k_1], y, \nu[1-k_1])$$

Since

$$a(1, 1_{\mathcal{C}_0}G^E_{R_0,\mathfrak{V}_0,Q,\mathfrak{U}}([2-k_1], y, \nu)) = \mathcal{D}(x_0, y, \nu)$$

for $(y, \nu) \in \mathcal{E}_{Q,\mathfrak{U}} \times \mathfrak{X}$, the formula (4.4.10.a) follows from (4.4.10.c), (4.4.10.e) and (4.4.10.f).■

Remark 4.4.11. This residue formula has a flavor similar to the work of Bellaiche [Be12] in which it is proved that the standard p-adic L-function attached to a θ-critical point is divisible by a similar product of p-adic Log's.

Remark 4.4.12. It is also possible to define a two variable Rankin-Selberg p-adic L-function interpolating the critical values $D_p(f_x, f_y, k_x - 1)$ by replacing $\Theta.E(\kappa, \kappa')$ by $E^{ord}(\kappa)$ in our construction where for $\kappa = [k].\psi_\kappa$ and $k \in \mathbf{Z}_{\geq 2}$ we have

$$E^{ord}(\kappa) = \frac{L(1-k, \psi)}{2} + \sum_{n=1}^{\infty}(\sum_{\substack{d|n \\ (n,p)=1}} \kappa(d)d^{-1})q^n.$$

Since $E(\kappa)$ has a pole at $\kappa = [0]$, this two-variable p-adic L-function would have a pole along the hypersuface defined by $\kappa_x = \kappa_y$. It should be easy to compute the corresponding residue and obtain a formula similar to the one of Hida's Theorem 3 [Hi93, p. 228].

References

[AIP] F. Andreatta, A. Iovita and V. Pilloni, *Families of Siegel modular forms*, to appear in *Annals of Math.* .

[Be12] J. Bellaiche, *Critical p-adic L-functions*, Invent. Math. 189 (2012), no. 1, 1–60.

[BO] P. Berthelot and A. Ogus , *Notes on crystalline cohomology*, Princeton University Press ; University of Tokyo Press, Tokyo, 1978. vi+243 pp

[Co] R. Coleman, *p-adic Banach spaces and families of modular forms.* Invent. Math. 127 (1997), no. 3, pp 417–479.

[Co96] R. Coleman, *Overconvergent and classical modular forms*, Invent. math. 124 (1996), pp 215–241.

[CM98] R. Coleman and B. Mazur, *The Eigencurve*, Galois representations in arithmetic algebraic geometry (Durham, 1996), vol. 254 of London Math. Soc. Lect. Note Ser. 1-113. Cambridge Univ. Press, Cambridge, 1998.

[CGJ] R. Coleman, F. Gouvea and N. Jochnowitz, *E_2, Θ and overconvergence*, IMRN International Mathematics Research Notices 1995, No. 1

[DR] H. Darmon and V. Rotger, *Diagonal cycles and Euler systems I: A p-adic Gross-Zagier Formula*, to appear to the *Ann. Sc. Ecole Norm. Sup.*

[Ha85] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties. I*, inventiones math. 82 (1985), pp 151-189

[Ha86] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties. II*, Compositio math., tome 60 (1986), pp 323–378.

[Har66] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin,

[Hi85] H. Hida, *A p-adic measure attached to the zeta functions associated with two elliptic modular forms. I*, Invent. math. 79, 159–195 (1985)

[Hi88] H. Hida, *A p-adic measure attached to the zeta functions associated with two elliptic modular forms. II*, Annales de l'Institut Fourier, tome 38 n3, pp 1–83 (1988).

[Hi93] H. Hida, *Elementary theory of L-functions and Eisenstein series*, Cambridge Univ. Press, London Math. Soc. Student Texts 26, (1993).

[Ka70] N. Katz, *Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin*, Inst. Hautes tudes Sci. Publ. Math. No. 39 (1970), 175232.

[Ka72] N. Katz, *p-adic Properties of Modular Schemes and Modular Forms*, International Summer School on Modular Functions, Antwerp (1972).

[Na92] M. Nappari, *Holomorphic Forms Canonically Attached to Nearly Holomorphic Automorphic Forms* Thesis Brandeis University (1992).

[Pi10] V. Pilloni, *Formes modulaires surconvergentes* , to appear in the Annales de l'Institut Fourier.

[Sh04] G. Shimura, *Arithmeticity in the Theory of Automorphic Forms*, American Mathematical Society, Mathematical Surveys and Monographs, Vol. 82, (2004).

[Sh76] G. Shimura, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. 29 (1976), no. 6, 783-804.

[Sh73] G. Shimura, *IIntroduction to the arithmetic theory of automorphic functions*, Kan Memorial Lectures, No. 1. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. xiv+267 pp.

[SU] C. M. Skinner and E. Urban, *p-adic families of nearly holomorphic forms and Applications*, in preparation.

[Ur11] E. Urban, *Eigenvarieties for reductive groups*, Annals of Math. 174 (2011), pp 1685–1784.

[Ur13] E. Urban, *On the ranks of Selmer groups of elliptic curves over* $\mathbf{Q}$, in *Automorphic Representations and L-Functions*, Proceedings of the International Colloquium held at the Tata Institute of Fundamental Research, pp 651– 680 (2013)

[Ti11] J. Tilouine, *Companion forms and classicity in the $GL_2(\mathbf{Q})$-case*, Number theory, 119141, Ramanujan Math. Soc. Lect. Notes Ser., 15, Ramanujan Math. Soc., Mysore, 2011.

[Wi] A. Wiles, *On ordinary ?-adic representations associated to modular forms*, Invent. Math. 94 (1988), no. 3, 529573.

[Xi] R. Harron and L. Xiao, *Gauss-Manin connections for p-adic families of nearly overconvergent modular forms*, arXiv:1308.1732.

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

E-mail address: urban@math.jussieu.fr, urban@math.columbia.edu