Midterm review - Complex variables.

Ex 1. Determine for which values $z \in \mathbb{C}$, the following series is convergent $\sum_{n=1}^{\infty} \frac{1}{n^z}$. Show it is an open set Ω of $\mathbb{C}$ and prove that

$$z \mapsto \zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is an holomorphic function on Ω.

Ex 2 A real valued function h on an open set $\Omega \subset \mathbb{C}$ is said harmonic if it is twice differentiable with continuous second order derivatives and satisfies the equation $\Delta h = 0$ where Δ is the Laplacian differential operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

a) Show that if h is the real part of an harmonic function on Ω, then it is harmonic.

b) Conversely show that in case Ω is a disc and h is harmonic then it is the real part of an holomorphic function on Ω.

c) Let h be a non negative real valued harmonic function on $\mathbb{C}$. Show that h is constant.

Ex 3 Using a half circle contour, compute the following integral

$$\int_{\infty}^{\infty} \frac{\cos x}{1+x^2} dx$$

(Hint : express $\frac{1}{1+x^2}$ in terms of $\frac{1}{x+i}$ and $\frac{1}{x-i}$.)

Ex 4 Compute the following integral :

$$\int_C \frac{\sin z}{z^4} dz$$

with C be the unit circle of center 0 with the counterclockwise orientation.

Ex 5 Let Ω be a bounded open subset of $\mathbb{C}$ and $\phi : \Omega \to \Omega$ be an holomorphic function such that there exists $z_0 \in \Omega$ such that $\phi(z_0) = z_0$ and $\phi'(z_0) = 1$. Considering the iteration $\phi^k = \phi \circ \cdots \circ \phi$ (where ϕ appears $k$times) and using the Cauchy inequalities prove that $\phi(z) = z$ for all $z \in \Omega$.